Abstract. In this paper, we provide necessary and sufficient conditions for the oscillation of every solution of the system of delay differential equations

\[ x^{(m)}(t) + Px(t - \tau) = 0, \]

where \( P \in \mathbb{R}^{s \times s} \), \( \tau \in \mathbb{R}^+ \) and \( m \) is an odd positive integer. Furthermore, we proved sufficient conditions for the oscillation of every solution of the system of delay differential equations

\[ x^{(m)}(t) + \sum_{i=1}^{n} P_i x(t - \tau_i) = 0, \]

where \( P_i \in \mathbb{R}^{s \times s} \), \( \tau_i \in \mathbb{R}^+ \) for \( i = 1, 2, ..., n \) and \( m \) is an odd positive integer.

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1. INTRODUCTION

Recently there has been a lot of studies concerning the oscillatory behaviour of differential equations, see [1, 9] and the reference cited therein. In [8], Ladas and Sficas established that a theorem for the oscillatory behaviour of all solutions for the following differential equation

\[ x'(t) + px(t - \tau) = 0, \]

where \( p, \tau \in \mathbb{R} \). (See also [4]).
In [9], Ferreira and Ladas obtained that the oscillatory behaviour of all solutions of linear autonomous system of differential equation

\[ x'(t) + Px(t - \tau) = 0, \]

where \( P \in \mathbb{R}^{s \times s} \) and \( \tau \in \mathbb{R}^+ \). Furthermore, they obtained in [9], sufficient conditions for the oscillation of all solutions of the differential equation

\[ x'(t) + \sum_{i=1}^{n} P_i x(t - \tau_i) = 0, \]

where \( P_i \in \mathbb{R}^{s \times s} \) and \( \tau_i \in \mathbb{R}^+ \) for \( i = 1, 2, ..., n \). (See also [4]).

Our aim of this paper is to study the oscillatory behaviour of odd order delay differential equations. In section 2, we extract necessary and sufficient conditions for the oscillation of all solutions of the system of differential equation

\[ x^{(m)}(t) + Px(t - \tau) = 0, \]

where \( P \in \mathbb{R}^{s \times s}, \tau \in \mathbb{R}^+ \) and \( m \) is an odd positive integer. In section 3, we obtain sufficient conditions for the oscillation of all solutions of the system of differential equation

\[ x^{(m)}(t) + \sum_{i=1}^{n} P_i x(t - \tau_i) = 0, \]

where \( P_i \in \mathbb{R}^{s \times s}, \tau_i \in \mathbb{R}^+ \) for \( i = 1, 2, ..., n \) and \( m \) is an odd positive integer. For an \( s \times s \) matrix \( P \) the logarithmic norm of \( P \) is denoted by \( \mu(P) \) and is defined to be

\[ \mu(P) = \max_{\|\xi\|=1} (P\xi, \xi) \]

where \(( , )\) is an inner product in \( \mathbb{R}^s \) and \( \|\xi\| = (\xi, \xi)^{1/2} \).

By a solution of the equation (1.1) we mean a function \( x \in C([t_1 - \tau, \infty), \mathbb{R}] \), for some \( t_1 \geq t_0 \), such that \( x \) is continuously differentiable on \([t_1, \infty)\) and \( x \) satisfies equation (1.1) for \( t \geq t_1 \). A solution of the equation (1.2) with \( x(t) = [x_1(t), x_2(t), ..., x_s(t)]^T \) is said to oscillate if every component \( x_i(t) \) of the solution has arbitrary large zeros. Otherwise the solution is called nonoscillatory.

We need the following lemma.

**Lemma 1.1.** Assume that \( P_i \in \mathbb{R}^{s \times s} \) and \( \tau_i \in \mathbb{R}^+ \) for \( i = 1, 2, ..., n \). Then the following statements are equivalent.

(a) Every solution of (1.5) oscillates componentwise.

(b) The characteristic equation of equation (1.5)

\[ \det \left( \lambda^m I + \sum_{i=1}^{n} P_i e^{-\lambda \tau_i} \right) = 0 \]

has no real roots. Here \( I \) is the \( s \times s \) identity matrix.
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Proof. The proof of Lemma 1.1 is obtained by using Theorem 5.1.1 in [4]. (See also [3]).

2. NecEssary And SuFFicient Conditions For Oscillation Of (1.4)

In this section we obtain necessary and sufficient condition for the oscillation of equation (1.4). The conditions will be given in terms of the \( \tau \in \mathbb{R}^+ \) and \( P \in \mathbb{R}^{s \times s} \) matrix.

**Theorem 2.1.** Let \( P \in \mathbb{R}^{s \times s} \), \( \tau \in [0, \infty) \) and \( m \) is an odd positive integer. Then every solution of equation (1.4) oscillates (componentwise) if and only if one of the following conditions holds:

(i) If \( \tau = 0 \), then \( P \) has no eigenvalues \( (-\infty, \infty) \).

(ii) If \( \tau > 0 \), then \( P \) has no eigenvalues in the interval \( (-\infty, (\frac{m}{\tau^e})^m) \).

**Proof.** (i) If \( \tau = 0 \), then the characteristic equation of (1.4) becomes

\[
    f(\lambda) = \det [\lambda^m I + P] = 0,
\]

which can also be written as

\[
    f(\lambda) = \det [-\lambda^m I - P] = 0.
\]

Take the function \( v(\lambda) \) as

\[
    v(\lambda) = -\lambda^m.
\]

If \( \lambda \in \mathbb{R} \) then \( v(\lambda) \in (-\infty, \infty) \). Therefore by Lemma 1.1, equation (2.1) has no real roots if and only if \( P \) has no eigenvalues in \( (-\infty, \infty) \).

(ii) If \( \tau > 0 \), then the characteristic equation of (1.4) becomes

\[
    f(\lambda) = \det [\lambda^m I + Pe^{-\lambda \tau}] = 0,
\]

which can also be written as

\[
    f(\lambda) = \det [-\lambda^m e^{\lambda \tau} I - P] = 0.
\]

Take the function \( v(\lambda) \) as

\[
    v(\lambda) = -\lambda^m e^{\lambda \tau}
\]

It is clear that \( v'(-\frac{m}{\tau^e}) = 0 \) and \( v''(-\frac{m}{\tau^e}) < 0 \). Thus, if \( \lambda \in \mathbb{R} \), then \( v(\lambda) \in (-\infty, (\frac{m}{\tau^e})^m) \). Therefore by Lemma 1.1, equation (2.2) has no real roots if and only if \( P \) has no eigenvalues in \( (-\infty, (\frac{m}{\tau^e})^m) \).

**Remark 2.1.** For the case \( m = 1 \) in equation (1.4), in [9] Ferreira and Ladas obtained the oscillatory results for equation (1.2). (See also [4])

**Remark 2.2.** For the case \( s = 1 \) and \( m = 1 \) in equation (1.4), in [4] Ladas and Sficas investigated the oscillatory behaviour of equation (1.1). (See also [8])

**Corollary 2.2.** Let \( P \in \mathbb{R}^{s \times s} \), \( \tau \in [0, \infty) \) and \( m \) is an even positive integer. Then every solution of equation (1.4) oscillates (componentwise) if and only if \( P \) has no eigenvalues in \( (-\infty, 0] \).
Proof. It is obtained by using similar method of the proof of Theorem 2.1.

3. SUFFICIENT CONDITIONS FOR OSCILLATION OF (1.5)

In this section we obtain sufficient conditions for the oscillation of all solutions of the linear equation with the matrix coefficients of $P_1, P_2, ..., P_m$

$$x^{(m)}(t) + \sum_{i=1}^{n} P_i x(t - \tau_i) = 0.$$ 

The conditions will be given in terms of the $\tau_i$ and logarithmic norm of the matrices $P_i$ for each $i = 1, 2, ..., m$.

Lemma 3.1. Let $P_i \in \mathbb{R}^{s \times s}$, $\tau_i \in [0, \infty)$ for $i = 1, 2, ..., n$ and let $m$ is an odd positive integer. Suppose also that

$$\sum_{i=1}^{n} \mu(-P_i)e^{-\gamma \tau_i} < 0 \text{ for } \gamma \in \mathbb{R}^+$$

and

$$\inf_{\gamma < 0} \left[ \frac{1}{\gamma^m} \sum_{i=1}^{n} \mu(-P_i)e^{-\gamma \tau_i} \right] > 1.$$ 

Then every solution of equation (1.5) oscillates (componentwise).

Proof. Assume, for the sake of contradiction, that (1.5) has a nonoscillatory solution. Then, by Lemma 1.1, the characteristic equation (1.6) has a real root $\gamma_0$. Therefore there exists a vector $u \in \mathbb{R}^s$ with $\|u\| = 1$ such that

$$\left( \gamma_0^m I + \sum_{i=1}^{n} P_i e^{-\gamma_0 \tau_i} \right) u = 0.$$ 

So

$$\gamma_0^m = \left( - \sum_{i=1}^{n} P_i e^{-\gamma_0 \tau_i} u, u \right) = \sum_{i=1}^{n} (-P_i u, u) e^{-\gamma_0 \tau_i} \leq \sum_{i=1}^{n} \mu(-P_i)e^{-\gamma_0 \tau_i}.$$ 

Clearly for $\gamma_0 > 0$ we have the following

$$0 \leq \sum_{i=1}^{n} \mu(-P_i)e^{-\gamma_0 \tau_i}$$

which contradicts to (3.1). Now we consider the case $\gamma_0 < 0$. Therefore we get

$$1 \geq \frac{1}{\gamma_0^m} \sum_{i=1}^{n} \mu(-P_i)e^{-\gamma_0 \tau_i}$$

which contradicts to (3.2) and completes the proof. \qed
Theorem 3.2. Let $P_i \in R^{s \times s}$, $\tau_i \in [0, \infty)$ and $\mu(-P_i) \leq 0$ for $i = 1, 2, \ldots, n$ and let $m$ is an odd positive integer. Then every solution of equation (1.5) oscillates (componentwise) provided that one of the following two conditions is satisfied:

\[
\sum_{i=1}^{n} -\mu(-P_i)(\tau_i)^m > \frac{m^m}{e^m}
\]

and

\[
\left[ \prod_{i=1}^{n} (-\mu(-P_i)) \right]^{\frac{1}{n}} \left( \sum_{i=1}^{n} \tau_i^m \right) > \frac{m^m}{e^m}
\]

Proof. We employ Lemma 3.1. As $\mu(-P_i) \leq 0$ for $i = 1, 2, \ldots, n$, (3.1) is satisfied and so it suffices to establish (3.2). First, assume that (3.3) holds. Then, we get

\[
\sup_{\gamma<0} \gamma^m e^{-\gamma \tau_i} = -\left( \frac{e\tau_i}{m} \right)^m,
\]

hence for $i = 1, 2, \ldots, n$,

\[
\frac{1}{\gamma^m} e^{-\gamma \tau_i} \mu(-P_i) \geq -\mu(-P_i) \left( \frac{e\tau_i}{m} \right)^m,
\]

so, for $\gamma < 0$,

\[
\frac{1}{\gamma^m} \sum_{i=1}^{n} e^{-\gamma \tau_i} \mu(-P_i) \geq \sum_{i=1}^{n} -\mu(-P_i) \left( \frac{e\tau_i}{m} \right)^m.
\]

So, from this inequality and (3.3), we have (3.2).

Now, we assume that (3.4) holds. By using the arithmetic-geometric mean inequality, we have for $\gamma < 0$,

\[
\frac{1}{\gamma^m} \sum_{i=1}^{n} e^{-\gamma \tau_i} \mu(-P_i) \geq \sum_{i=1}^{n} -\mu(-P_i) \left( \frac{e\tau_i}{m} \right)^m.
\]
So, from this inequality and (3.4), we have (3.2).

**Remark 3.1.** For the case $m = 1$ in equation (1.5), in [9] Ferreira and Ladas obtained the oscillatory results for equation (1.3).

**Remark 3.2.** For the case $s = 1$ and $m = 1$ in equation (1.5), in [8] Ladas and Sficas obtained the oscillatory results for the equation

$$x'(t) + \sum_{i=1}^{n} p_i x(t - \tau_i) = 0,$$

where $p_i, \tau_i \geq 0$ for $i = 1, 2, ..., n$.

**References**


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