Sharp Bounded for Numerical Radius and Operator Norm Inequalities

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Abstract

Let \((H, <.,.>)\) be a complex Hilbert space and \(B(H)\) denote the \(\mathcal{C}^*\)-algebra of all bounded linear operators on \(H\). In this paper, we introduce a new upper and lower bounded for numerical radius and operator norm inequalities.

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1. Introduction

Let \((H, <.,.>)\) be a complex Hilbert space. The numerical range of an operator \(A\) is the subset of the complex numbers \(\mathcal{C}\) given by \(w(A) = \{ <Ax, x>, x \in H, \|x\| = 1 \}\);

(i) The numerical range of an operator is convex;
(ii) The spectrum of an operator is contained in the closure of its numerical range;

(iii) If $A$ is self-adjoint, then $\omega(A)$ is real. [1,2,9,10]

The numerical radius $\omega(A)$ of an operator $A$ on $H$ is defined by $\omega(A) = \sup \{ \|\lambda\| : \lambda \in \omega(A) \}$. It is well known that $\omega(.)$ is a norm on the Banach algebra $B(H)$ of all bounded linear operators acting on $H$ and the following inequality holds true $\omega(A) \leq \|A\| \leq 2\omega(A)$. [4,5]

We recall some classical results involving the numerical radius of two linear operators $A,B$.

**Theorem 1.** If $A,B$ are two bounded linear operators on the Hilbert space $(H, \langle .., .. \rangle)$,

then $\omega(AB) \leq 4\omega(A)\omega(B)$. In the case that $AB = BA$, then $\omega(AB) \leq 2\omega(A)\omega(B)$.

The following results are also well known.[1,11]

**Theorem 2.** If $A$ is a unitary operator that commutes with another operator $B$,

then $\omega(AB) \leq \omega(B)$. [1,11]

If $A$ is an isometry and $AB = BA$, then $\omega(AB) \leq \omega(B)$ also holds true.[1]

**Theorem 3.** Let $A$ be a normal operator commuting with $B$, then $\omega(AB) \leq \omega(A)\omega(B)$. [1,8]

### 2. Sharp Bounded For Numerical Radius And Operator Norm Inequalities

**Theorem 4.** If $X$ is a normed space and $x,y \in X$ such that $\|x\|\|y\| \neq 0$ then for every $0 \leq p \leq 1$ or $p \geq 2$ we have
\[
2 \left[ \left( \|x\|^p + \|y\|^p \right)^{\frac{2}{p}} + \left( 2^p - 4 \right) \left( \|x\|^r + \|y\|^r \right) \right] \leq \left\| x+y \right\|^2 + \left\| x-y \right\|^2 \] [3]

**Theorem 5.** If \( A \in \mathcal{B}(H) \), then \( \| AA^* A + A^* A \| \leq 4\omega^j(A) \). [6]

**Theorem 6.** For every \( A, B, C, D \in \mathcal{B}(H) \) and \( r, s \geq 1 \), we have

\[
\left\| B^* A + D^* C \right\| \leq \left\| (A^* A)^j + (C^* C)^j \right\| \left\| (B^* B)^j + (D^* D)^j \right\| \] [7]

**Notice.** If we put in theorem 6, \( B^* = A^*, D^* = -A, C = A^* \) and \( r = s = 1 \), then we have follows inequality.

**Corollary 7.** For every \( A \in \mathcal{B}(H) \) we have

\[
\| AA^* A - A^* A \| \leq \left\| A^* A + AA^* \right\| .
\]

Also its obtainable by [11].

**Theorem 8.** (Faryad-Khosravi Theorem) For every \( A \in \mathcal{B}(H) \) such that \( \| A^* A - AA^* \| < \| A^* A + AA^* \| \), we have \( \| A \| < 2^\frac{3}{2} \omega(A) \).

**Proof:** \( \mathcal{B}(H) \) is a normed space, so put \( x = AA^* \) and \( y = A^* A \) in theorem 4, then for every \( p \geq 2 \) or \( 0 \leq p \leq 1 \) we have

\[
\left( \| AA^* A + A^* A \|^2 + \| AA^* - A^* A \|^2 \right)^{\frac{1}{2}} \geq 2' \left( \| AA^* \|^{\frac{2}{p}} + \| A^* A \|^{\frac{2}{p}} \right) + \\
(2^p - 4) \left( \| AA^* \|^{\frac{2}{p}} + \| A^* A \|^{\frac{2}{p}} \right) = 2^p \left( \| A \|^{2p} + \| A^* \|^{2p} \right) + (2^p - 4) \left( \| A \|^{r} + \| A^* \|^{r} \right) \\
= 2^p \left( \| A \|^{r} + 2^p \| A \|^{r} - 4 \| A \|^{r} \right) = 4^p \| A \|^{p} .\]

Therefore,

\[
4^p \| A \|^{p} \leq \| AA^* A + A^* A \|^2 + \| AA^* - A^* A \|^2 .
\]
\[ \leq 2\|AA^* + A^*A\|^2 \leq 32 \omega^4(A). \text{ Therefore, } \|A\| \leq 2^\frac{3}{\omega}(A). \]

**Notice.** Since for every \( A \in \mathcal{B}(H) \) such that \( \|A^*A - AA^*\| < \|A^*A + AA^*\| \), \( 2^{\frac{3}{2}}\omega(A) < 2\omega(A) \), so \( 2^{\frac{3}{2}}\omega(A) \) is better upper bounded than \( 2\omega(A) \) for \( \|A\| \).

**Corollary 9.** For every \( A, B \in \mathcal{B}(H) \) such that \( \|A^*A - AA^*\| < \|A^*A + AA^*\| \) and \( \|B^*B - BB^*\| < \|B^*B + BB^*\| \), we have \( \omega(AB) < 2\sqrt{2}\omega(A)\omega(B) \).

**Proof:** We have \( \omega(AB) \leq \|AB\| \leq \|A\|\|B\| \)

\[ \leq 2^{\frac{1}{2}}2^{\frac{1}{2}}\omega(A)\omega(B) \]

\[ = 2\sqrt{2}\omega(A)\omega(B). \text{ (By Faryad-Khosravi Theorem)} \]

**Corollary 10.** For every \( A \in \mathcal{B}(H) \) such that \( \|A^*A - AA^*\| < \|A^*A + AA^*\| \) we have \( \frac{\sqrt{2}}{4}\|A^*A\| < \omega^2(A) \).

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**References**


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