Comments on 'Fixed Point Theorems for $\varphi$-Contraction in Probabilistic Metric Space'

Abderrahim Mbarki

National school of Applied Sciences
P.O. Box 669, Oujda University, Morocco
MATSİ Laboratory
ambarki@ensa.univ-oujda.ac.ma

Abdelhak Benbrik

Department of Mathematics
Oujda university, 60000 Oujda, Morocco
MATSİ Laboratory
benbrik@sciences.univ-oujda.ac.ma

Abedelmalek Ouahab

Department of Mathematics
Oujda university, 60000 Oujda, Morocco
MATSİ Laboratory
ouahab05@yahoo.fr

Wafa Ahid

Department of Mathematics
Oujda university, 60000 Oujda, Morocco
MATSİ Laboratory
ahid.wafa@gmail.com

Tahiri Ismail

Department of Mathematics
Oujda university, 60000 Oujda, Morocco
MATSİ Laboratory
tahiri.ismail@mail.com
Abstract
In this work we have shown that an affirmative answer was already given in [1, 5] to the question raised in [4] and have extended a fixed point theorem by L. Ćirić [4] to a larger class of PM spaces. In the final part of the paper we have shown that the result can be yet improved by a common fixed point theorem for a semigroups of \(\varphi\)-probabilistic contractions.

Mathematics Subject Classification: Primary 47H10. Secondary 54H25, 34B15

Keywords: Fixed point, Probabilistic metric space, \(\varphi\)-probabilistic contractions

1 Introduction
The notion of \(\varphi\)-probabilistic contractions were first defined and studied by Mbarki et al. [1, 5]. Moreover, in [5] he found that the \(\varphi'\)-contraction mappings are a particular type of \(\varphi\)-probabilistic contractions and gave the relationship between \(\varphi\) and \(\varphi'\).

In this paper, we have shown that an affirmative answer was already given in [1, 5] to the question "Whether the Banach fixed point principle for \(k\)-probabilistic contractions is also true for \(\varphi\)-probabilistic contractions without the hypothesis that \(\varphi \in \{ \varphi : [0, \infty) \to [0, \infty) \mid \sum_{i=0}^{\infty} \varphi^i(t) < +\infty \text{ for all } t > 0 \} \) ?" raised by L. Ćirić in [4]. This is done with the help of "A Picard iterates of \(\varphi\)-probabilistic contractions is a Cauchy sequence iff it is bounded sequence". In particular, we extend a recent result of L. Ćirić [4] who formulated a new general class of \(\varphi\)-probabilistic contractions.

2 Basic concepts and lemmas

we briefly recall some definitions and known results in probabilistic metric space. As in [6] a nonnegative real function \(f\) defined on \([0, \infty]\) is called a distance distribution function (briefly, a d.d.f) if it is nondecreasing, left continuous on \((0, \infty)\), with \(f(0) = 0\) and \(f(\infty) = 1\). The set of all d.d.f’s will be denoted by \(\Delta^+\); and the set of all \(f \in \Delta^+\) for which \(\lim_{s \to \infty} f(s) = 1\) by \(D^+\).

Example 2.1 For \(a \in [0, \infty]\), the unit step at \(a\) is the function \(\epsilon_a\) defined as

\[
\epsilon_a(x) = \begin{cases} 
0, & \text{if } x \leq a, \text{ for } 0 \leq a < \infty \\
1, & \text{if } x > a
\end{cases}
\]
and
\[ \epsilon_\infty(x) = \begin{cases} 0, & \text{if } 0 \leq x < \infty, \\ 1, & \text{if } x = \infty \end{cases} \]

**Definition 2.2** We say that \( \tau \) is a triangle function on \( \Delta^+ \) if assigns a d.d.f. in \( \Delta^+ \) to every pair of d.d.f’s in \( \Delta^+ \times \Delta^+ \) and satisfies the following conditions:

\[
\begin{align*}
\tau(F,G) &= \tau(G,F), \\
\tau(F,G) &\leq \tau(K,H) \quad \text{whenever } F \leq K, G \leq H, \\
\tau(F,\epsilon_0) &= F, \\
\tau(\tau(F,G),H) &= \tau(F,\tau(G,H)).
\end{align*}
\]

A t-norm is a binary operation on \([0,1]\) which is associative, commutative, nondecreasing in each place and has 1 as identity. Among the most important Examples of t-norms we point out:

\[ T_L(a,b) = \max\{a+b-1,0\}, \quad T_p(a,b) = ab \quad \text{and} \quad T_M(a,b) = \min(a,b), \]

and for any t-norm \( T \) we have \( T \leq T_M \). If more \( T \) is left-continuous the operation \( \tau_T : \Delta^+ \times \Delta^+ \to \Delta^+ \) such that

\[ \tau_T(f,g)(t) = \sup\{T(f(u),g(v)) : u + v = t\}, \]

is a triangle function.

**Lemma 2.3** [6] If \( T \) is continuous, then \( \tau_T \) is continuous.

If \( T \) is a t-norm, \( x \in [0,1] \) and \( n \in \mathbb{N} \) then we shall write

\[ T^n(x) = \begin{cases} 1 & \text{if } n = 0, \\
T(T^{n-1}(x),x) & \text{otherwise.} \end{cases} \]

**Definition 2.4** A t-norm \( T \) is of H-type if the family \( (T^n(x))_{n \in \mathbb{N}} \) is equicontinuous at the point \( x = 1 \), i.e.,

\[ \forall \epsilon \in (0,1) \exists \lambda \in (0,1) : t > 1 - \lambda \Rightarrow T^n(t) > 1 - \epsilon \quad \text{for all } n \geq 1. \]

A trivial Example of a t-norm of H-type is \( T_M \) for more Examples (see, e.g., [2]).

**Definition 2.5** A probabilistic metric space (briefly, PM space) is a triple \( (M,F,\tau) \) where \( M \) is a nonempty set , \( F \) is a function from \( M \times M \) into \( \Delta^+ \), \( \tau \) is a triangle function , and the following conditions are satisfied for all \( p, q, r \) in \( M \),

(i) \( F_{pq} = \epsilon_0 \) iff \( p = q \),
(ii) \( F_{pq} = F_{qp} \),
(iii) \( F_{pq} \geq \tau(F_{pr},F_{rq}) \).

If \( \tau = \tau_T \) for some t-norm \( T \), then \( (M,F,\tau) \) is called a Menger space.
Let \((M, F)\) be a probabilistic semimetric space (i.e. (i) and (ii) are satisfied). The \((\epsilon, \lambda)\)-topology in \((M, F)\) is generated by the family of neighborhoods
\[
\mathcal{N} = \{ N_p(\epsilon, \lambda) : p \in M, \epsilon > 0 \text{ and } \lambda > 0 \},
\]
where
\[
N_p(\epsilon, \lambda) = \{ q \in M : F_{pq}(\epsilon) > 1 - \lambda \},
\]
and if the triangle function \(\tau\) is continuous, then the \((\epsilon, \lambda)\)-topology is a Hausdorff topology [6].

Here and in the sequel, when we speak about a probabilistic metric space \((M, F, \tau)\), we always assume that \(\tau\) is continuous and \(M\) be endowed with the \((\epsilon, \lambda)\)-topology.

**Definition 2.6** Let \((M, F, \tau)\) be a PM space. Then
(i) A sequence \((x_n)\) in \(M\) is said to be convergent to \(x \in M\) (we write \((x_n) \to x\)) if for any given \(\epsilon > 0\) and \(\lambda > 0\), there exists a positive integer \(N = N(\epsilon, \lambda)\) such that \(F_{x_nx}(\lambda) > 1 - \epsilon\) whenever \(n \geq N\).
(ii) A sequence \((x_n)\) in \(M\) is said to be strong Cauchy sequence if for any \(\epsilon > 0\) and \(\lambda > 0\), there exists a positive integer \(N = N(\epsilon, \lambda)\) such that \(F_{x_nx_m}(\lambda) > 1 - \epsilon\) whenever \(n, m \geq N\).
(iii) A PM space \((M, F, \tau)\) is said to be complete if each Cauchy sequence in \(M\) is convergent to some point in \(M\).

**Definition 2.7** Let \(A\) be a nonempty subset of a PM space \((X, F, \tau)\). The probabilistic diameter of \(A\) is the function \(D_A\) defined on \([0, \infty]\) by
\[
D_A(s) = \left\{ \begin{array}{ll}
\lim_{t \to s^-} \varphi_A(t) & \text{for } 0 \leq s < \infty \\
1 & \text{for } s = \infty,
\end{array} \right.
\]
where
\[
\varphi_A(t) = \inf \{ F_{pq}(t) | p, q \text{ in } A \}.
\]
It is immediate that \(D_A\) is in \(\triangle^+\) for any \(A \subseteq X\), and for all \(p, q\) in \(A\), \(F_{pq} \geq D_A\). A nonempty set \(A\) in a PM space is bounded if \(D_A\) is in \(D^+\).

## 3 \(\varphi\)-probabilistic contraction mapping

Throughout this paper, \((M, F)\) be a probabilistic semimetric space and \(f\) is a selfmap on \(M\). Power of \(f\) are defined by \(f^0x = x\) and \(f^{n+1}x = f(f^n x)\), \(n \geq 0\). When there is no risk of ambiguity, we will use the notation \(x_n = f^n x\), in particular \(x_0 = x, x_1 = fx\). The set \(\{ f^n x : n = 1.2.3... \}\) is called an orbit
The letter $\Psi$ denotes the set of all function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that

$\phi(0) = 0, \quad \phi(t) < t$ and $\liminf_{r \rightarrow t^+} \phi(r) < t, \quad \forall t > 0.$

We denote by $\Phi$ the set of functions $\phi : [0, \infty) \rightarrow [0, \infty)$ such that

$\phi(0) = 0, \quad \phi(t) < t$ and $\limsup_{r \rightarrow t} \phi(r) < t, \quad \forall t > 0.$

Clearly, $\Phi \subset \Psi$.

The letter $\Omega$ will be reserved for the set of functions satisfying:

$(\Omega_1)$ $\delta : [0, \infty] \rightarrow [0, \infty]$ is lower semi-continuous from the left, nondecreasing and $\delta(0) = 0$;
$(\Omega_2)$ For each $t \in (0, \infty)$, $\delta(t) > t$ and $\delta(\infty) = \infty$.

Given a function $\varphi : [0, \infty) \rightarrow [0, \infty)$ such that $\varphi(t) < t$ for $t > 0$, and a selfmap $f$ of a probabilistic semimetric space $(M, F)$, we say that $f$ is $\varphi$-probabilistic contraction if

$$F_{fpfq}(\varphi(t)) \geq F_{pq}(t).$$

for all $p, q \in M$ and $t > 0$.

Follows [5], we also have the following Definition

**Definition 3.1** Let $(M, F, \tau)$ be a PM space. For $\delta \in \Omega$, a mapping $f : M \rightarrow M$ is called $\delta$-probabilistic contraction in the sense of Mbarki if

$$F_{fpfq}(\delta(t)) \geq F_{pq}(\delta(t)).$$

for all $p, q \in M$ and $t > 0$.

Next, we show the following

**Lemma 3.2** Every $\varphi$-probabilistic contraction with $\varphi \in \Phi$ is $\delta$-probabilistic contraction in the sense of Mbarki

**Proof.** Let $f$ be a $\varphi$-probabilistic contraction with $\varphi \in \Phi$. By [3, Lemma 1], there exists a strictly increasing and continuous function $\phi : [0, \infty) \rightarrow [0, \infty)$ such that

$\varphi(t) < \phi(t) < t$

for all $t > 0$. Hence, it is easy to check that $f$ is a $\delta$-probabilistic contraction in the sense of Mbarki where $\delta$ defined as

$$\delta(t) = \begin{cases} \phi^{-1}(t), & \text{if } 0 \leq t < \lim_{t \rightarrow \infty} \phi(t), \\ +\infty, & \text{if } t \geq \lim_{t \rightarrow \infty} \phi(t) \end{cases}$$

We shall make frequent use of the followings Lemmas
Lemma 3.3 [4] If a function $\varphi \in \Psi$, then
\[ \lim_{n \to \infty} \varphi^n(t) = 0 \text{ for all } t > 0. \]

Lemma 3.4 [4] Let $(M, F, \tau)$ be a PM space where $\text{Ran} F \subset D^+$. Let $x, y \in M$, if there exist $\varphi \in \Psi$ such that
\[ F_{xy}(\varphi(t)) = F_{xy}(t) \text{ for all } t > 0, \]
then $x = y$.

4 Fixed point theorems

We begin with two auxiliary results concerning the orbit of $\varphi$-probabilistic contraction mappings.

Lemma 4.1 Let $(M, F, \tau)$ be a PM space such that $\text{Ran} F \subset D^+$. Every Cauchy sequences is bounded sequence.

Proof. Let $\{x_n\}$ be a Cauchy sequence. Given $\epsilon > 0$, then for $t > 0$ there is $N$ such that
\[ F_{x_n x_m}(t) > 1 - \epsilon, \quad (3) \]
whenever $n, m \geq N$.

Since $\text{Ran} F \subset D^+$, there exists $t' > t$ such that
\[ F_{x_n x_m}(t') > 1 - \epsilon \text{ for all } n, m < N. \quad (4) \]

So from (3) and (4), we have
\[ F_{x_n x_m}(t') \geq F_{x_n x_m}(t') > 1 - \epsilon, \]
for all $n, m \in \mathbb{N}$. So
\[ \varphi_{O(x)}(t') > 1 - \epsilon. \]

Next, for $s > t'$
\[ \varphi_{O(x)}(s') \geq \varphi_{O(x)}(t') > 1 - \epsilon. \]

for all $s'$ such that $s > s' > t'$. Letting $s' \to s$ we obtain
\[ D_{O(x)}(s) > 1 - \epsilon. \]
\( \varphi \)-Contraction in probabilistic metric space

Since this for an arbitrary \( \epsilon > 0 \), there is \( s > 0 \) such that

\[
D_{O_f}(s) > 1 - \epsilon.
\]

Hence

\[
D_{O_f}(s) \to 1 \text{ as } s \to \infty.
\]

This completes the proof.

Conversely, we have the following

**Lemma 4.2** Let \((M, F, \tau)\) be a PM space where \( \text{Ran} F \subset D^+ \) and \( f \) is a \( \varphi \)-probabilistic contraction mapping on \( M \) with \( \varphi \in \Psi \). If the orbit \( O_f(x) \) for some \( x \in M \) is bounded, then \( \{f^n(x)\} \) is a Cauchy sequence.

**Proof.** Let \( n, m \in \mathbb{N} \) such that \( m > n \) and \( t > 0 \).

\[
F_{x_n x_m}(\varphi^n(t)) \geq F_{x_{n-1} x_{m-1}}(\varphi^{n-1}(t)) \\
\;
\geq \cdots \\
\geq F_{x_0 x_{m-n}}(t) \\
\geq D_{O_f}(t).
\]

Let \( \lambda > 0 \) and \( \epsilon \in (0,1) \) be given, since \( D_{O_f}(t) \to 1 \) as \( t \to \infty \) there exist \( t_0 > 0 \) such that

\[
D_{O_f}(t_0) > 1 - \epsilon.
\]

Since \( \varphi^n(t_0) \to 0 \) as \( n \to \infty \), there is \( N \in \mathbb{N} \) such that

\[\varphi^n(t_0) < \lambda \text{ whenever } n \geq N,\]

then

\[
F_{x_n x_m}(\lambda) \geq F_{x_n x_m}(\varphi^n(t_0)) \\
\geq D_{O_f}(t_0) \\
> 1 - \epsilon.
\]

Thus we proved that for each \( \lambda > 0 \) and \( \epsilon \in (0,1) \) there exists a positive integer \( N \) such that

\[
F_{x_n x_m}(\lambda) > 1 - \epsilon \text{ for all } n, m \geq N.
\]

This means that \( \{x_n\} \) is a Cauchy sequence.

As consequence of Lemma 4.1 we have

**Lemma 4.3** Let \((M, F, T)\) be a Menger space where \( \text{Ran} F \subset D^+ \) and \( f \) is a \( \varphi \)-probabilistic contraction mapping on \( M \) with \( \varphi \in \Psi \). If the \( t \)-norm \( T \) is the \( H \)-type, then for all \( x \in M \), the orbit \( O_f(x) \) is bounded.
Proof. Using the same arguments as in the proof of [4, Theorem 12], we show that \( \{x_n\} \) is a Cauchy sequence. Hence and by Lemma 4.1, we concluded that \( \mathcal{O}_f(x) \) is bounded.

Next, recall the main result of [1]

**Theorem 4.4** Let \((M, F, \tau)\) be a complete PM space where \( \text{Ran} F \subset D^+ \) and \( f \) is a \( \delta \)-probabilistic contraction mapping on \( M \) in the sense of Mbarki. If the orbit \( \mathcal{O}_f(x) \) for some \( x \in M \) is bounded, then \( f \) has a unique fixed point \( z \), moreover, the sequence \( \{f^n x\} \) converges to \( z \).

As consequences of Theorem 4.4, Lemma 3.2 and Lemma 4.3, we have the following

**Corollary 4.5** Let \((M, F, T)\) be a complete Menger space where \( \text{Ran} F \subset D^+ \) under a t-norm \( T \) of H-type and \( f \) is a \( \varphi \)-probabilistic contraction mapping on \( M \) with \( \phi \in \Phi \). Then \( f \) has a unique fixed point \( z \), moreover, the sequence \( \{f^n x\} \) converges to \( z \).

In view of above Corollary it is very much clear that Theorem 4.4 give an affirmative answer raised by L. Ćirić in [4]. We also have the following result.

**Theorem 4.6** Let \((M, F, \tau)\) be a complete PM space where \( \text{Ran} F \subset D^+ \) and \( f \) is a \( \varphi \)-probabilistic contraction mapping on \( M \) with \( \varphi \in \Psi \). If the orbit \( \mathcal{O}_f(x) \) for some \( x \in M \) is bounded, then \( f \) has a unique fixed point \( z \), moreover, the sequence \( \{f^n x\} \) converges to \( z \).

Proof. Let \( x \in M \) such that \( \mathcal{O}_f(x) \) is a bounded sequence, by Lemma 4.2 \( \{x_n\} \) is a Cauchy sequence. Since \((M, F, \tau)\) is complete, \( \{x_n\} \) converges to some \( z \in M \).

Now we shall show that \( z \) is a fixed point of \( f \).

Let \( t > 0 \), then

\[
F_{x_n f z}(\varphi(t)) \geq F_{x_{n-1} z}(t),
\]

therefore

\[
F_{x_n f z}(t) \geq F_{x_{n-1} z}(t),
\]

letting \( n \to \infty \), we get \( z = f z \).

To complete the proof we need to show that \( z \) is unique. Indeed, let \( u \) be another fixed point of \( f \) and \( t > 0 \) then

\[
F_{u z}(t) \geq F_{u z}(\varphi(t)) \text{ and } F_{f u f z}(\varphi(t)) \geq F_{u z}(t),
\]

thus \( F_{u z}(\varphi(t)) = F_{u z}(t) \). Hence by Lemma 3.4 \( u = z \).
Remark 4.7 Note that the hypothesis "PM space \((M, F, \tau)\) has the property that \(\text{Ran} F \subset D^+\)" is a necessary condition for the uniqueness of fixed points when they exist. Indeed consider \(M = \{p, q\}\) and \(F_{pq} = \frac{1}{2}e_0 + \frac{1}{2}e_\infty\), then the identity function on \(M\) is probabilistic contraction mapping on \(M\) with two fixed points.

- The condition-hypothesis that there exist \(x \in M\) such that \(O_f(x)\) is bounded it is necessary condition of the existence of fixed point as the following Sherwood’s Example [7] shows

Example 4.8 Let \(G\) be the distribution function defined by

\[
G(t) = \begin{cases} 
0, & \text{if } t \leq 4, \\
1 - \frac{1}{n}, & \text{if } 2^n < t \leq 2^{n+1} \quad n > 1.
\end{cases}
\]

Consider the set \(M = \{1, 2, \ldots, n, \ldots\}\) and define \(F\) on \(M \times M\) as follows

\[
F_{n,m+n}(t) = \begin{cases} 
0, & \text{if } t \leq 0, \\
T_L^m(G(2^n t), G(2^{n+1} t), \ldots, G(2^{m+n} t)), & \text{if } t > 0.
\end{cases}
\]

Then \((X; F; T_L)\) is a complete PM space and the mapping \(g(n) = n + 1\) is \(\varphi\)-contractive with \(\varphi(t) = \frac{1}{2}t\). But \(g\) is fixed point free mapping. Since there does not exist \(n\) in \(X\), such that \(O_g(x)\) is bounded.

As direct consequences of Theorem 4.6 and Lemma 4.3, we obtain the following.

Corollary 4.9 [4, Theorem 12]. Let \((M, F, T)\) be a complete Menger space where \(\text{Ran} F \subset D^+\) under a \(t\)-norm \(T\) of \(H\)-type and \(f\) is a \(\varphi\)-probabilistic contraction mapping on \(M\). Then \(f\) has a unique fixed point \(z\), moreover, the sequence \(\{f^n x\}\) converges to \(z\).

5 Common fixed point Theorem

Let \(S\) be a semigroup of selfmaps on \((M, F, \tau)\). For any \(x \in M\), the orbit of \(x\) under \(S\) starting at \(x\) is the set \(O(x)\) defined to be \(\{x\} \cup Sx\), where \(Sx\) is the set \(\{g(x) : g \in S\}\). We say that \(S\) is left reversible if, for any \(f, g\) in \(S\), there are \(a, b\) such that \(fa = gb\). It is obvious that left reversibility is equivalent to the statement that any two right ideals of \(S\) have nonempty intersection. Finally, we say that \(S\) is \(\varphi\)-probabilistic contraction if there exists a function \(\varphi\) such that for each \(g\) in \(S\), \(g\) is \(\varphi\)-probabilistic contraction.

Theorem 5.1 Suppose \(S\) is a left reversible semigroup of selfmaps on \(M\) such that the following conditions (i) and (ii) are satisfied
Prove. It follows from Theorem 4.6 that each $g$ in $S$ has a unique fixed point $z_g$ in $M$ and for any $x \in M$, the sequence of iterates $(g^n x)$ converges to $z_g$. So, to complete the proof it suffices to show that $z_f = z_g$ for any $f, g \in S$.

Let $n$ be an arbitrary positive integer. The left reversibility of $S$ shows that are $a_n$ and $b_n$ in $S$ such that $f^n a_n = g^n b_n$, then

$$F_{z_f z_g} \geq \tau(F_{z_f f^n a_n x}, F_{g^n b_n x z_g}), \quad (5)$$

and

$$F_{z_f f^n a_n x} \geq \tau(F_{z_f f^n x}, F_{f^n f^n a_n x}). \quad (6)$$

Next we shall show that $F_{f^n x f^n a_n x} \to \epsilon_0$ as $n \to \infty$.

Let $\lambda > 0$ and $\epsilon \in (0, 1)$ be given, since $\mathcal{O}(x)$ is bounded, then there is $t > 0$ such that

$$D_{\mathcal{O}(x)}(t) > 1 - \epsilon$$

and since $\varphi^n(t) \to 0$ as $n \to \infty$ there exists a positive integer $N$ such that

$$\varphi^n(t) > \lambda \text{ whenever } n \geq N.$$

So

$$F_{f^n x f^n a_n x}(\lambda) \geq F_{f^n x f^n a_n x}(\varphi^n(t)) \geq F_{x a_n x}(t) \geq D_{\mathcal{O}(x)}(t) > 1 - \epsilon.$$

This means that $F_{f^n x f^n a_n x} \to \epsilon_0$ as $n \to \infty$. Letting $n \to \infty$ in the inequality (6) we get $F_{z_f f^n a_n x} \to \epsilon_0$.

Likewise, we also have $F_{g^n b_n x z_g} \to \epsilon_0$, which implies that, as $n \to \infty$ in (5) we obtain that $z_f = z_g$. This completes the proof of Theorem 5.1.

References


[5] Mbarki, A., Quelques aspects de la théorie du point fixe pour les semigroupes, Thése de Doctorat en Sciences, Faculté des Sciences, Oujda, Maroc, No. 36/01.


Received: October, 2012