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Comments on 'Fixed Point Theorems for φ-Contraction in Probabilistic Metric Space'

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Abstract

In this work we have shown that an affirmative answer was already given in [1, 5] to the question raised in [4] and have extended a fixed point theorem by L. Ćirić [4] to a larger class of PM spaces. In the final part of the paper we have shown that the result can be yet improved by a common fixed point theorem for a semigroups of φ-probabilistic contractions.

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1 Introduction

The notion of φ-probabilistic contractions were first defined and studied by Mbarki et al. [1, 5]. Moreover, in [5] he found that the φ'- contraction mappings are a particular type of φ-probabilistic contractions and gave the relationship between φ and φ'.

In this paper, we have shown that an affirmative answer was already given in [1, 5] to the question " Whether the Banach fixed point principle for k-probabilistic contractions is also true for φ-probabilistic contractions without the hypothesis that $\varphi \in \left\{\varphi : [0,\infty) \to [0,\infty) \middle| \sum_{i=0}^{\infty} \varphi^i(t) < +\infty \text{ for all } t > 0\right\}$? " raised by L. Ćirić in [4]. This is done with the help of " A Picard iterates of φ-probabilistic contractions is a Cauchy sequence iff it is bounded sequence". In particular, we extend a recent result of L. Ćirić [4] who formulated a new general class of φ-probabilistic contractions.

2 Basic concepts and lemmas

we briefly recall some definitions and known results in probabilistic metric space . As in [6] a nonnegative real function f defined on $[0,\infty]$ is called a distance distribution function (briefly, a d.d.f) if it is nondecreasing, left continuous on $(0,\infty)$, with $f(0) = 0$ and $f(\infty) = 1$. The set of all d.d.f's will be denoted by $\triangle^+$; and the set of all $f \in \triangle^+$ for which $\lim_{s\to\infty} f(s) = 1$ by D^+.

Example 2.1 *For $a \in [0,\infty]$, the unit step at a is the function ϵ_a defined as*

$$\epsilon_a(x) = \begin{cases} 0, & \textit{if} \quad x \le a, \quad \textit{for } 0 \le a < \infty \\ 1, & \textit{if} \quad x > a \end{cases}$$

and

$$\epsilon_\infty(x) = \begin{cases} 0, & \text{if } 0 \le x < \infty, \\ 1, & \text{if } \quad x = \infty \end{cases}$$

Definition 2.2 *We say that τ is a triangle function on Δ^+ if assigns a d.d.f. in Δ^+ to every pair of d.d.f's in $\Delta^+ \times \Delta^+$ and satisfies the following conditions:*

$$\begin{aligned} \tau(F,G) &= \tau(G,F), \\ \tau(F,G) &\le \tau(K,H) \qquad \text{whenever } F \le K, G \le H, \\ \tau(F,\epsilon_0) &= F, \\ \tau(\tau(F,G),H) &= \tau(F,\tau(G,H)). \end{aligned}$$

A t-norm is a binary operation on $[0,1]$ which is associative, commutative, nondecreasing in each place and has 1 as identity. Among the most important Examples of t-norms we point out:

$$T_L(a,b) = \max\{a+b-1,0\}, \quad T_p(a,b) = ab \quad \text{and} \quad T_M(a,b) = Min(a,b),$$

and for any t-norm T we have $T \le T_M$. If more T is left-continuous the operation $\tau_T : \Delta^+ \times \Delta^+ \to \Delta^+$ such that

$$\tau_T(f,g)(t) = sup\{T(f(u),g(v)) : u+v=t\},$$

is a triangle function.

Lemma 2.3 [6] *If T is continuous, then τ_T is continuous.*

If T is a t-norm, $x \in [0,1]$ and $n \in \mathbb{N}$ then we shall write

$$T^n(x) = \begin{cases} 1 & \text{if } n = 0, \\ T(T^{n-1}(x),x) & \text{otherwise.} \end{cases}$$

Definition 2.4 *A t-norm T is of H-type if the family $(T^n(x))_{n\in\mathbb{N}}$ is equicontinuous at the point $x=1$, i.e.,*

$$\forall \epsilon \in (0,1)\ \exists \lambda \in (0,1) : \ t > 1-\lambda \Rightarrow T^n(t) > 1-\epsilon \ \text{ for all } \ n \ge 1.$$

A trivial Example of a t-norm of H-type is T_M for more Examples (see, e.g., [2]).

Definition 2.5 *A probabilistic metric space (briefly,PM space) is a triple (M,F,τ) where M is a nonempty set , F is a function from $M \times M$ into $\triangle^+$, τ is a triangle function , and the following conditions are satisfied for all p, q, r in M,*
(i) $F_{pq} = \epsilon_0$ *iff* $p = q$,
(ii) $F_{pq} = F_{qp}$,
(iii) $F_{pq} \ge \tau(F_{pr}, F_{rq})$.
If $\tau = \tau_T$ for some t-norm T, then (M,F,τ) is called a Menger space.

Let (M, F) be a probabilistic semimetric space (i.e. *(i)* and *(ii)* are satisfied). The (ϵ, λ)-topology in (M, F) is generated by the family of neighborhoods

$$\mathcal{N} = \{N_p(\epsilon, \lambda) : p \in M, \epsilon > 0 \text{ and } \lambda > 0\},$$

where

$$N_p(\epsilon, \lambda) = \{q \in M : F_{pq}(\epsilon) > 1 - \lambda\},$$

and if the triangle function τ is continuous, then the (ϵ, λ)-topology is a Hausdorff topology [6].
Here and in the sequel, when we speak about a probabilistic metric space (M, F, τ), we always assume that τ is continuous and M be endowed with the (ϵ, λ)-topology.

Definition 2.6 *Let (M, F, τ) be a PM space. Then*
(i) *A sequence (x_n) in M is said to be convergent to $x \in M$ (we write $(x_n) \to x$) if for any given $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n x}(\lambda) > 1 - \epsilon$ whenever $n \geq N$.*
(ii) *A sequence (x_n) in M is said to be strong Cauchy sequence if for any $\epsilon > 0$ and $\lambda > 0$, there exists a positive integer $N = N(\epsilon, \lambda)$ such that $F_{x_n x_m}(\lambda) > 1 - \epsilon$ whenever $n, m \geq N$.*
(iii) *A PM space (M, F, τ) is said to be complete if each Cauchy sequence in M is convergent to some point in M.*

Definition 2.7 *Let A be a nonempty subset of a PM space (X, F, τ). The probabilistic diameter of A is the function $D_A defined$ on $[0, \infty]$ by*

$$D_A(s) = \begin{cases} \lim_{t \to s^-} \varphi_A(t) & \text{for} \quad 0 \leq s < \infty \\ 1 & \text{for} \quad s = \infty, \end{cases}$$

where

$$\varphi_A(t) = \inf\{F_{pq}(t) | p, q \quad in \quad A\}.$$

It is immediate that D_A is in $\triangle^+$ for any $A \subseteq X$, and for all p, q in A, $F_{pq} \geq D_A$. A nonempty set A in a PM space is bounded if D_A is in D^+.

3 φ-probabilistic contraction mapping

Throughout this paper, (M, F) be a probabilistic semimetric space and f is a selfmap on M. Power of f are defined by $f^0 x = x$ and $f^{n+1} x = f(f^n x)$, $n \geq 0$. When there is no risk of ambiguity, we will use the notation $x_n = f^n x$, in particular $x_0 = x$, $x_1 = fx$. The set $\{f^n x : n = 1.2.3...\}$ is called an orbit

(starting at x) and denoted $\mathcal{O}_f(x)$.
The letter Ψ denotes the set of all function $\varphi : [0,\infty) \to [0,\infty)$ such that

$$\varphi(0) = 0, \quad \varphi(t) < t \text{ and } \liminf_{r\to t^+} \varphi(r) < t \quad \forall t > 0.$$

We denote by Φ the set of functions $\phi : [0,\infty) \to [0,\infty)$ such that

$$\phi(0) = 0, \quad \phi(t) < t \text{ and } \limsup_{r\to t} \phi(r) < t \quad \forall t > 0.$$

Clearly, $\Phi \subset \Psi$.

The letter Ω will be reserved for the set of functions satisfying:
(Ω_1) $\delta : [0,\infty] \to [0,\infty]$ is lower semi-continuous from the left, nondecreasing and $\delta(0) = 0$;
(Ω_2) For each $t \in (0,\infty)$, $\delta(t) > t$ and $\delta(+\infty) = +\infty$.

Given a function : $\varphi : [0,\infty) \to [0,\infty)$ such that $\varphi(t) < t$ for $t > 0$, and a selfmap f of a probabilistic semimetric space (M, F), we say that f is φ-probabilistic contraction if

$$F_{fpfq}(\varphi(t)) \geq F_{pq}(t). \tag{1}$$

for all $p, q \in M$ and $t > 0$,

Follows [5], we also have the following Definition

Definition 3.1 *Let (M, F, τ) be a PM space. For $\delta \in \Omega$, a mapping $f : M \to M$ is called δ-probabilistic contraction in the sense of Mbarki if*

$$F_{fpfq}(t) \geq F_{pq}(\delta(t)). \tag{2}$$

for all $p, q \in M$ and $t > 0$.

Next, we show the following

Lemma 3.2 *Every φ-probabilistic contraction with $\varphi \in \Phi$ is δ-probabilistic contraction in the sense of Mbarki*

Proof. Let f be a φ-probabilistic contraction with $\varphi \in \Phi$. By [3, Lemma 1], there exists a strictly increasing and continuous function $\phi : [0,\infty) \to [0,\infty)$ such that

$$\varphi(t) < \phi(t) < t$$

for all $t > 0$. Hence, it is easy to check that f is a δ-probabilistic contraction in the sense of Mbarki where δ defined as

$$\delta(t) = \begin{cases} \phi^{-1}(t), & \text{if} \quad 0 \leq t < \lim_{t\to\infty} \phi(t), \\ +\infty, & \text{if} \quad t \geq \lim_{t\to\infty} \phi(t) \end{cases}$$

We shall make frequent use of the followings Lemmas

Lemma 3.3 [4] *If a function $\varphi \in \Psi$, then*

$$\lim_{n\to\infty} \varphi^n(t) = 0 \text{ for all } t > 0.$$

Lemma 3.4 [4] *Let (M, F, τ) be a PM space where $RanF \subset D^+$. Let $x, y \in M$, if there exist $\varphi \in \Psi$ such that*

$$F_{xy}(\varphi(t)) = F_{xy}(t) \text{ for all } t > 0,$$

then $x = y$.

4 Fixed point theorems

We begin with two auxiliary results concerning the orbit of φ-probabilistic contraction mappings.

Lemma 4.1 *Let (M, F, τ) be a PM space such that $RanF \subset D^+$. Every Cauchy sequences is bounded sequence.*

Proof. Let $\{x_n\}$ be a Cauchy sequence. Given $\epsilon > 0$, then for $t > 0$ there is N such that

$$F_{x_n x_m}(t) > 1 - \epsilon, \tag{3}$$

whenever $n, m \geq N$.
Since $RanF \subset D^+$, there exists $t^{'} > t$ such that

$$F_{x_n x_m}(t^{'}) > 1 - \epsilon \text{ for all } n, m < N. \tag{4}$$

So from (3) and (4), we have

$$\begin{aligned} F_{x_n x_m}(t^{'}) &\geq F_{x_n x_m}(t^{'}) \\ &> 1 - \epsilon, \end{aligned}$$

for all $n, m \in \mathbb{N}$. So

$$\varphi_{\mathcal{O}(x)}(t^{'}) > 1 - \epsilon.$$

Next, for $s > t^{'}$

$$\begin{aligned} \varphi_{\mathcal{O}(x)}(s^{'}) &\geq \varphi_{\mathcal{O}(x)}(t^{'}) \\ &> 1 - \epsilon. \end{aligned}$$

for all $s^{'}$ such that $s > s^{'} > t^{'}$. Letting $s^{'} \to s$ we obtain

$$D_{\mathcal{O}(x)}(s) > 1 - \epsilon.$$

Since this for an arbitrary $\epsilon > 0$, there is $s > 0$ such that

$$D_{\mathcal{O}(x)}(s) > 1 - \epsilon.$$

Hence

$$D_{\mathcal{O}(x)}(s) \to 1 \text{ as } s \to \infty.$$

This completes the proof.

Conversely, we have the following

Lemma 4.2 *Let Let (M, F, τ) be a PM space where $RanF \subset D^+$ and f is a φ-probabilistic contraction mapping on M with $\varphi \in \Psi$. If the orbit $\mathcal{O}_f(x)$ for some $x \in M$ is bounded, then $\{f^n(x)\}$ is a Cauchy sequence.*

Proof. Let $n, m \in \mathbb{N}$ such that $m > n$ and $t > 0$.

$$\begin{array}{rcl} F_{x_n x_m}(\varphi^n(t)) & \geq & F_{x_{n-1} x_{m-1}}(\varphi^{n-1}(t)) \\ & \vdots & \\ & \geq & F_{x_0 x_{m-n}}(t) \\ & \geq & D_{\mathcal{O}_f(x)}(t). \end{array}$$

Let $\lambda > 0$ and $\epsilon \in (0, 1)$ be given, since $D_{\mathcal{O}_f(x)}(t) \to 1$ as $t \to \infty$ there exist $t_0 > 0$ such that

$$D_{\mathcal{O}_f(x)}(t_0) > 1 - \epsilon.$$

Since $\varphi^n(t_0) \to 0$ as $n \to \infty$, there is $N \in \mathbb{N}$ such that

$$\varphi^n(t_0) < \lambda \text{ whenever } n \geq N,$$

then

$$\begin{array}{rcl} F_{x_n x_m}(\lambda) & \geq & F_{x_n x_m}(\varphi^n(t_0)) \\ & \geq & D_{\mathcal{O}_f(x)}(t_0) \\ & > & 1 - \epsilon. \end{array}$$

Thus we proved that for each $\lambda > 0$ and $\epsilon \in (0, 1)$ there exists a positive integer N such that

$$F_{x_n x_m}(\lambda) > 1 - \epsilon \text{ for all } n, m \geq N.$$

This means that $\{x_n\}$ is a Cauchy sequence.

As consequence of Lemma 4.1 we have

Lemma 4.3 *Let (M, F, T) be a Menger space where $RanF \subset D^+$ and f is a φ-probabilistic contraction mapping on M with $\varphi \in \Psi$. If the t-norm T is the H-type, then for all $x \in M$, the orbit $\mathcal{O}_f(x)$ is bounded.*

Proof. Using the same arguments as in the proof of [4, Theorem 12], we show that $\{x_n\}$ is a Cauchy sequence. Hence and by Lemma 4.1, we concluded that $\mathcal{O}_f(x)$ is bounded.

Next, recall the main result of [1]

Theorem 4.4 *Let Let (M, F, τ) be a complete PM space where $RanF \subset D^+$ and f is a δ-probabilistic contraction mapping on M in the sense of Mbarki. If the orbit $\mathcal{O}_f(x)$ for some $x \in M$ is bounded, then f has a unique fixed point z, moreover, the sequence $\{f^n x\}$ converges to z.*

As consequences of Theorem 4.4, Lemma 3.2 and Lemma 4.3, we have the following

Corollary 4.5 *Let (M, F, T) be a complete Menger space where $RanF \subset D^+$ under a t-norm T of H-type and f is a φ-probabilistic contraction mapping on M with $\phi \in \Phi$. Then f has a unique fixed point z, moreover, the sequence $\{f^n x\}$ converges to z.*

In view of above Corollary it is very much clear that Theorem 4.4 give an affirmative answer raised by L. Ćirić in [4]. We also have the following result.

Theorem 4.6 *Let Let (M, F, τ) be a complete PM space where $RanF \subset D^+$ and f is a φ-probabilistic contraction mapping on M with $\varphi \in \Psi$. If the orbit $\mathcal{O}_f(x)$ for some $x \in M$ is bounded, then f has a unique fixed point z, moreover, the sequence $\{f^n x\}$ converges to z.*

Proof. Let $x \in M$ such that $\mathcal{O}_f(x)$ is a bounded sequence, by Lemma 4.2 $\{x_n\}$ is a Cauchy sequence. Since (M, F, τ) is complete, $\{x_n\}$ converges to some $z \in M$.
Now we shall show that z is a fixed point of f.
Let $t > 0$, then

$$F_{x_n fz}(\varphi(t)) \geq F_{x_{n-1}z}(t),$$

therefore

$$F_{x_n fz}(t) \geq F_{x_{n-1}z}(t),$$

letting $n \to \infty$, we get $z = fz$.
To complete the proof we need to show that z is unique. Indeed, let u be another fixed point of f and $t > 0$ then

$$F_{uz}(t) \geq F_{uz}(\varphi(t)) \text{ and } F_{fufz}(\varphi(t)) \geq F_{uz}(t),$$

thus $F_{uz}(\varphi(t)) = F_{uz}(t)$. Hence by Lemma 3.4 $u = z$.

Remark 4.7 *Note that the hypothesis "PM space (M, F, τ) has the property that $RanF \subset D^+$" is a necessary condition for the uniqueness of fixed points when they exist. Indeed consider $M = \{p, q\}$ and $F_{pq} = \frac{1}{2}\epsilon_0 + \frac{1}{2}\epsilon_\infty$, then the identity function on M is probabilistic contraction mapping on M with two fixed points.*
- The condition-hypothesis that there exist $x \in M$ such that $\mathcal{O}_f(x)$ is bounded it is necessary condition of the existence of fixed point as the following Sherwood's Example [7] shows

Example 4.8 *Let G be the distribution function defined by*

$$G(t) = \begin{cases} 0, & \text{if} \quad t \leq 4, \\ 1 - \frac{1}{n}, & \text{if} \quad 2^n < t \leq 2^{n+1} \quad n > 1. \end{cases}$$

Consider the set $M = \{1, 2, ..., n, ..\}$ and define F on $M \times M$ as follows

$$F_{n,m+n}(t) = \begin{cases} 0, & \text{if} \quad t \leq 0, \\ T_L^m(G(2^n t), G(2^{n+1}t), ..., G(2^{n+m}t)), & t > 0. \end{cases}$$

Then $(X; F; T_L)$ is a complete PM space and the mapping $g(n) = n + 1$ is φ-contractive with $\varphi(t) = \frac{1}{2}t$. But g is fixed point free mapping. Since there does not exist n in X, such that $\mathcal{O}_g(x)$ is bounded.

As direct consequences of Theorem 4.6 and Lemma 4.3, we obtain the following.

Corollary 4.9 [4, Theorem 12]. *Let (M, F, T) be a complete Menger space where $RanF \subset D^+$ under a t-norm T of H-type and f is a φ-probabilistic contraction mapping on M. Then f has a unique fixed point z, moreover, the sequence $\{f^n x\}$ converges to z.*

5 Common fixed point Theorem

Let S be a semigroup of selfmaps on (M, F, τ). For any $x \in M$, the orbit of x under S starting at x is the set $\mathcal{O}(x)$ defined to be $\{x\} \cup Sx$, where Sx is the set $\{g(x) : g \in S\}$. We say that S is left reversible if, for any f, g in S, there are a, b such that $fa = gb$. It is obvious that left reversibility is equivalent to the statement that any two right ideals of S have nonempty intersection. Finally, we say that S is φ-probabilistic contraction if there exists a function φ such that for each g in S, g is φ-probabilistic contraction.

Theorem 5.1 *Suppose S is a left reversible semigroup of selfmaps on M such that the following conditions (i) and (ii) are satisfied*

i. There exists x in M such that the orbit $\mathcal{O}(x)$ is bounded;
ii. S is φ-probabilistic contraction with $\varphi \in \Psi$;
then S have a unique common fixed point z and, moreover, the sequence $\{g^n x\}$ converges to z for each g in S.

Proof. It follows from Theorem 4.6 that each g in S has a unique fixed point z_g in M and for any $x \in M$, the sequence of iterates $(g^n x)$ converges to z_g. So, to complete the proof it suffices to show that $z_f = z_g$ for any $f, g \in S$.
Let n be an arbitrary positive integer. The left reversibility of S shows that are a_n and b_n in S such that $f^n a_n = g^n b_n$, then

$$F_{z_f z_g} \geq \tau(F_{z_f f^n a_n x}, F_{g^n b_n x z_g}), \tag{5}$$

and

$$F_{z_f f^n a_n x} \geq \tau(F_{z_f f^n x}, F_{f^n x f^n a_n x}). \tag{6}$$

Next we shall show that $F_{f^n x f^n a_n x} \to \epsilon_0$ as $n \to \infty$.
Let $\lambda > 0$ and $\epsilon \in (0,1)$ be given, since $\mathcal{O}(x)$ is bounded, then there is $t > 0$ such that

$$D_{\mathcal{O}(x)}(t) > 1 - \epsilon$$

and since $\varphi^n(t) \to 0$ as $n \to \infty$ there exists a positive integer N such that

$$\varphi^n(t) > \lambda \text{ whenever } n \geq N.$$

So

$$\begin{array}{rcl} F_{f^n x f^n a_n x}(\lambda) & \geq & F_{f^n x f^n a_n x}(\varphi^n(t)) \\ & \geq & F_{x a_n x}(t) \\ & \geq & D_{\mathcal{O}(x)}(t) \\ & > & 1 - \epsilon. \end{array}$$

This means that $F_{f^n x f^n a_n x} \to \epsilon_0$ as $n \to \infty$. Letting $n \to \infty$ in the inequality (6) we get $F_{z_f f^n a_n x} \to \epsilon_0$.
Likewise, we also have $F_{g^n b_n x z_g} \to \epsilon_0$, which implies that, as $n \to \infty$ in (5) we obtain that $z_f = z_g$. This completes the proof of Theorem 5.1.

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