Effect of Thermal Gradient on Natural Frequencies of Tapered Rectangular Plate

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Abstract
In this paper, an analysis is presented about the effect of bi-parabolic temperature variations on mechanical vibrations of non-homogeneous rectangular plate which is clamped along the boundary. It is considered that thickness of plate varies linearly in x direction. Also, exponential variation in poisson ratio is taken due to non-homogeneity present in plate’s material. The governing differential equation is solved by the Rayleigh-Ritz method to calculate first two modes of frequencies for different values of taper constant, thermal gradient, aspect ratio and non-homogeneity constant. All the numeric values of both the modes of frequency are given in tabular form.

Mathematics Subject Classification 70J30

Keywords: Vibration, frequency, thermal gradient

1 Introduction
Tapered plates i.e. plates with variable thickness along with thermal condition are extensively used in modern technology i.e. naval structure, aircraft etc. which provide a number of attractive features such as material saving, high strength, reliability and also meet the desirability of economy. Since most of machines or
their parts work under the influence of temperature, therefore the changes in mechanical or physical properties of plate’s materials can’t be neglected and analysis of their vibrational behaviour need special attention.
In available literature, sufficient work has been done on the effect of one dimensional temperature variation on vibrations of plates of variable thickness but negligible work had been done with two dimensional temperature variations along with non-homogeneity constant variation.
Here, the effect of bi-parabolic temperature variations on the vibrations of non-homogeneous rectangular plate is discussed. Values of natural frequencies are calculated for first two modes of vibrations for various values of taper constant, thermal gradient, aspect ratio & non-homogeneity constant by using Rayleigh Ritz method.

2 Equation of motion

Differential equation of motion for rectangular plate of variable thickness with variable thickness is given by (1):

\[
[D_1 (W_{xxxx} + 2W_{xyy} + W_{yyyy}) + 2D_{1x}(W_{xxx} + W_{xyy}) + 2D_{1y}(W_{yyy} + W_{yyy}) + D_{1xx}(W_{xx} + \nu W_{yy}) + D_{1yy}(W_{yy} + \nu W_{xx}) + 2(1 - \nu)D_{1,xy}W_{xy} - \rho p^2 g W = 0
\]  

Here W, g, ρ & ν are deflection function, thickness of plate, density and poisson ratio of plate material respectively. Also, D₁ is flexural rigidity of rectangular plate is defined as

\[
D_1 = \frac{Eg^3}{12(1 - \nu^2)}
\]  

(Here, a comma ‘,’ in the suffix of W shows partial differentiation of W with respect to associate variable)

Bi-parabolic temperature variations is taken as

\[
\tau = \tau_0 (1 - (x/a)^2)(1 - (y/b)^2)
\]  

where \( \tau \) denotes the temperature excess above the reference temperature at any point on the plate and \( \tau_0 \) denotes the temperature excess above the reference temperature at \( x=y=0 \) and “a” & “b” denote the length and breadth of rectangular plate respectively.

The temperature dependence of the modulus of elasticity for most of engineering material is taken as follows

\[
E = E_0 (1 - \gamma \tau)
\]  

where \( E_0 \) is the value of the Young’s modulus at reference temperature i.e. \( \tau = 0 \) and \( \gamma \) is the slope of the variation of E. After substituting value of \( \tau \) from eq. (3), eq. (4) become

\[
E = E_0 \left(1 - \alpha \left(1 - \frac{x^2}{a^2}\right)\left(1 - \frac{y^2}{b^2}\right)\right)
\]  

where \( \alpha = \gamma \tau_0 \) (0 ≤ \( \alpha \) < 1) thermal gradient.
The thickness variation in x-direction is considered as below:-
\[ g = g_0(1 + \beta x/a) \]  \hspace{1cm} (6)
where \( \beta \) is taper constant in x-direction and \( g=g_0 \) at \( x=y=0 \).
As the non-homogeneity in the material occurs, poisson ratio varies exponentially in x-direction:-
\[ v = v_0 e^{\alpha_1 x/a} \]  \hspace{1cm} (7)
where \( v_0 \) denotes poisson ratio at reference temperature i.e. \( \tau = 0 \).
Since the expansion of \( e^{\alpha_1 x/a} \) contains infinite terms of increasing powers of \( \alpha_1 x/a \), therefore to make convenient and accurate calculations (up to four places of decimal), authors consider first six terms of \( e^{\alpha_1 x/a} \) s.t.
\[ v = v_0 e^{\alpha_1 x/a} = v_0 \left[ 1 + (\alpha_1 x/a)/1! + (\alpha_1 x/a)^2/2! + (\alpha_1 x/a)^3/3! + (\alpha_1 x/a)^4/4! + (\alpha_1 x/a)^5/5! \right] \]  \hspace{1cm} (8)
Put the value of \( E, g, \) \& \( v \) from equation (5), (6), (8) in the equation (2), one obtain
\[ D_1 = \frac{E_0 \left( 1 - a \left( 1 - \frac{x^2}{a^2} \right) \left( 1 - \frac{y^2}{b^2} \right) \right) (g_0(1 + \beta x/a))^{3/2}}{12(1 - v_0^2 e^{2\alpha_1 x/a})} \]  \hspace{1cm} (9)

3 Boundary Conditions And Corresponding Deflection Function
Since the plate is assumed as clamped at all the four edges, so the boundary conditions are
\[ \begin{align*}
W &= W_x = 0 \hspace{0.5cm}, x = 0, a \\
W &= W_y = 0 \hspace{0.5cm}, y = 0, b
\end{align*} \]
And corresponding two-term deflection function is taken as follows:
\[ W = [(x/a)(y/b)(1-x/a)(1-y/b)]^2[C_1 + C_2(x/a)(y/b)(1-x/a)(1-y/b)] \]

4 Methodology
Rayleigh-Ritz technique is applied to solve the frequency equation. This method is based on principal of conservation of energy i.e. maximum strain energy \( (E_p) \) must be equal to the maximum kinetic energy\( (E_k) \). So it is necessary for the problem under consideration that [2]
\[ \delta (E_p - E_k) = 0 \]  \hspace{1cm} (10)
where \[ E_k = \frac{1}{2} \rho p^2 \int_0^a \int_0^b gW^2 dy dx \]  \hspace{1cm} (11)
and
\[ E_p = \frac{1}{2} \int_0^a \int_0^b D_1 \left( (W_{xx})^2 + (W_{yy})^2 + 2\nu W_{xx}W_{yy} + 2(1 - \nu)W_{xy}^2 \right) dy dx \]  

(12)

Assuming the non-dimensional variables as

\[ X = x/a, Y = y/a \]  

(13)

On using equation (13) in equation (11) & (12), one get

\[ E_k^* = \frac{1}{2\rho a^2 g_0} \int_0^{b/a} (1 + \beta X)W^2 dY dX \]  

(14)

\[ E_p^* = Q \int_0^1 \left\{ \left( 1 - \alpha(1 - X^2) (1 - a/b Y^2) \right) \left( 1 + \beta X \right)^3 / \left( 1 - v_0 e^{a_1 X} \right) \right\} \left\{ (W_{XX})^2 + (W_{YY})^2 + 2v_0 e^{a_1 X} W_{XX}W_{YY} + 2(1 - v_0 e^{a_1 X}) (W_{XY})^2 \right\} dY dX \]  

(15)

where \( Q = E_0 g_0^3 / 24a^2 \)

After substituting the values of \( E_k^* \) and \( E_p^* \) from eq. (14) and (15) in eq.(10), one get

\[ \left( E_p^* - \lambda^2 E_k^* \right) = 0 \]  

(16)

Here \( \lambda^2 = \frac{12\rho a^4}{E_0 g_0^2} \), is a frequency parameter. Eq. (16) consists two unknown constants i.e. \( C_1 \) & \( C_2 \) arising due to the substitution of value of deflection function (W). These two constants are to be determined as follows

\[ \frac{\partial(E_p^*-\lambda^2 E_k^*)}{\partial C_n} = 0, n=1, 2 \]  

(17)

On simplifying eq. (17), one gets

\[ D_{n1} C_1 + D_{n2} C_2 = 0, n=1, 2 \]  

(18)

where \( D_{n1}, D_{n2} \) involve parametric constant and the frequency parameter \( \lambda^2 \).

For a non-trivial solution, the determinant of the coefficient of equation (18) must be zero. So one gets, the frequency equation as

\[ \begin{vmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{vmatrix} = 0 \]  

(19)

Equation (19) is a quadratic equation in \( \lambda^2 \) which provides two values of \( \lambda^2 \). From these two values of \( \lambda^2 \), one can easily obtained the two modes of vibrations of frequencies i.e. \( \lambda_1 \) (Model1) & \( \lambda_2 \) (Model2).

5 Results and Discussion

Computations have been made for calculating frequency for first two modes of vibrations at different values of taper constants (\( \beta \)), aspect ratio (\( a/b \)), thermal gradient (\( \alpha \)) & non-homogeneity constant (\( \alpha_1 \)) by using MATHEMATICA (a software). It is considered that plate is made up of Duralium (an alloy of Aluminium) and the following parameters are used in calculations:

\[ E_0 = 7.08 \times 10^{10} \text{N/M}^2, \rho = 2.80 \times 10^3 \text{Kg/M}^3, \nu_0=0.345 \text{ & } h_0=0.01 \text{m} \]
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Results are given in tabular form for various combinations of parameters.

**Table (1): Frequency Vs Non Homogeneity Constant for a/b=1.5**

<table>
<thead>
<tr>
<th>$\alpha$</th>
<th>$\beta = \alpha = 0.0$</th>
<th>$\beta = \alpha = 0.2$</th>
<th>$\beta = \alpha = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mode 1</td>
<td>Mode 2</td>
<td>Mode 1</td>
</tr>
<tr>
<td>0.0</td>
<td>64.77</td>
<td>255.99</td>
<td>68.17</td>
</tr>
<tr>
<td>0.2</td>
<td>65.74</td>
<td>259.80</td>
<td>69.21</td>
</tr>
<tr>
<td>0.4</td>
<td>66.99</td>
<td>264.80</td>
<td>70.58</td>
</tr>
<tr>
<td>0.6</td>
<td>68.65</td>
<td>271.49</td>
<td>72.42</td>
</tr>
<tr>
<td>0.8</td>
<td>70.87</td>
<td>280.55</td>
<td>74.91</td>
</tr>
<tr>
<td>1.0</td>
<td>73.85</td>
<td>293.01</td>
<td>78.27</td>
</tr>
</tbody>
</table>

From table 1, it is observed that frequency increases continuously as non-homogeneity constant increases from 0.0 to 1.0 for both the modes of vibrations for different combinations of taper constant and thermal gradient i.e.

i) $\beta = \alpha = 0.0$  
ii) $\beta = \alpha = 0.2$  
iii) $\beta = \alpha = 0.6$

As the combined values of $\beta$ & $\alpha$ increases from 0.0 to 0.6, values of frequency also increases.

**Table (2): Frequency Vs Taper Constant for a/b=1.5 and $\alpha = 0.2$**

<table>
<thead>
<tr>
<th>$\beta$</th>
<th>$\alpha = 0.0$</th>
<th>$\alpha = 0.2$</th>
<th>$\alpha = 0.6$</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mode 1</td>
<td>Mode 2</td>
<td>Mode 1</td>
</tr>
<tr>
<td>0.0</td>
<td>61.73</td>
<td>244.04</td>
<td>62.65</td>
</tr>
<tr>
<td>0.2</td>
<td>68.17</td>
<td>269.39</td>
<td>69.21</td>
</tr>
<tr>
<td>0.4</td>
<td>74.83</td>
<td>295.60</td>
<td>76.10</td>
</tr>
<tr>
<td>0.6</td>
<td>81.68</td>
<td>322.48</td>
<td>82.96</td>
</tr>
<tr>
<td>0.8</td>
<td>88.66</td>
<td>349.86</td>
<td>90.06</td>
</tr>
<tr>
<td>1.0</td>
<td>95.74</td>
<td>377.64</td>
<td>97.27</td>
</tr>
</tbody>
</table>

It is evident from table 2 that frequency increases continuously as taper constant increases from 0.0 to 1.0 for different values of non-homogeneity constant i.e.

i) $\alpha_1 = 0.0$  
ii) $\alpha_2 = 0.2$  
iii) $\alpha_3 = 0.6$

On comparing the above three cases, one can note that as the value of non-homogeneity constant increases, frequency also increases for both the two modes.
of vibrations.

**Table (3):-Frequency Vs Aspect Ratio for \(\beta = 0.2\) and \(\alpha = 0.2\)**

<table>
<thead>
<tr>
<th>a/b</th>
<th>(\alpha_1 = 0.0)</th>
<th>(\alpha_1 = 0.2)</th>
<th>(\alpha_1 = 0.6)</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Mode 1</td>
<td>Mode 2</td>
<td>Mode 1</td>
</tr>
<tr>
<td>0.25</td>
<td>25.76</td>
<td>105.16</td>
<td>26.23</td>
</tr>
<tr>
<td>0.5</td>
<td>27.77</td>
<td>111.23</td>
<td>28.26</td>
</tr>
<tr>
<td>0.75</td>
<td>32.34</td>
<td>127.33</td>
<td>32.88</td>
</tr>
<tr>
<td>1.0</td>
<td>40.44</td>
<td>158.29</td>
<td>41.09</td>
</tr>
<tr>
<td>1.25</td>
<td>52.42</td>
<td>205.81</td>
<td>53.23</td>
</tr>
<tr>
<td>1.5</td>
<td>68.17</td>
<td>269.39</td>
<td>69.21</td>
</tr>
</tbody>
</table>

From table 3, it is clearly seen that frequency increases continuously as aspect ratio increases from 0.25 to 1.5 for different values of non-homogeneity constant as shown below:

i) \(\alpha_1 = 0.0\)  
ii) \(\alpha_1 = 0.2\)  
iii) \(\alpha_1 = 0.6\)

After compared above three cases, one can easily get that frequency increases as non-homogeneity constant increase for both the modes of vibrations.

**Conclusion**

On comparing the results of present paper with [4], it is interesting to observe that the values of frequencies in present paper (bi-parabolic temperature variations) are slightly less than [4] (bi-linear temperature variations) for the corresponding values of parameters. Therefore, authors advised the practitioners or researchers that they must go through the findings of the present paper so that they can get the desired frequencies by proper choice of parameters.

**References**


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