Some Characterizations of Timelike Curves
According to Bishop Frame in Minkowski 3-Space

Hüseyin Kocayiğit, Ali Özdemir, Muhammed Çetin, Buket Arda

Celal Bayar University, Faculty of Science and Arts
Mathematics Department, Manisa, Turkey

huseyin.kocayigit@cbu.edu.tr, ali.ozdemir@cbu.edu.tr
mat.mcetin@hotmail.com, buket-arda-18@hotmail.com

Abstract. In this study, we give some characterizations of timelike curves according to Bishop Frame in Minkowski 3-space $E^{3}_{1}$ by using Laplacian operator and Levi-Civita connection.

Mathematics Subject Classification : 53A04, 14H45, 53B30, 53A35

Keywords : Bishop Frame, slant helix, Laplacian operator

1. Introduction

It is well-known that a curve of constant slope or general helix is defined by the property that the tangent of the curve makes a constant angle with a fixed straight line which is called the axis of the general helix. A necessary and sufficient condition for a curve to be a general helix is that the ratio of curvature to torsion be constant [9]. The study of these curves in $E^{3}$ has been given by many mathematicians. Moreover, İlarslan studied the characterizations of helices in Minkowski 3-space $E^{3}_{1}$ and found differential equations according to Frenet vectors characterizing the helices in $E^{3}_{1}$ [14]. Then, Kocayiğit obtained general differential equations which characterize the Frenet curves in Euclidean 3-space $E^{3}$ and Minkowski 3-space $E^{3}_{1}$ [10].

Analogue to the helix curve, Izumiya and Takeuchi have defined a new special curve called the slant helix in Euclidean 3-space $E^{3}$ by the property that the principal normal of a space curve $γ$ makes a constant angle with a fixed direction [19]. The spherical images of tangent indicatrix and binormal indicatrix
of a slant helix have been studied by Kula and Yaylı [15]. They obtained that the spherical images of a slant helix are spherical helices. Moreover, Kula et al. studied the relations between a general helix and a slant helix [16]. They have found some differential equations which characterize the slant helix.

Position vectors of slant helices have studied by Ali and Turgut [2]. Also, they have given the generalization of the concept of a slant helix in the Euclidean $n$-space $E^n$ [3].

Furthermore, Chen and Ishikawa classified biharmonic curves, the curves for which $\Delta H = 0$ holds in semi-Euclidean space $E^v_1$ where $\Delta$ is Laplacian operator and $H$ is mean curvature vector field of a Frenet curve [8]. Later, Kocayiğit and Hacısalihoğlu studied biharmonic curves and 1-type curves i.e., the curves for which $H H = \lambda \Delta$ holds, where $\lambda$ is constant, in Euclidean 3-space $E^3$ [13] and Minkowski 3-space $E^3_1$ [12]. They showed the relations between 1-type curves and circular helix and the relations between biharmonic curves and geodesics. Moreover, slant helices have been studied by Bükçü and Karacan according to Bishop Frame in Euclidean 3-space [4] and Minkowski space [5,6]. Characterizations of space curves according to Bishop Frame in Euclidean 3-space $E^3$ have been given in [11].

In this paper, we give some characterizations of timelike curves according to Bishop Frame in Minkowski 3-Space $E^3_1$ by using Laplacian operator. We find the differential equations characterizing timelike curves with Bishop Frame.

2. Preliminaries

Let $\mathbb{R}^3 = \{(x_1, x_2, x_3) : x_1, x_2, x_3 \in \mathbb{R}\}$ be a 3-dimensional vector space, and let $\hat{x} = (x_1, x_2, x_3)$ and $\hat{y} = (y_1, y_2, y_3)$ be two vectors in $\mathbb{R}^3$. The Lorentz scalar product of $\hat{x}$ and $\hat{y}$ is defined by $\langle \hat{x}, \hat{y} \rangle_L = -x_1 y_1 + x_2 y_2 + x_3 y_3$.

$E^3_1 = (\mathbb{R}^3, \langle \hat{x}, \hat{y} \rangle_L)$ is called 3-dimensional Lorentzian space, Minkowski 3-Space or 3-dimensional Semi-Euclidean space. The vector $\hat{x}$ in $E^3_1$ is called a spacelike vector, null vector or a timelike vector if $\langle \hat{x}, \hat{x} \rangle_L > 0$ or $x = 0$,

$\langle \hat{x}, \hat{x} \rangle_L = 0$, $x \neq 0$ or $\langle \hat{x}, \hat{x} \rangle_L < 0$, respectively [7]. Similarly a curve $\gamma$ is called spacelike, timelike or null if $\langle \gamma', \gamma' \rangle_L > 0$, $\langle \gamma', \gamma' \rangle_L < 0$ or $\langle \gamma', \gamma' \rangle_L = 0$, respectively. For $\vec{x} \in E^3_1$, the norm of the vector $\vec{x}$ is defined by $\| \vec{x} \|_L = \sqrt{\langle \vec{x}, \vec{x} \rangle_L}$, and $\vec{x}$ is called a unit vector if $\| \vec{x} \|_L = 1$. For any vectors $\vec{x}, \vec{y} \in E^3_1$, Lorentzian cross product of $\vec{x}$ and $\vec{y}$ is defined by

$\vec{x} \wedge_L \vec{y} = (x_2 y_3 - x_3 y_2, x_3 y_1 - x_1 y_3, x_1 y_2 - x_2 y_1)$

Denoted the moving Frenet frame along a space curve $\gamma$ by $\{\vec{T}, \vec{N}, \vec{B}\}$.
where $\vec{T}, \vec{N}$ and $\vec{B}$ are tangent, principal normal and binormal vector of $\gamma$, respectively. If $\gamma$ is a timelike curve, then this set of orthogonal unit vectors, known as the Frenet frame, has the following properties

\[
\vec{T}' = \kappa \vec{N} \\
\vec{N}' = \kappa \vec{T} + \tau \vec{B} \\
\vec{B}' = -\tau \vec{N}
\]

where $\langle \vec{T}, \vec{T} \rangle_L = -1, \langle \vec{N}, \vec{N} \rangle_L = 1, \langle \vec{B}, \vec{B} \rangle_L = 1$.

The parallel transport frame or Bishop frame is an alternative approach to defining a moving frame that is well-defined even when the timelike curve has vanishing second derivative. We can transport an orthonormal frame along a timelike curve simply by parallel transporting each component of the frame. The parallel transport frame is based on the observation that, while $\vec{T}(s)$ for a given timelike curve model is unique, we may choose convenient arbitrary basis $\left(\vec{N}_1(s), \vec{N}_2(s)\right)$ for the remainder of the frame, so long as it is in the normal plane perpendicular to $\vec{T}(s)$ at each point. If the derivatives of $\left(\vec{N}_1(s), \vec{N}_2(s)\right)$ depend only on $\vec{T}(s)$ and not each other, we can make $\vec{N}_1(s)$ and $\vec{N}_2(s)$ vary smoothly throughout the path regardless of the curvature. We therefore have the alternative frame equations (1) [17,1].

Denote by $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$ the moving Bishop frame along the timelike curve $\gamma(s) : I \subset \mathbb{R} \rightarrow E^3_1$ in the Minkowski 3-space $E^3_1$. For an arbitrary timelike curve $\gamma(s)$ in the space $E^3_1$, the following Bishop formula are given by

\[
\begin{bmatrix}
\nabla_{\gamma} \vec{T} \\
\nabla_{\gamma} \vec{N}_1 \\
\nabla_{\gamma} \vec{N}_2
\end{bmatrix} = \begin{bmatrix}
0 & k_1 & k_2 \\
k_1 & 0 & 0 \\
k_2 & 0 & 0
\end{bmatrix} \begin{bmatrix}
\vec{T} \\
\vec{N}_1 \\
\vec{N}_2
\end{bmatrix}
\]

(See [18]). The relations between $\kappa, \tau, \theta$ and $k_1, k_2$ are given as follows

\[
\kappa(s) = \sqrt{k_1^2 + k_2^2} \\
\theta(s) = \arg \tan \left( \frac{k_2}{k_1} \right), \quad k_1 \neq 0.
\]

So that $k_1$ and $k_2$ effectively correspond to Cartesian coordinate system for the polar coordinates $\kappa, \theta$ with $\theta = \int \tau(s) ds$. The orientation of the parallel transport frame includes the arbitrary choice of integration constant $\theta_0$, which disappears from $\tau$ due to the differentiation [18].

A regular timelike curve $\gamma : I \rightarrow E^3_1$ is called a slant helix if unit vector $\vec{N}_1(s)$ of $\gamma$ makes a constant angle $\theta$ with a fixed unit vector $\vec{U}$; that is,
\[ \left( \ddot{N}_t(s), \ddot{U} \right) = \text{const.}, \text{ for all } s \in I. \]

Let, \( \gamma: I \rightarrow E^3_1 \) be a unit speed timelike curve with nonzero nature curvatures \( k_1, k_2 \). Then \( \gamma \) is a slant helix if and only if \( k_1 / k_2 \) is constant [4].

Let, \( \nabla \) denotes the Levi-Civita connection given by \( \nabla_{\gamma} = \frac{d}{ds} \) where \( s \) is the arclength parameter of the timelike curve \( \gamma \). The Laplacian operator of \( \gamma \) is defined by
\[
\Delta = -\nabla_{\gamma}^2 = -\nabla_{\gamma} \nabla_{\gamma} \quad (2)
\]
(See [12]).

3. Characterizations of Timelike Curves with respect to Bishop Frame in Minkowski 3-Space \( E^3_1 \)

In this section we will give the characterizations of the timelike curves according to Bishop frame in Minkowski 3-space \( E^3_1 \). Furthermore, we will obtain the general differential equations which characterize the timelike curves according to the vectors \( \vec{T}, \vec{N}_1, \vec{N}_2 \) in \( E^3_1 \).

**Theorem 3.1.** Let \( \gamma \) be a unit speed timelike curve in Minkowski 3-space \( E^3_1 \) with Bishop frame \( \lbrace \vec{T}, \vec{N}_1, \vec{N}_2 \rbrace \), curvature \( k_1 \) and torsion \( k_2 \). The differential equation characterizing \( \gamma \) according to the tangent vector \( \vec{T} \) is given by
\[
\lambda_4 \nabla_{\gamma}^3 \vec{T} + \lambda_5 \nabla_{\gamma}^2 \vec{T} + \lambda_2 \nabla_{\gamma} \vec{T} + \lambda_1 \vec{T} = 0
\]
where
\[
\lambda_4 = f, \\
\lambda_5 = -g, \\
\lambda_2 = k_1 k_2 z + t + k_2 k_1^3 - k_1 k_2^3, \\
\lambda_1 = 3 f h + (k_1^2 + k_2^2) g,
\]
and
\[
f = \left( \frac{k_1}{k_2} \right) k_2^2, \quad g = k_2 k_1 - k_1 k_2, \quad h = -k_1 k_1' - k_2 k_2', \quad z = k_2 k_2' - k_1 k_1', \quad t = k_2 k_2' - k_1 k_1'
\]

**Proof.** Let \( \gamma \) be a unit speed timelike curve Bishop frame \( \lbrace \vec{T}, \vec{N}_1, \vec{N}_2 \rbrace \) and \( k_1, k_2 \) be the curvature and torsion of the curve, respectively. By differentiating \( \vec{T} \) three times with respect to \( s \) we find the followings,
\[
\nabla_{\gamma} \vec{T} = k_1 \vec{N}_1 + k_2 \vec{N}_2 \quad (3)
\]
\[
\nabla_{\gamma}^2 \vec{T} = (k_1^2 + k_2^2) \vec{T} + k_1 \vec{N}_1 + k_2 \vec{N}_2 \quad (4)
\]
Characterizations of timelike curves

\( \nabla^3_y \vec{T} = 3(k_1'k_1 + k_2'k_2) \vec{T} + (k_3^3 + k_1k_2^2 + k_1') \vec{N}_1 + (k_3^3 + k_2^3k_2 + k_2') \vec{N}_2. \) \hfill (5)

From (3) and (4) we have
\[ \vec{N}_1 = \frac{k_1(k_1^2 + k_2^2)}{k_2k_1 - k_2k_1} \vec{T} + \frac{k_2}{k_2k_1 - k_2k_1} \nabla \gamma \vec{T} - \frac{k_2}{k_2k_1 - k_2k_1} \nabla^2 \vec{T}, \] \hfill (6)

and
\[ \vec{N}_2 = \frac{k_1(k_1^2 + k_2^2)}{k_1k_2 - k_1k_2} \vec{T} + \frac{k_1}{k_1k_2 - k_1k_2} \nabla \gamma \vec{T} - \frac{k_1}{k_1k_2 - k_1k_2} \nabla^2 \vec{T}. \] \hfill (7)

By substituting (6) and (7) in (5) we get
\[ f \nabla \gamma \vec{T} - g \nabla^2 \vec{T} + (k_1k_2z + t - k_1k_2^3 + k_3k_1^3) \nabla \gamma \vec{T} + (3fh + (k_1^2 + k_2^2)g) \vec{T} = 0 \] \hfill (8)

where
\[ f = \left( \frac{k_1}{k_2} \right)^2, \quad g = k_1k_2 - k_2k_1, \quad h = -k_1k_1' - k_2k_2', \quad z = k_2k_2' - k_1k_1', \quad t = k_2'k_1 - k_1k_2'. \]

Defining
\[ \lambda_3 = f, \]
\[ \lambda_2 = -g, \]
\[ \lambda_1 = k_1k_2z + t + k_2k_2^3 - k_1k_1^3, \]
\[ \lambda_4 = 3fh + (k_1^2 + k_2^2)g, \]

from (8) we get
\[ \lambda_3 \nabla \gamma \vec{T} + \lambda_2 \nabla^2 \vec{T} + \lambda_1 \nabla \gamma \vec{T} + \lambda_4 \vec{T} = 0 \]

which is desired equation.

**Theorem 3.2.** Let \( \gamma \) be a unit speed timelike curve Bishop frame \( \{ \vec{T}, \vec{N}_1, \vec{N}_2 \} \), curvature \( k_1 \) and torsion \( k_2 \). The differential equation characterizing \( \gamma \) according to the vector \( \vec{N}_1 \) is given by
\[ \nabla^3 \vec{N}_1 + \beta_1 \nabla^2 \vec{N}_1 + \beta_2 \nabla \gamma \vec{N}_1 + \beta_3 \vec{N}_1 = 0 \]

where
\[ \beta_1 = -2 \frac{k_1'k_1 - k_2'}{k_1}, \]
\[ \beta_2 = 2 \frac{(k_1')^2 + k_1k_2^2}{k_1} \frac{k_1}{k_1' - k_2^2}, \]
\[ \beta_3 = \frac{k_1^2k_2}{k_2} - k_1k_1'. \]

**Proof.** Let \( \gamma \) be a unit speed timelike curve Bishop frame \( \{ \vec{T}, \vec{N}_1, \vec{N}_2 \} \) and \( k_1 \), \( k_2 \) be curvature and torsion of the curve, respectively. Differentiating \( \vec{N}_1 \) three times with respect to \( s \) gives
\[ \nabla \gamma \vec{N}_1 = k_1 \vec{T} \]  
(9)

\[ \nabla^2 \gamma \vec{N}_1 = k_1^2 \vec{T} + k_1 k_2 \vec{N}_1 + k_2 \vec{N}_2 \]  
(10)

\[ \nabla^3 \gamma \vec{N}_1 = (k_1^3 + k_1^2 k_2^2) \vec{T} + 3k_1 k_2 \vec{N}_1 + (2k_2 k_1 + k_1 k_2^2) \vec{N}_2. \]  
(11)

From (9) and (10) we have

\[ \vec{T} = \frac{1}{k_1} \nabla \gamma \vec{N}_1 \]  
(12)

and

\[ \vec{N}_2 = \frac{1}{k_1 k_2} \nabla^2 \gamma \vec{N}_1 - \frac{k_1}{k_1 k_2} \nabla \gamma \vec{N}_1 - \frac{k_2}{k_1 k_2} \vec{N}_2. \]  
(13)

respectively. By substituting (12) and (13) in (11) we obtain

\[ \nabla^3 \gamma \vec{N}_1 + \left( -2 \frac{k_1}{k_1 k_2} \frac{k_2}{k_2} \right) \nabla^2 \gamma \vec{N}_1 + \left( 2 \frac{(k_2)^2}{k_1} + \frac{k_1 k_2^2}{k_1} - \frac{k_2^2}{k_1} - k_2 \right) \nabla \gamma \vec{N}_1 + \left( \frac{k_1 k_2^2}{k_1} - k_2 \right) \vec{N}_1 = 0 \]  
(14)

Writing

\[ \beta_3 = -2 \frac{k_1}{k_1 k_2} \frac{k_2}{k_2}, \]

\[ \beta_2 = 2 \frac{(k_1)^2}{k_1} \frac{k_2}{k_1 k_2} - \frac{k_2}{k_1} - k_2, \]

\[ \beta_1 = \frac{k_1 k_2^2}{k_2} - k_1 k_2, \]

from (14) we get

\[ \nabla^3 \gamma \vec{N}_1 + \beta_3 \nabla^2 \gamma \vec{N}_1 + \beta_2 \nabla \gamma \vec{N}_1 + \beta_1 \vec{N}_1 = 0 \]

which is desired equation.

If \( \gamma \) is a slant helix in \( E^3 \), then \( \frac{k_1}{k_2} = \text{const.} \), that is \( \frac{k_1}{k_2} = \frac{k_1}{k_2} \). In this case, we have \( \beta_1 = 0 \). Therefore, we give the following corollary.

**Corollary 3.1.** Let \( \gamma \) be a slant helix in \( E^3 \) with Bishop frame \( \{ \vec{T}, \vec{N}_1, \vec{N}_2 \} \), curvature \( k_1 \) and torsion \( k_2 \). The differential equation characterizing \( \gamma \) according to the vector \( \vec{N}_1 \) is given by

\[ \nabla^3 \gamma \vec{N}_1 + \left( -3 \frac{k_1}{k_1 k_2} \right) \nabla^2 \gamma \vec{N}_1 + \left( 3 \frac{(k_2)^2}{k_1} - \frac{k_2}{k_1} - k_2 \right) \nabla \gamma \vec{N}_1 = 0 \]

**Theorem 3.3.** Let \( \gamma \) be a unit speed timelike curve in Minkowski 3-space \( E^3 \) with Bishop frame \( \{ \vec{T}, \vec{N}_1, \vec{N}_2 \} \), curvature \( k_1 \) and torsion \( k_2 \). The differential
equation characterizing $\gamma$ according to the vector $\vec{N}_2$ is given by

$$\nabla^3_\gamma \vec{N}_2 + \eta_3 \nabla^2_\gamma \vec{N}_2 + \eta_2 \nabla_\gamma \vec{N}_2 + \eta_1 \vec{N}_2 = 0$$

where

$$\eta_3 = -2 \frac{k_2}{k_2} - \frac{k_1}{k_1},$$

$$\eta_2 = 2 \frac{(k_2)^2}{k_2} + \frac{k_1 k_2}{k_1 k_2} - \frac{k_2}{k_2} - k_1^2 - k_2^2,$$

$$\eta_1 = \frac{k_1 k_2^2}{k_1} - k_2 k_2.$$  

**Proof.** Let $\gamma$ be a unit speed timelike curve with Bishop frame $\{\vec{T}, \vec{N}_1, \vec{N}_3\}$ and $k_1, k_2$ be curvature and torsion of the curve, respectively. By differentiating $\vec{N}_2$ three times with respect to $s$ we find the followings,

$$\nabla_\gamma \vec{N}_2 = k_2 \vec{T} \tag{15}$$

$$\nabla^2_\gamma \vec{N}_2 = k_2 \vec{T} + k_1 k_2 \vec{N}_1 + k_2^2 \vec{N}_2 \tag{16}$$

$$\nabla^3_\gamma \vec{N}_2 = (k_2^3 + k_1 k_2^2 + k_2^3) \vec{T} + (2k_1 k_2 + k_1 k_2) \vec{N}_1 + 3k_1 k_2^2 \vec{N}_2. \tag{17}$$

From (15) and (16) we have

$$\vec{T} = \frac{1}{k_2} \nabla_\gamma \vec{N}_2 \tag{18}$$

and

$$\vec{N}_1 = \frac{1}{k_1 k_2} \nabla^2_\gamma \vec{N}_2 - \frac{k_2}{k_1 k_2} \nabla_\gamma \vec{N}_2 - \frac{k_2}{k_1} \vec{N}_2. \tag{19}$$

By substituting (18) and (19) in (17) we get

$$\nabla^3_\gamma \vec{N}_2 + \left( -2 \frac{k_2}{k_2} - \frac{k_1}{k_1} \right) \nabla^2_\gamma \vec{N}_2 + \left( 2 \frac{(k_2)^2}{k_2} + \frac{k_1 k_2}{k_1 k_2} - \frac{k_2}{k_2} - k_1^2 - k_2^2 \right) \nabla_\gamma \vec{N}_2 + \left( \frac{1}{k_1 k_2} - k_1 k_2 \right) \vec{N}_2 = 0 \tag{20}$$

writing

$$\eta_3 = -2 \frac{k_2}{k_2} - \frac{k_1}{k_1},$$

$$\eta_2 = 2 \frac{(k_2)^2}{k_2} + \frac{k_1 k_2}{k_1 k_2} - \frac{k_2}{k_2} - k_1^2 - k_2^2,$$

$$\eta_1 = \frac{k_1 k_2^2}{k_1} - k_2 k_2.$$  

from (20) we get

$$\nabla^3_\gamma \vec{N}_2 + \eta_3 \nabla^2_\gamma \vec{N}_2 + \eta_2 \nabla_\gamma \vec{N}_2 + \eta_1 \vec{N}_2 = 0$$

which is desired equation.
If $\gamma$ is a slant helix in $E^3_1$ i.e., $\frac{k_1}{k_2} = \frac{k_2}{k_1}$, then, we obtain the following corollary.

**Corollary 3.2.** Let $\gamma$ be a slant helix in $E^3_1$ with Bishop frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$, curvature $k_1$, torsion $k_2$. The differential equation characterizing $\gamma$ according to the vector $\vec{N}_2$ is given by

$$\nabla^3_{\gamma} \vec{N}_2 - \frac{3}{k_2} \nabla^2_{\gamma} \vec{N}_2 + \left( -\frac{k_1}{k_2} - k_1^2 - k_2^2 + 3 \frac{(k_2)^2}{k_2^2} \right) \nabla_{\gamma} \vec{N}_2 = 0$$

### 4. Timelike Curves with Harmonic 1-type $\vec{T}, \vec{N}_1, \vec{N}_2$ Vectors According to Bishop Frame in Minkowski 3-Space $E^3_1$

In this section we will give the characterizations of the timelike curves with Harmonic 1-type $\vec{T}, \vec{N}_1, \vec{N}_2$ vectors in Minkowski 3-space $E^3_1$.

**Definition 4.1.** A regular timelike curve $\gamma$ in $E^3_1$ said to have harmonic tangent vector if

$$\Delta \vec{T} = 0, \quad (21)$$

holds. Further, a regular timelike curve $\gamma$ in $E^3_1$ said to have harmonic 1-type tangent vector if

$$\Delta \vec{T} = \lambda \vec{T}, \quad \lambda \in \mathbb{R}, \quad (22)$$

holds.

First we prove the following theorem.

**Theorem 4.1.** Let $\gamma$ be a unit speed timelike curve Bishop frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$. Then, $\gamma$ is of harmonic 1-type tangent vector if and only if the curvature $k_1$ and torsion $k_2$ of the curve $\gamma$ satisfy the followings,

$$\lambda = -k_1^2 - k_2^2, \quad k_1 = c_1, \quad k_2 = c_2. \quad (23)$$

where $\lambda, c_1, c_2$ are constants.

**Proof.** Let $\gamma$ be a unit speed timelike curve tangent vector $\vec{T}$ and let $\Delta$ be the Laplacian associated with $\nabla$. One can use (3) and (4) to compute

$$\Delta \vec{T} = -(k_1^2 + k_2^2) \vec{T} - k_1 \vec{N}_1 - k_2 \vec{N}_2 \quad (24)$$

We assume that timelike curve $\gamma$ is of harmonic 1-type tangent vector. Substituting (24) in (22) we have (23).

Conversely, if the equations (23) satisfy for the constant $\lambda$, then it is easy to show that $\gamma$ is of harmonic 1-type tangent vector.
Corollary 4.1. Let $\gamma$ be a unit speed timelike curve. Then, $\gamma$ is of harmonic 1-type tangent vector if and only if $\gamma$ is a slant helix, with constant curvature and constant torsion.

Corollary 4.2. Let $\gamma$ be a unit speed timelike curve in $E^3_1$ with Bishop frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$. Then, $\gamma$ has harmonic tangent vector, if and only if $k_1(s) = k_2(s) = 0$.

Let, now consider the characterization of the timelike curve $\gamma$ according to the vector $\vec{N}_1$. Similar to the Definition 4.1, we can give the following definition.

Definition 4.2. A regular timelike curve $\gamma$ in $E^3_1$ said to have harmonic vector $\vec{N}_1$ if

$$\Delta \vec{N}_1 = 0,$$  \hspace{1cm} (25)

holds. Further, a regular timelike curve $\gamma$ in $E^3_1$ said to have harmonic 1-type vector $\vec{N}_1$ if

$$\Delta \vec{N}_1 = \lambda \vec{N}_1, \hspace{0.5cm} \lambda \in \mathbb{R}$$  \hspace{1cm} (26)

holds.

Theorem 4.2. Let $\gamma$ be a unit speed timelike curve. Then, $\gamma$ is of harmonic 1-type vector $\vec{N}_1$ if and only if the curvature $k_1$ and the torsion $k_2$ of the curve $\gamma$ satisfy the followings,

$$\lambda = -k_2^3, \hspace{0.5cm} k_1 = \text{const.}, \hspace{0.5cm} k_2 = 0.$$  \hspace{1cm} (27)

Proof. Let $\gamma$ be a unit speed timelike curve and let $\Delta$ be the Laplacian associated with $\nabla$. One can use (9) and (10) to compute

$$\Delta \vec{N}_1 = -k_1 \vec{T} - k_1^2 \vec{N}_1 - k_1 k_2 \vec{N}_2.$$  \hspace{1cm} (28)

We assume that the timelike curve $\gamma$ is of harmonic 1-type vector $\vec{N}_1$. Substituting (28) in (26) we have (27).

Conversely, if the equations (27) satisfy for the constant $\lambda$, then it is easy to show that timelike curve $\gamma$ is of harmonic 1-type vector $\vec{N}_1$.

Corollary 4.3. Let $\gamma$ be a unit speed timelike curve in $E^3_1$ with Bishop frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$. Then, $\gamma$ has harmonic vector $\vec{N}_1$, if and only if $k_1(s) = 0$.

Finally, let give the characterization of $\gamma$ timelike curve with according to the vector $\vec{N}_2$. 

Definition 4.3. A regular timelike curve $\gamma$ in $E^3_1$ said to have harmonic vector $\vec{N}_2$ if
\[ \Delta \vec{N}_2 = 0, \] (29)
holds. Further, a regular timelike curve $\gamma$ in $E^3_1$ said to have harmonic 1-type vector $\vec{N}_2$ if
\[ \Delta \vec{N}_2 = \lambda \vec{N}_2, \quad \lambda \in \mathbb{R}, \] (30)
holds.

Theorem 4.3. Let $\gamma$ be a unit speed timelike curve in $E^3_1$ with Bishop frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$. Then, $\gamma$ is of harmonic 1-type vector $\vec{N}_2$ if and only if the curvature $k_1$ and the torsion $k_2$ of the curve $\gamma$ satisfy the followings,
\[ \lambda = -k_2^2, \quad k_2 = \text{const.}, \quad k_1 = 0 \] (31)
Proof. Let $\gamma$ be a unit speed timelike curve and let $\Delta$ be the Laplacian associated with $\nabla$. One can use (15) and (16) to compute
\[ \Delta \vec{N}_2 = -k_2^2 \vec{T} - k_2 k_1 \vec{N}_1 - k_2 \vec{N}_2 \] (32)
We assume that the timelike curve $\gamma$ is of harmonic 1-type vector $\vec{N}_2$. Substituting (32) in (30) we have (31).

Conversely, if the equations (31) satisfy for the constant $\lambda$, then it is easy to show that timelike curve $\gamma$ is of harmonic 1-type vector $\vec{N}_2$.

Corollary 4.4. Let $\gamma$ be a unit speed timelike curve. Then, $\gamma$ has harmonic vector $\vec{N}_2$, if and only if $k_2(s) = 0$.

Let now consider the general characterizations of a Bishop timelike curve $\gamma$ according to the Laplacian operator $\Delta$. Then, by considering the vectors $\vec{T}, \vec{N}_1$ and $\vec{N}_2$ we obtain the followings.

Theorem 4.4. Let $\gamma$ be a unit speed timelike curve in $E^3_1$ with Bishop frame $\{\vec{T}, \vec{N}_1, \vec{N}_2\}$. Then,
\[ \Delta \vec{T} + \lambda \vec{N}_2 + \mu \vec{T} = 0, \] (33)
holds along $\gamma$ for the constants $\lambda$ and $\mu$, if and only if $\gamma$ is a slant helix, with curvature and the torsion
\[ k_1 = ck_2, \quad k_2 = \mp \sqrt{\frac{\mu}{c^2 + 1}}, \]
where $c$ is constant.
**Proof.** Assume that (33) holds along $\gamma$. Then from the equalities (3), (4) and (33) we have

$$\mu - k_1^2 - k_2^2 = 0, \quad \lambda k_1 - k_1' = 0, \quad \lambda k_2 - k_2' = 0$$

(34)

The second and third equation of the system (34) gives that $k_1 / k_2$ is constant, i.e., $\gamma$ is a slant helix. Furthermore, from the equations of the system (34) we get

$$k_1 = ck_2$$

(35)

and

$$k_2 = \mp \sqrt{\frac{\mu}{c^2 + 1}}$$

(36)

where $c$ is constant.

Conversely, if $\gamma$ is a slant helix with curvature $k_1$ and torsion $k_2$ given by (35) and (36), respectively, it is easily seen that (34) holds.

**Theorem 4.5.** Let $\gamma$ be a unit speed timelike curve and $\mu$ be a nonzero constant. Then,

$$\Delta \vec{N}_1 + \mu \vec{N}_1 = 0,$$

(37)

holds along the curve $\gamma$ if and only if,

$$k_1 = \mp \sqrt{\mu}, \quad k_2 = 0$$

(38)

**Proof.** Assume that (37) holds along timelike curve $\gamma$. Then from the equality (28) we have

$$\mu - k_1^2 = 0, \quad k_1 k_2 = 0$$

(39)

from the equations of the system (39) we get

i) If $k_1 \neq 0$, $k_1 = \mp \sqrt{\mu}$ and $k_2 = 0$.

ii) If $k_1 = 0$, $\mu = 0$

Conversely, if (39) holds then (37) is satisfied.

**Theorem 4.6.** Let $\gamma$ be a unit speed timelike curve and $\rho$ be a nonzero constant. Then,

$$\Delta \vec{N}_2 + \rho \vec{N}_2 = 0$$

(40)

holds along the timelike curve $\gamma$ if and only if

$$k_2 = \mp \sqrt{\rho}, \quad k_1 = 0.$$ 

(41)

**Proof.** Assume that (40) holds along the timelike curve $\gamma$. Then from the equality (32) we have

$$\rho - k_2^2 = 0, \quad k_1 k_2 = 0.$$ 

(42)

From the equations of the system (42) we get

i) If $k_2 \neq 0$, $k_2 = \mp \sqrt{\rho}$ and $k_1 = 0$.

ii) If $k_2 = 0$, $\rho = 0$
Conversely if (43) and (44) hold, then it is easily seen that (40) is satisfied.

References


Characterizations of timelike curves


Received: October, 2012