Weyl and Weyl Type Theorems for Class $A_k^*$ and Quasi Class $A_k^*$ Operators

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Abstract

Recently many mathematicians are working on finding operators which have Bishop’s property, SVEP and which satisfy Weyl’s theorem, where mostly these operators are normaloids. In our previous papers, we have defined quasi class $A_k^*$ and m-quasi k-paranormal operators and showed that they have these properties but are not normaloids. In this paper, we define quasi class $A_k^*$ operators for positive integers $k$ and show that restriction of quasi class $A_k^*$ operators to an invariant subspace are class $A_k^*$, they have Bishop’ property, (H) property and satisfy Weyl and other Weyl type theorems but are not normaloids. We also define algebraically quasi class $A_k^*$ operators and prove that spectral mapping theorem for Weyl spectrum and for essential approximate point spectrum hold. Also if $T$ is an algebraically quasi class $A_k^*$ operator, then $T$ is polaroid, Generalised Weyl’s theorem holds for $T$ and other Weyl type theorems are discussed. We further prove that tensor product of two quasi class $A_k^*$ operators is closed.

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1 Introduction and Preliminaries

Let $B(H)$ be the Banach algebra of all bounded linear operators on a non-zero complex Hilbert space $H$. By an operator $T$, we mean an element in $B(H)$. If $T$ lies in $B(H)$, then $T^*$ denotes the adjoint of $T$ in $B(H)$. The ascent of $T$ denoted by $p(T)$, is the least non-negative integer $n$ such that $\ker T^n = \ker T^{n+1}$. The
descent of $T$ denoted by $q(T)$, is the least non-negative integer $n$ such that $\text{ran}(T^n) = \text{ran}(T^{n+1})$. $T$ is said to be of finite ascent if $p(T - \lambda) < \infty$, for all $\lambda \in C$. If $p(T)$ and $q(T)$ are both finite, then $p(T) = q(T)$ ( [14], Proposition 38.3) Moreover, $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$ precisely when $\lambda$ is a pole of the resolvent of $T$. An operator $T$ is said to have the single valued extension property (SVEP) at $\lambda_0 \in C$, if for every open neighborhood $U$ of $\lambda_0$, the only analytic function $f : U \to X$ which satisfies the equation $(\lambda I - T)f(\lambda) = 0$ for all $\lambda \in U$ is the function $f \equiv 0$. An operator $T$ is said to have SVEP, if $T$ has SVEP at every point $\lambda \in C$.

An operator $T$ is called a Fredholm operator if the range of $T$ denoted by $\text{ran}(T)$ is closed and both $\ker T$ and $\ker T^*$ are finite dimensional and is denoted by $T \in \Phi(H)$. An operator $T$ is called upper semi-Fredholm operator, $T \in \Phi_+(H)$, if $\text{ran}(T)$ is closed and $\ker T$ is finite dimensional. An operator $T$ is called lower semi-Fredholm operator, $T \in \Phi_-(H)$, if $\ker T^*$ is finite dimensional. The index of a semi-Fredholm operator is an integer defined as $\text{ind}(T) = \dim \ker T - \dim \ker T^*$. An upper semi-Fredholm operator, with index less than or equal to 0 is called upper semi-Weyl operator and is denoted by $T \in \Phi_+(H)$. A lower semi-Fredholm operator with index greater than or equal to 0 is called lower semi-Weyl operator and is denoted by $T \in \Phi_-(H)$. A Fredholm operator of index 0 is called Weyl operator. The spectrum of $T$ is defined as $\sigma(T) = \{ \lambda \in C : T - \lambda I \text{ is not invertible} \}$. The Weyl spectrum of $T$ is defined as $w(T) = \{ \lambda \in C : T - \lambda I \text{ is not Weyl} \}$. The set of all isolated eigenvalues of finite multiplicity of $T$ is denoted by $\pi_{00}(T)$. We say that Weyl's theorem holds for $T$ [9] if $T$ satisfies the equality $\sigma(T) - w(T) = \pi_{00}(T)$. The approximate point spectrum of $T$ is defined as $\sigma_a(T) = \{ \lambda \in C : T - \lambda I \text{ is not bounded below} \}$. The essential spectrum of $T$ is defined as $\sigma_e(T) = \{ \lambda \in C : T - \lambda I \text{ is not Fredholm} \}$.

An upper(lower) semi-Fredholm operator with finite ascent is called upper(lower) semi-Browder operator and is denoted by $T \in B_+(H)(T \in B_-(H))$. A Fredholm operator with finite ascent and descent is called Browder operator. Clearly, the class of all Browder operators is contained in the class of all Weyl operators. Similarly the class of all upper(lower) semi-Browder operators is contained in the class of all upper(lower) semi-Weyl operators. The Browder spectrum of $T$ is defined as $\sigma_b(T) = \{ \lambda \in C : T - \lambda I \text{ is not Browder} \}$. The set of all isolated eigenvalues of finite multiplicity of $T$ in $\sigma_a(T)$ is denoted by $\pi_{00}^a(T)$ and $p_{00}(T)$ is defined as $p_{00}(T) = \sigma(T) - \sigma_b(T)$. The essential approximate point spectrum of $T$ is defined as $\sigma_{ea}(T) = \{ \lambda \in C : T - \lambda I \notin \Phi_+(H) \}$. We say that a-Weyl's theorem holds for $T$ [19], if $T$ satisfies the equality $\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}^a(T)$. We say that $T$ satisfies property(w) if $\sigma_a(T) - \sigma_{ea}(T) = \pi_{00}(T)$ and $T$ satisfies property(b) if $\sigma_a(T) - \sigma_{ea}(T) = p_{00}(T)$.

For an operator $T$ and a non-negative integer $n$, define $T_{[n]}$ to be the restriction of $T$ to $R(T^n)$ viewed as a map from $R(T^n)$ into $R(T^n)$. In particular,
If for some integer \( n \), \( R(T^n) \) is closed and \( T_{[n]} \) is an upper (resp. a lower) semi-Fredholm operator, then \( T \) is called an upper (resp. lower) semi-B-Fredholm operator. Moreover if \( T_{[n]} \) is a Fredholm operator, then \( T \) is called a B-Fredholm operator. A semi-B-Fredholm operator is an upper or a lower semi-B-Fredholm operator. The index of a semi-B-Fredholm operator \( T \) is the index of semi-Fredholm operator \( T_{[d]} \), where \( d \) is the degree of the stable iteration of \( T \) and defined as \( d = \inf\{n \in N; \text{for all } m \in N, m \geq n \Rightarrow (R(T^n) \cap N(T)) \subset (R(T^m) \cap N(T)) \} \). \( T \) is called a B-Weyl operator if it is B-Fredholm of index 0. An operator \( T \) is Drazin invertible, if it has finite ascent and descent. \( E(T) \) denotes the isolated eigenvalues of \( T \) with no restriction on multiplicity. The B-Weyl spectrum \( \sigma_{BW}(T) \) of \( T \) is defined by

\[
\sigma_{BW}(T) = \{ \lambda \in C : T - \lambda I \text{ is not a } B - \text{Weyl operator} \}.
\]

We say that \( T \) satisfies generalized Weyl’s theorem if \( \sigma(T) - \sigma_{BW}(T) = E(T) \). By [7], if Generalized Weyl’s theorem holds for \( T \), then Weyl’s theorem holds for \( T \).

An operator \( T \) is called normaloid if \( r(T) = \|T\| \), where \( r(T) = \sup\{\|\lambda\| : \lambda \in \sigma(T)\} \). An operator \( T \) is called hereditarily normaloid, if every part of it is normaloid. An operator \( T \) is called polaroid if \( \text{iso } \sigma(T) \subseteq \pi(T) \), where \( \pi(T) \) is the set of poles of the resolvent of \( T \) and \( \text{iso } \sigma(T) \) is the set of all isolated points of \( \sigma(T) \). An operator \( T \) is said to be isoloid if every isolated point of \( \sigma(T) \) is an eigenvalue of \( T \). An operator \( T \) is said to be reguloid if for every isolated point \( \lambda \) of \( \sigma(T) \), \( \lambda - T \) is relatively regular. An operator \( T \) is known as relatively regular if and only if \( \ker T \) and \( T(X) \) are complemented. Also Polaroid \( \Rightarrow \text{reguloid } \Rightarrow \text{isoloid} \).

If \( \lambda \in \text{iso } \sigma(T) \), the spectral projection (or Riesz idempotent) \( E_\lambda \) of \( T \) with respect to \( \lambda \) is defined by \( E_\lambda = \frac{1}{2\pi i} \int_{\partial D}(z - T)^{-1} \, dz \), where \( D \) is a closed disk with centre at \( \lambda \) and radius small enough such that \( D \cap \sigma(T) = \{\lambda\} \). Then \( E_\lambda^2 = E_\lambda \), \( E_\lambda T = TE_\lambda \), \( \sigma(T_{E_\lambda H}) = \{\lambda\} \) and \( \ker(T - \lambda) \subset E_\lambda H \).

An operator \( T \in B(H) \) satisfies (Bishop’s) property \((\beta)\) if, for every open subset \( U \) of complex plane \( C \) and every sequence of analytic functions \( f_n : U \rightarrow H \) with the property that \( (T - \lambda)f_n(\lambda) \rightarrow 0 \) as \( n \rightarrow \infty \) uniformly on all compact subsets of \( U \), \( f_n(\lambda) \rightarrow 0 \) as \( n \rightarrow \infty \) locally uniformly on \( U \).

In this paper, we first prove some spectral properties of class \( A_k^* \) operators and then prove that if \( T \) is quasi class \( A_k^* \), then \( T \) has (H) property, Bishop’s property, \( T \) is not normaloid, Weyl’s theorem hold for \( T \) and \( f(T) \) for all positive integers \( k \) and \( f \in H(\sigma(T)) \) and if \( T^* \) has SVEP, then a-Weyl’s theorem hold for \( T \) and \( f(T) \) for all positive integers \( k \) and \( f \in H(\sigma(T)) \), where \( H(\sigma(T)) \) is the space of all analytic functions on an open neighborhood of spectrum of \( T \). We further prove that if \( T \) is an algebraically quasi class \( A_k^* \) operator for a positive integer \( k \), then spectral mapping theorem and spectral mapping theorem for essential approximate point spectrum hold for \( T \), \( T \) is polaroid, Generalised Weyl’s theorem holds for \( T \) and other Weyl type theorems are discussed.
2 Spectral properties of class $A_k^*$ operators

In this section, using matrix representation first we prove that the restriction of class $A_k^*$ operators to an invariant subspace is class $A_k^*$, and if $T$ is class $A_k^*$ then Weyl’s theorem hold for $T$, $T^*$ and $f(T)$ for $f \in \mathcal{H}(\sigma(T))$ and if $T^*$ has SVEP, then a-Weyl’s theorem hold for $T$, $T^*$ and $f(T)$ for $f \in \mathcal{H}(\sigma(T))$.

**Definition 2.1.** An operator $T \in B(H)$ is defined to be of class $A_k^*$, if $|T^k|^{\frac{2}{k}} \geq |T^*|^2$ for some positive integer $k$. If $k=1$, class $A_k^*$ operator coincides with hyponormal operator.

**Definition 2.2.** [20] An operator $T$ is called $k^*$-paranormal for a positive integer $k$, if for every unit vector $x$ in $H$, $\|T^k x\| \geq \|T^* x\|^k$.

**Theorem 2.3.** If $T$ is class $A_k^*$ for some positive integer $k$, then $T$ is $k^*$-paranormal.

Proof. consider $\|T^k x\|^2 \geq \langle (T^k)^{\frac{2}{k}} x, x \rangle \geq \langle T^* T x, x \rangle = \|T^* x\|^2$. Hence $T$ is $k^*$-paranormal.

**Theorem 2.4.** If $T$ is class $A_k^*$ operator for a positive integer $k$ and $M$ is an invariant subspace of $T$, then the restriction $T|_M$ is class $A_k^*$.

Proof. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $H$ onto $M$ and $T_1 = T|_M$. Then $TP = PTP$ and $T_1 = (PTP)|_M$. Since $T$ is of class $A_k^*$ operator and $P$ is a projection on $M$, $P(|T^k|^{\frac{2}{k}} - |T^*|^2) P \geq 0$. By Hansen’s inequality[12],

$$P(|T^k|^{\frac{2}{k}}) P \leq (PT^k TP)^{\frac{1}{k}} = \begin{pmatrix} |T_1^k|^{\frac{2}{k}} & 0 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} |T_1^k|^{\frac{2}{k}} & 0 \\ 0 & 0 \end{pmatrix}.$$ 

Hence $\begin{pmatrix} |T_1^k|^{\frac{2}{k}} & 0 \\ 0 & 0 \end{pmatrix} \geq P(|T^k|^{\frac{2}{k}}) P \geq P |T^*|^2 P = \begin{pmatrix} |T_1^*|^2 + |T_2^*|^2 & 0 \\ 0 & 0 \end{pmatrix}.$$

Hence $|T_1^k|^{\frac{2}{k}} - |T_1^*|^2 \geq |T_2^*|^2 \geq 0$. Hence $T_M$ is a class $A_k^*$ operator on $M$. 

**Theorem 2.5.** If $T$ is class $A_k^*$ operator for a positive integer $k$, $0 \neq \lambda \in \sigma_p(T)$ and $T$ is of the form $T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix}$ on $\ker(T - \lambda) \oplus \text{ran}(T - \lambda)^*$, then $T_2 = 0$ and $T_3$ is class $A_k^*$.

Proof. Let $P$ be the orthogonal projection of $H$ onto $\ker(T - \lambda)$. Since $T$ is class $A_k^*$, $T$ satisfies $|T^k|^{\frac{2}{k}} - |T^*|^2 \geq 0$, where $k$ is a positive integer.

Hence $P(|T^k|^{\frac{2}{k}} - |T^*|^2) P \geq 0$, where $P |T^*|^2 P = \begin{pmatrix} |\lambda|^2 + T_2 T_2^* & 0 \\ 0 & 0 \end{pmatrix}$ and
\[
\begin{pmatrix}
|\lambda|^2 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
|\lambda|x^k & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
P \mid T^k \mid^2 P & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
|\lambda|^2 + T_k^* & 0 \\
0 & 0
\end{pmatrix}.
\]

Hence by Hansen’s inequality, we get
\[
\begin{pmatrix}
|\lambda|^2 & 0 \\
0 & 0
\end{pmatrix} = \begin{pmatrix}
P \mid T^k \mid^2 P & 0 \\
0 & 0
\end{pmatrix} \geq \begin{pmatrix}
T_k^* & 0 \\
0 & 0
\end{pmatrix} \geq \begin{pmatrix}
0 & 0 \\
0 & 0
\end{pmatrix}.
\]

Therefore \( T_2 T_k^* = 0 \) and hence \( T_2 = 0 \). Also
\[
0 \leq |T_k|^2 - TT^* = \begin{pmatrix}
0 & 0 \\
0 & |T_k|^2 - |T_3|^2
\end{pmatrix}
\]
and hence \( T_3^* \) is class \( A_k^* \).

**Corollary 2.6.** If \( T \) is class \( A_k^* \) operator for a positive integer \( k \) and \((T - \lambda)x = 0 \) for \( \lambda \neq 0 \) and \( x \in H \), then \((T - \lambda)^* x = 0 \).

**Corollary 2.7.** If \( T \) is class \( A_k^* \) operator for a positive integer \( k \), \( 0 \neq \lambda \in \sigma_p(T) \), then \( T \) is of the form \( T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix} \) on \( \ker(T - \lambda) \oplus \text{ran}(T - \lambda)^* \), where \( T_3 \) is class \( A_k^* \) and \( \ker(T_3 - \lambda) = \{0\} \).

Since class \( A_k^* \) operators are \( k^* \) paranormal operators, by [20] class \( A_k^* \) operators are normaloids and by [18] we have the following results.

**Theorem 2.8.** If \( T \) is class \( A_k^* \) operator for a positive integer \( k \), then

1. for \( \lambda \in C \), if \( \sigma(T) = \lambda \) then \( T = \lambda \).
2. \( T \) is polaroid.
3. \( T \) is isoloid.
4. if \( \lambda \in \sigma(T) \) is an isolated point, then \( E_\lambda H = \ker(T - \lambda) \) and hence \( \lambda \) is an eigenvalue of \( T \).
5. if \( \lambda \neq 0 \) be an isolated point in \( \sigma(T) \), then \( E_\lambda \) is self-adjoint and satisfies \( E_\lambda H = \ker(T - \lambda) = \ker(T - \lambda)^* \).
6. \( T \) has SVEP, \( p(\lambda I - T) \leq 1 \) for all \( \lambda \in C \) and \( T^* \) is reguloid.
7. Weyl’s theorem holds for \( T \) and \( T^* \). If in addition, \( T^* \) has SVEP, then a-Weyl’s theorem holds for both \( T \) and \( T^* \) and for \( f(T) \) for every \( f \in H(\sigma(T)) \).

**Theorem 2.9.** If \( T \) is a class \( A_k^* \) operator for some positive integer \( k \), then \( f(w(T)) = w(f(T)) \) for every \( f \in H(\sigma(T)) \).

**Proof.** Since \( T \) is of finite ascent by theorem 2.8, by [14], Proposition 38.5] \( \text{ind}(T - \lambda) \neq 0 \) for all complex numbers \( \lambda \). Therefore by theorem 5 of [13], \( f(w(T)) = w(f(T)) \) for every \( f \in H(\sigma(T)) \). \( \square \)
Theorem 2.10. If $T \in B(H)$ is class $A_k^*$ operator for a positive integer $k$, then Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Proof. By theorem 2.8, $T$ is isoloid and Weyl’s theorem holds for $T$. By lemma of [17], $f(\sigma(T) - \pi_{00}(T)) = \sigma(f(T)) - \pi_{00}(f(T))$, for every $f \in H(\sigma(T))$. By theorem 2.9, $f(w(T)) = w(f(T))$, for every $f \in H(\sigma(T))$. Hence $\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T) - \pi_{00}(T)) = f(w(T)) = w(f(T))$. Hence Weyl’s theorem holds for $f(T)$, for every $f \in H(\sigma(T))$. \hfill $\square$

3 Spectral properties of Quasi class $A_k^*$ operators

In this section, we define quasi class $A_k^*$ operators, give an example and give another simple example to show that it is not anormaloid. We also give a matrix representation and show that restriction of quasi class $A_k^*$ operators to an invariant subspace is class $A_k^*$, quasi class $A_k^*$ operators are polaroids and have (H) property, Bishop’s property, satisfy Weyl and other Weyl type theorems.

Definition 3.1. An operator $T \in B(H)$ is defined to be of quasi class $A_k^*$, if $T^*\left(|T|^k - |T|^2\right)T \geq 0$ for some positive integer $k$.

Example 3.2. Let $H$ be the direct sum of a denumerable number of copies of two dimensional Hilbert space $R \times R$. Let $A$ and $B$ be two operators on $R \times R$ with $0 \leq A \leq B$. For any fixed positive integer $n$, define an operator $T = T_{A,B,n}$ on $H$ as $T((x_1, x_2, \ldots)) = (0, A(x_1), A(x_2), \ldots A(x_n), B(x_{n+1}), \ldots)$ Its adjoint $T^*$ is $T^*((x_1, x_2, \ldots)) = (A(x_2), A(x_3), \ldots A(x_{n+1}), B(x_{n+2}), \ldots)$ For $n \geq k$, $T_{A,B,n}$ is of quasi class $A_k^*$ if and only if $A$ and $B$ satisfies

$A(A^{k-i}B^2A^{k-i})^{\frac{i}{2}} \geq A^4$ for $i = 1, 2, \ldots k$. If $A = \begin{pmatrix} 1 & 0 \\ 0 & 2 \end{pmatrix}$ and $B^2 = \begin{pmatrix} 17 & 1 \\ 1 & 5 \end{pmatrix}$, then $T_{A,B,n}$ is of quasi class $A_k^*$, for every positive integer $k$.

The following example shows that quasi class $A_k^*$ operators are different classes for different positive integers $k$ and they are not a subclass of normaloids.

Example 3.3. Let $H = C^2$ and $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$. Then $T$ is quasi class $A_2^*$ but $T$ is not quasi $A_1^*$. Also $T$ is not a normaloid.

Since $S \geq 0$ implies $T^*ST \geq 0$, the following result is trivial.

Theorem 3.4. If $T$ belongs to class $A_k^*$, for some positive integer $k \geq 1$, then $T$ belongs to quasi class $A_k^*$.
Theorem 3.5. Let $T \in B(H)$ be a quasi class $A_k^*$ operator for a positive integer $k$ and $T$ not has a dense range. Let $T$ has the following representation: $T = \begin{pmatrix} T_1 & T_2 \\ 0 & T_3 \end{pmatrix}$ on $H = \overline{\text{ran}(T)} \oplus \ker(T^*)$. Then $T_1$ is class $A_k^*$ operator on $\overline{\text{ran}(T)}$ and $T_3 = 0$. Furthermore, $\sigma(T) = \sigma(T_1) \cup \{0\}$.

Proof. Let $P$ be the orthogonal projection of $H$ onto $\overline{\text{ran}(T)}$. Then $TP = PTP$. Since $T$ is quasi class $A_k^*$ operator, $P \left( |T^k|^2 - |T^*|^2 \right) P \geq 0$. By Hansen’s inequality [12], $P |T^k|^2 P \leq (P |T^k|^2 P)\frac{\lambda}{\tau}$, which has no interior points. Therefore $\sigma(T) = \sigma(T_1) \cup \{0\}$. \hfill $\square$

Corollary 3.6. Let $T \in B(H)$ be a quasi class $A_k^*$ operator for a positive integer $k$ and $T$ not have a dense range. Then $T$ has the following representation $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ on $H = \overline{\text{ran}(T)} \oplus \ker T^*$, where $T_1$ is class $A_k^*$ operator.

Theorem 3.7. If $T \in B(H)$ is quasi class $A_k^*$ operator for a positive integer $k$, then $T$ is isoloid.

Proof. Let $T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix}$ on $H = \overline{\text{ran}(T)} \oplus \ker T^*$. Let $\lambda_0$ be an isolated point of $\sigma(T)$. Then either $\lambda_0 = 0$ or $0 \neq \lambda_0 \in \text{iso}(\sigma(T_1))$. By Theorem 3.5, $T_1$ is isoloid. If $\lambda_0 \in \text{iso}(\sigma(T_1))$, then $\lambda_0 \in \sigma_p(T_1)$ and hence $\lambda_0 \in \sigma_p(T)$. On the contrary, if $\lambda_0 = 0$, then $T_1$ is invertible. Hence for any $x \neq 0$ in $H$, $(\lambda_0 - 1)T_1 x + x = 0$. Hence $\lambda_0 = 0 \in \sigma(T_1)$. Therefore $T$ is isoloid. \hfill $\square$

Theorem 3.8. If $T$ is quasi class $A_k^*$ operator for a positive integer $k$ and $M$ is an invariant subspace of $T$, then the restriction $T_{|M}$ is class $A_k^*$.

Proof. Let $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$ be the orthogonal projection of $H$ onto $M$ and $T_1 = T_{|M}$. Then $TP = PTP$ and $T_1 = (PTP)_{|M}$. Since $T$ is of quasi class $A_k^*$, \hfill $\square$
operator and \( P \) is a projection on \( M \), \( P(|T^k|^\frac{3}{2} - |T^s|^2)P \geq 0 \).
Hence as in theorem 2.4, we get \( |T^k|^\frac{3}{2} - (|T_1^s|^2 + |T_2^s|^2) \geq 0 \).
Hence \( T_1 \) is a class \( A^*_k \) operator on \( M \).

\[ \text{Theorem 3.9.} \quad \text{If} \ T \ \text{is quasi class} \ A^*_k \ \text{operator for a positive integer} \ k, \ 0 \neq \lambda \in \sigma_p(T) \ \text{and} \ T \ \text{is of the form} \ T = \begin{pmatrix} \lambda & T_2 \\ 0 & T_3 \end{pmatrix} \ \text{on} \ \ker(T - \lambda) \oplus \overline{\text{ran}(T - \lambda)^*}, \]
then \( T_2 = 0 \) and \( T_3 \) is quasi class \( A^*_k \).

\[ \text{Proof.} \ \text{Let} \ P \ \text{be the orthogonal projection of} \ H \ \text{onto} \ \ker(T - \lambda). \]
Since \( T \) is quasi class \( A^*_k \), \( T \) satisfies \( T^s(|T^k|^\frac{3}{2} - |T^s|^2)T \geq 0 \), where \( k \) is a positive integer. Hence \( P \left(|T^k|^\frac{3}{2} - |T^s|^2\right)P \geq 0 \) where \( P \left|T^s\right|^2P = \begin{pmatrix} |\lambda|^2 & 0 \\ 0 & 0 \end{pmatrix} \).
and \( P \left|T^k\right|^2P = \begin{pmatrix} |\lambda|^{2(k)} & 0 \\ 0 & 0 \end{pmatrix} \).
Therefore, \( \left(|\lambda|^2 \ 0 \right) = \left(P \left|T^k\right|^2P \right)^\frac{3}{2} \geq \left(P \left|T^s\right|^2P \right) \geq P \left|T^s\right|^2P = \begin{pmatrix} |\lambda|^2 + |T^s|^2 & 0 \\ 0 & 0 \end{pmatrix} \).
Therefore \( T_2 = 0 \) and hence \( T_3 \) is quasi class \( A^*_k \).

\[ \text{Corollary 3.10.} \quad \text{If} \ T \ \text{is quasi class} \ A^*_k \ \text{operator for a positive integer} \ k \ \text{and} \ (T - \lambda)x = 0 \ \text{for} \ \lambda \neq 0 \ \text{and} \ x \in H, \ \text{then} \ (T - \lambda)^*x = 0. \]

\[ \text{Corollary 3.11.} \quad \text{If} \ T \ \text{is quasi class} \ A^*_k \ \text{operator for a positive integer} \ k, \ 0 \neq \lambda \in \sigma_p(T), \ \text{then} \ T \ \text{is of the form} \ T = \begin{pmatrix} \lambda & 0 \\ 0 & T_3 \end{pmatrix} \ \text{on} \ \ker(T - \lambda) \oplus \overline{\text{ran}(T - \lambda)^*}, \]
where \( T_3 \) is quasi class \( A^*_k \) and \( \ker(T_3 - \lambda) = \{0\} \).

\[ \text{Theorem 3.12.} \quad \text{If} \ T \ \text{is quasi class} \ A^*_k \ \text{operator for a positive integer} \ k \ \text{and for} \ \lambda \in C, \ \sigma(T) = \lambda \ \text{then} \ T = \lambda, \ \text{if} \ \lambda \neq 0 \ \text{and} \ T - \lambda \ \text{is nilpotent, if} \ \lambda = 0. \]

\[ \text{Proof.} \ \text{By corollary 3.6,} \ T = \begin{pmatrix} T_1 & T_2 \\ 0 & 0 \end{pmatrix} \ \text{on} \ H = \overline{\text{ran}(T)} \oplus \ker T^*, \ \text{where} \ T_1 \ \text{is class} \ A^*_k \ \text{operator with} \ \sigma(T) = \sigma(T_1) \cup 0. \ \text{If} \ \lambda = 0, \ \text{then} \ \sigma(T_1) = 0. \ \text{Hence by theorem 2.8,} \ T_1 = 0. \ \text{Hence} \ T^2 = 0. \ \text{Hence} \ T - \lambda \ \text{is nilpotent.} \ \text{If} \ \lambda \neq 0, \ \text{then} \ T \ \text{is an invertible quasi class} \ A^*_k \ \text{operator and hence class} \ A^*_k \ \text{with} \ \sigma(T) = \lambda. \ \text{Hence again by theorem 2.8,} \ T = \lambda. \]

\[ \text{Theorem 3.13.} \quad \text{If} \ T \ \text{is quasi class} \ A^*_k \ \text{operator for some positive integer} \ k, \ \text{then} \ T \ \text{is polaroid.} \]

\[ \text{Proof.} \ \text{If} \ \lambda \in \text{iso} \ \sigma(T) \ \text{using the spectral projection of} \ T \ \text{with respect to} \ \lambda, \ \text{we can write} \ T = T_1 \oplus T_2 \ \text{where} \ \sigma(T_1) = \{\lambda\} \ \text{and} \ \sigma(T_2) = \sigma(T) - \{\lambda\}. \ \text{Since} \ T_1 \ \text{is class} \ A^*_k \ \text{operator and} \ \sigma(T_1) = \{\lambda\}, \ \text{by theorem 2.8,} \ T_1 = \lambda. \ \text{Since} \ \lambda \notin \sigma(T_2), \ \text{T}_2 - \lambda I \ \text{is invertible.} \ \text{Hence both} \ T_1 - \lambda I \ \text{and} \ T_2 - \lambda I \ \text{and hence} \ T - \lambda I \ \text{have finite ascent and descent.} \ \text{Hence} \ \lambda \ \text{is a pole of the resolvent of} \ T. \ \text{Hence} \ T \ \text{is polaroid.} \]
Corollary 3.14. If $T$ is quasi class $A_k^*$ operator for some positive integer $k$, then $T$ is reguloid.

Theorem 3.15. If $T$ is a quasi class $A_k^*$ operator for a positive integer $k$ and $\lambda \in \sigma(T)$ is an isolated point, then the Riesz idempotent operator $E_\lambda$ with respect to $\lambda$ satisfies $E_\lambda H = \ker(T - \lambda)$. Hence $\lambda$ is an eigenvalue of $T$.

Proof. Since $\ker(T - \lambda) \subseteq E_\lambda H$, it is enough to prove that $E_\lambda H \subseteq \ker(T - \lambda)$. Now $\sigma(T|_{E_\lambda H}) = \{\lambda\}$ and $T|_{E_\lambda H}$ is class $A_k^*$. Therefore by theorem 2.8, $T|_{E_\lambda H} = \lambda$. Hence $E_\lambda H = \ker(T - \lambda)$.

T is said to have property (H) if $H_0(\lambda I - T) = \ker(\lambda I - T)$, where $H_0(T) = \{x \in X : lim_{n \to \infty} \|T^n x\|^{1/n} = 0\}$. By [15], $E_\lambda H = H_0(\lambda I - T)$. Hence by theorem 3.15, quasi class $A_k^*$ operators have (H) property. Hence by theorem 2.5, theorem 2.6 and theorem 2.8 of Aiena [2], we get the following results.

Theorem 3.16. If $T$ is quasi class $A_k^*$ operator for some positive integer $k$, then $T$ has SVEP and $p(\lambda I - T) \leq 1$ for all $\lambda \in C$. Furthermore, both $T$ and $T^*$ are reguloid.

Corollary 3.17. If $T$ is quasi class $A_k^*$ operator for some positive integer $k$, then $T^*$ is isoloid.

Theorem 3.18. If $T$ is quasi class $A_k^*$ operator for some positive integer $k$, then Weyl’s theorem holds for $T$ and $T^*$; If in addition, $T^*$ has SVEP, then a-Weyl’s theorem holds for both $T$ and $T^*$.

Theorem 3.19. If $T$ is quasi class $A_k^*$ operator for some positive integer $k$ and $T^*$ has SVEP, then a-Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Braha and Tanahashi[8] has shown that k*-paranormal operators satisfy property ($\beta$). Since class $A_k^*$ operators are k*-paranormal operators, class $A_k^*$ operators satisfy property ($\beta$).

Theorem 3.20. If $T$ is quasi class $A_k^*$ operator for some positive integer $k$, then $T$ satisfies property ($\beta$).

Proof. Let $\{f_n(\lambda)\}$ be a sequence of analytic functions $f_n : U \to H$ on an open subset $U$ of $C$ such that $(T - \lambda)f_n(\lambda) \to 0$ as $n \to \infty$ uniformly on all compact subsets of $U$. Then by corollary 3.6
\[
\begin{pmatrix}
T_1 - \lambda & T_2 \\
0 & -\lambda
\end{pmatrix}
\begin{pmatrix}
f_{n_1} \\
f_{n_1}
\end{pmatrix}
= \begin{pmatrix}
(T_1 - \lambda)f_{n_1}(\lambda) + T_2 f_{n_2}(\lambda) \\
(-\lambda)f_{n_2}(\lambda)
\end{pmatrix} \to 0
\]
Hence $(T_1 - \lambda)f_{n_1}(\lambda) + T_2 f_{n_2}(\lambda) \to 0$ and $(-\lambda)f_{n_2}(\lambda) \to 0$ as $n \to \infty$. Hence $f_{n_2}(\lambda) \to 0$ locally uniformly on $U$. Hence $(T_1 - \lambda)f_{n_1}(\lambda) \to 0$. Since $T_1$ satisfies property ($\beta$), $f_{n_1}(\lambda) \to 0$ locally uniformly on $U$. Hence $f_n(\lambda) \to 0$ locally uniformly on $U$. Hence quasi class $A_k^*$ satisfies property ($\beta$).
Theorem 3.21. If $T$ is quasi class $A_k^*$ operator for some positive integer $k$, then $\text{ind}(T - \lambda I) \leq 0$ for all complex numbers $\lambda$.

Proof. Since $T$ is of finite ascent by lemma 3.16, by [14], Proposition 38.5 $\text{ind}(T - \lambda) \neq 0$ for all complex numbers $\lambda$. \hfill \Box

By theorem 3.7 and by the lemma of [17], we get the following result immediately.

Theorem 3.22. If $T$ is a quasi class $A_k^*$ operator for some positive integer $k$, then $\sigma(f(T)) = f(\sigma(T)) - \pi_{00}(f(T))$ for every $f \in H(\sigma(T))$.

By theorem 3.21 and by theorem 5 of [13], the following result is trivial.

Theorem 3.23. If $T$ is a quasi class $A_k^*$ operator for a positive integer $k$, then $\sigma(w(T)) = w(\sigma(T))$ for every $f \in H(\sigma(T))$.

Proof. By theorem 3.22, corollary 3.18 and theorem 3.23, for every $f \in H(\sigma(T))$, $\sigma(f(T)) - \pi_{00}(f(T)) = f(\sigma(T)) - \pi_{00}(f(T)) = f(w(T)) = w(f(T))$. Hence Weyl’s theorem holds for $f(T)$, for every $f \in H(\sigma(T))$. \hfill \Box

4 Tensor Product of Quasi class $A_k^*$ operators

Let $H$ and $K$ denote the Hilbert spaces. For given non-zero operators $T \in B(H)$ and $S \in B(K)$, $T \otimes S$ denotes the tensor product on the product space $H \otimes K$. Tensor product of two non-zero operators satisfy

1. $(T \otimes S)^*(T \otimes S) = T^*T \otimes S^*S$

2. $|T \otimes S|^p = |T|^p \otimes |S|^p$, for any positive real number $p$.

In this section, we find a necessary and sufficient condition for tensor product of two quasi class $A_k^*$ operators to be a quasi class $A_k^*$ operator.

Theorem 4.1. If $T \in B(H)$ and $S \in B(K)$ are non-zero operators. Then $T \otimes S$ is quasi class $A_k^*$ operator if and only if one of the following holds:

1. $T$ and $S$ are quasi class $A_k^*$ operators.

2. $T^2 = 0$ or $S^2 = 0$
Proof.

Consider \((T \otimes S)^* (|T \otimes S|^k)^{\frac{x}{2}} - |(T \otimes S)^*|^2 (T \otimes S)\)

\[= (T \otimes S)^* (|T^k \otimes S^k|^\frac{x}{2} - |T^*|^2 \otimes |S^*|^2) (T \otimes S)\]

\[= T^* (|T^k|^\frac{x}{2} - |T|^2) T \otimes S^* (|T^k|^\frac{x}{2} S + T^* |T^*|^2 T \otimes S^* (|T^k|^\frac{x}{2} - |S^*|^2) S)\]

\[+ T^* |T^*|^2 T \otimes S^* (|T^k|^\frac{x}{2} - |S^*|^2) S,\]

Hence, if either (i) \(T\) and \(S\) are quasi class \(A_k^*\) operators or (ii) \(T^2 = 0\) or \(S^2 = 0\), then \(T \otimes S\) is quasi class \(A_k^*\) operator. Conversely, suppose that \(T \otimes S\) is quasi class \(A_k^*\) operator. Then by the above equality,

\(T^* (|T^k|^\frac{x}{2} - |T|^2) T \otimes S^* (|T^k|^\frac{x}{2} S + T^* |T^*|^2 T \otimes S^* (|T^k|^\frac{x}{2} - |S^*|^2) S \geq 0\)

Therefore for every \(x \in H\) and \(y \in K\), \(\langle T^* (|T^k|^\frac{x}{2} - |T|^2) T x, x \rangle \langle S^* (|T^k|^\frac{x}{2} - |S|^2) S y, y \rangle \geq 0\).

It is sufficient to prove that either (i) or (ii) holds. Assume the contrary that, neither of \(T^2\) and \(S^2\) is the zero operator and \(T\) is not quasi class \(A_k^*\) operator. Then there exists \(x_k \in H\) such that \(\langle T^* (|T^k|^\frac{x}{2} - |T|^2) T x_k, x_k \rangle < 0\) and \(\langle T^* |T^*|^2 T x_k, x_k \rangle > 0\).

Let \(\alpha = \langle T^* (|T^k|^\frac{x}{2} - |T|^2) T x_k, x_k \rangle\) and \(\beta = \langle T^* |T^*|^2 T x_k, x_k \rangle\).

Then \((\alpha + \beta) \langle S^* (|S^k|^\frac{x}{2} - |S|^2) S y, y \rangle \geq \beta \langle S^* |S|^2 S y, y \rangle\)

Since \(\alpha + \beta < \beta\), this implies that \(\langle S^* (|S^k|^\frac{x}{2} - |S|^2) S y, y \rangle \geq 0\).

Hence \(S\) is quasi class \(A_k^*\) operator. Using Holder-McCarthy inequality,

\[\langle S^* |S^k|^\frac{x}{2} S y, y \rangle = \langle (S^* (k^2) S^k)^\frac{x}{2} S y, S y \rangle \leq \langle (S^* (k^2) S^k S y, S y) X \| S y \| \| S y \|^{\frac{2(k-1)}{k}} \]

\[= \| S^{k+1} y \|^{\frac{x}{2}} \| S y \|^{\frac{2(k-1)}{k}} \]

and \(\langle S^* |S^*|^2 S y, y \rangle = \langle S^* S^* S y, S y \rangle = \| S^* S y \|^2 \).

Therefore, \((\alpha + \beta) \| S^{k+1} y \|^{\frac{x}{2}} \| S y \|^{\frac{2(k-1)}{k}} \geq \beta \| S^* S y \|^2 \).

Since \(S\) is quasi class \(A_k^*\) operator, \(S\) has a decomposition of the form \(S = \begin{pmatrix} S_1 & S_2 \\ 0 & 0 \end{pmatrix} \) on \(H = \overline{\text{ran}(S) \oplus \ker (S^*)}\), where \(S_1\) is class \(A_k^*\) operator on \(\overline{\text{ran}(S)}\).

Hence \((\alpha + \beta) \| S_1^{k+1} \xi \|^{\frac{x}{2}} \| S_1 \xi \|^{\frac{2(k-1)}{k}} \geq \beta \| S_1^2 \xi \|^2 \) for all \(\xi \in \overline{\text{ran}(S)}\).

Since \(S_1\) is normaloid, \((\alpha + \beta) \| S_1 \|^{\frac{2(k+1)}{k}} \| S_1 \|^{\frac{2(k-1)}{k}} \geq \beta \| S_1 \|^4\) i.e \((\alpha + \beta) \| S_1 \|^4 \geq \beta \| S_1 \|^4\). Hence \(S_1 = 0\). Hence \(S^2 y = S_1 (S y) = 0\) for all \(y \in K\). This is a contradiction to that \(S^2\) is not a zero operator. Hence \(T\) must be quasi class \(A_k^*\) operator. In a similar manner, we can show that \(S\) is quasi class \(A_k^*\) operator. Hence the result. \(\square\)
5 Algebraically quasi class $A_k^*$ operators

In this section, we define algebraically quasi class $A_k^*$ and show that spectral mapping theorem for weyl spectrum, Generalised Weyl’s and other Weyl type theorems hold.

**Definition 5.1.** An operator $T$ is defined to be of algebraically quasi class $A_k^*$ for a positive integer $k$, if there exists a non-constant complex polynomial $p(t)$ such that $p(T)$ is of quasi class $A_k^*$.

**Theorem 5.2.** If $T$ is algebraically quasi class $A_k^*$ operator for some positive integer $k$ and $\sigma(T) = \mu_0$, then $T - \mu_0$ is nilpotent.

*Proof.* Since $T$ is algebraically quasi class $A_k^*$, there is a non-constant polynomial $p(t)$ such that $p(T)$ is quasi class $A_k^*$ for some positive integer $k$. Then applying theorem 3.12, $\sigma(p(T)) = p(\sigma(T)) = \{p(\mu_0)\}$ implies $p(T) - p(\mu_0)$ is nilpotent. Hence for some positive integer $m$, $(p(z) - p(\mu_0))^m = 0$. Let $p(z) - p(\mu_0) = a(z - \mu_0)^{k_0}(z - \mu_1)^{k_1}\cdots(z - \mu_t)^{k_t}$ where $\mu_j \neq \mu_s$ for $j \neq s$. Then $0 = (p(T) - p(\mu_0))^m = a(T - \mu_0)^{mk_0}(T - \mu_1)^{mk_1}\cdots(T - \mu_t)^{mk_t}$. Since $T - \mu_1, T - \mu_2, \cdots T - \mu_t$ are invertible, $(T - \mu_0)^{mk_0} = 0$. Hence $T - \mu_0$ is nilpotent. \qed

If $T$ is algebraically quasi class $A_k^*$ operator for some positive integer $k$, then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is quasi class $A_k^*$. By theorem 3.16, $p(T)$ is of finite ascent. Therefore $p(T)$ and hence $T$ has SVEP([16], Theorem 3.3.6).

**Theorem 5.3.** If $T$ is algebraically quasi class $A_k^*$ operator for some positive integer $k$, then Generalized Weyl’s theorem holds for $T$.

*Proof.* Assume that $\lambda \in \sigma(T) - \sigma_{BW}(T)$. Then $T - \lambda$ is B-Weyl and not invertible. Claim: $\lambda \in \partial \sigma(T)$ Assume the contrary that $\lambda$ is an interior point of $\sigma(T)$. Then there exists a neighborhood $U$ of $\lambda$ such that dim ker$(T - \mu) > 0$ for all $\mu$ in $U$. Hence by ([10], theorem 10) $T$ does not have SVEP which is a contradiction. Hence $\lambda \in \partial \sigma(T) - \sigma_{BW}(T)$. Therefore by punctured neighborhood theorem, $\lambda \in E(T)$.

Conversely suppose that $\lambda \in E(T)$. Using the Riesz idempotent $E_\lambda$ with respect to $\lambda$, we can represent $T$ as the direct sum $T = \begin{pmatrix} T_1 & 0 \\ 0 & T_2 \end{pmatrix}$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Then by theorem 5.2, $T_1 - \lambda$ is nilpotent. Since $\lambda \notin \sigma(T_2)$, $T_2 - \lambda$ is invertible. Hence both $T_1 - \lambda$ and $T_2 - \lambda$ have both finite ascent and descent. Hence $T - \lambda$ has both finite ascent and descent. Hence $T - \lambda$ is Drazin invertible. Therefore, by ([6], Lemma 4.1) $T - \lambda$ is B-Fredholm of index 0. Hence $\lambda \in \sigma(T) - \sigma_{BW}(T)$. Therefore $\sigma(T) - \sigma_{BW}(T) = E(T)$. \qed
Corollary 5.4. If $T$ is algebraically quasi class $A^*_k$ operator for some positive integer $k$, then Weyl’s theorem holds for $T$.

By ([4], Theorem 2.16) we get the following result.

Corollary 5.5. If $T$ is algebraically quasi class $A^*_k$ for some positive integer $k$, and $T^*$ has SVEP, then $a$-Weyl’s theorem and property $(w)$ hold for $T$.

Theorem 5.6. If $T$ is algebraically quasi class $A^*_k$ operator for some positive integer $k$, then $w(f(T)) = f(w(T))$ for every $f \in H(\sigma(T))$.

Proof. Suppose that $T$ is algebraically quasi class $A^*_k$ for some positive integer $k$, then $T$ has SVEP. Hence by ([14], Proposition 38.5) $\text{ind}(T - \lambda) \leq 0$ for all complex numbers $\lambda$. Now to prove the result it is sufficient to show that $f(w(T)) \subseteq w(f(T))$. Let $\lambda \in f(w(T))$. Suppose if $\lambda \notin w(f(T))$, then $f(T) - \lambda I$ is Weyl and hence $\text{ind}(f(T) - \lambda) = 0$. Let $f(z) - \lambda = (z - \lambda_1)(z - \lambda_2)\ldots(z - \lambda_n)g(z)$. Then $f(T) - \lambda = (T - \lambda_1)(T - \lambda_2)\ldots(T - \lambda_n)g(T)$ and $\text{ind}(f(T) - \lambda) = 0 = \text{ind}(T - \lambda_1) + \text{ind}(T - \lambda_2) + \cdots + \text{ind}(T - \lambda_n) + \text{ind}g(T)$. Since each of $\text{ind}(T - \lambda_i) \leq 0$, we get that $\text{ind}(T - \lambda_i) = 0$, for all $i = 1, 2, \ldots, n$. Therefore $T - \lambda_i$ is weyl for each $i = 1, 2, \ldots, n$. Hence $\lambda_i \notin w(T)$ and hence $\lambda \notin f(w(T))$, which is a contradiction. Hence the theorem.

By lemma of [17], theorem 5.4 and theorem 5.6, we get the following result.

Theorem 5.7. If $T$ is algebraically quasi class $A^*_k$ operator for some positive integer $k$, then Weyl’s theorem holds for $f(T)$, for every $f \in Hol(\sigma(T))$.

Theorem 5.8. If $T$ or $T^*$ is algebraically quasi class $A^*_k$ operator for some positive integer $k$, then $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

Proof. For $T \in B(H)$, by [19] the inclusion $\sigma_{ea}(f(T)) \subseteq f(\sigma_{ea}(T))$ holds for every $f \in H(\sigma(T))$ with no restrictions on $T$. Therefore, it is enough to prove that $f(\sigma_{ea}(T)) \subseteq \sigma_{ea}(f(T))$.

Suppose if $\lambda \notin \sigma_{ea}(f(T))$ then $f(T) - \lambda \in \Phi_+(H)$, that is $f(T) - \lambda$ is upper semi-Fredholm operator with index less than or equal to zero. Also $f(T) - \lambda = c(T - \alpha_1)(T - \alpha_2)\ldots(T - \alpha_n)g(T)$ where $g(T)$ is invertible and $c, \alpha_1, \alpha_2, \ldots, \alpha_n \in C$.

If $T$ is algebraically quasi class $A^*_k$ for some positive integer $k$, then there exists a non-constant polynomial $p(t)$ such that $p(T)$ is quasi class $A^*_k$. Then $p(T)$ has SVEP and hence $T$ has SVEP. Therefore $\text{ind}(T - \alpha_i) \leq 0$ and hence $T - \alpha_i \in \Phi_+(H)$ for each $i = 1, 2, \ldots, n$. Therefore $\lambda = f(\alpha_i) \notin f(\sigma_{ea}(T))$. Hence $\sigma_{ea}(f(T)) = f(\sigma_{ea}(T))$.

If $T^*$ is algebraically quasi class $A^*_k$ for some positive integer $k$, then there exists a non-constant polynomial $p(t)$ such that $p(T^*)$ is quasi class $A^*_k$. Then $p(T^*)$ has SVEP and hence $T^*$ has SVEP. Therefore $\text{ind}(T - \alpha_i) \geq 0$ for
Theorem 5.9. If $T$ is algebraically quasi class $A^*_k$ operator for some positive integer $k$, then $T$ is polaroid.

Proof. If $\lambda \in \text{iso}\, \sigma(T)$ using the spectral projection of $T$ with respect to $\lambda$, we can write $T = T_1 \oplus T_2$ where $\sigma(T_1) = \{\lambda\}$ and $\sigma(T_2) = \sigma(T) - \{\lambda\}$. Since $T_1$ is algebraically quasi class $A^*_k$ operator and $\sigma(T_1) = \{\lambda\}$, by theorem 5.2, $T_1 - \lambda I$ is nilpotent. Since $\lambda \notin \sigma(T_2)$, $T_2 - \lambda I$ is invertible. Hence both $T_1 - \lambda I$ and $T_2 - \lambda I$ and hence $T - \lambda I$ have finite ascent and descent. Hence $\lambda$ is a pole of the resolvent of $T$. Hence $T$ is polaroid.

Corollary 5.10. If $T$ is algebraically quasi class $A^*_k$ operator for some positive integer $k$, then $T$ is reguloid.

Corollary 5.11. If $T$ is algebraically quasi class $A^*_k$ operator for some positive integer $k$, then $T$ is isoloid.

If $T^*$ has SVEP, then by ([3], Lemma 2.15) $\sigma_{ea}(T) = \sigma(T)$ and by ([1], corollary 2.45) $\sigma(T) = \sigma_a(T)$. Hence we get the following result.

Corollary 5.12. If $T$ is algebraically quasi class $A^*_k$ for some positive integer $k$ and if in addition $T^*$ has SVEP, then a-Weyl’s theorem holds for $f(T)$ for every $f \in H(\sigma(T))$.

Corollary 5.13. If $T^*$ is algebraically quasi class $A^*_k$ for some positive integer $k$, then $w(f(T)) = f(w(T))$.

By ([3], Theorem 2.17), we get the following results.

Corollary 5.14. If $T$ is algebraically quasi class $A^*_k$ for some positive integer $k$, and $T^*$ has SVEP then property (b) hold for $T$.

Corollary 5.15. If $T$ is algebraically class $A^*_k$ for some positive integer $k$, Weyl’s theorem, a-Weyl’s theorem, property(w) and property(b) hold for $T^*$.

References


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