Poincare Bifurcation of a Kind of Hamiltonian System under Polynomial Perturbation¹

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Abstract

It was proved that the sharp upper bound of the number of zeros of Abelian integrals and the number of limit cycles bifurcated from Poincare bifurcation were $B(n)$. An explicit $B(n)$ is derived for the number of zeros of Abelian integrals $I(h) = \oint_{\Gamma(h)} f(x, y) dy - g(x, y) dx$ on the open interval $(0, \infty)$, where $\Gamma(h)$ is an oval lying on the algebraic curve $H(x, y) = x^{2a}/A + y^{2b}/B = h$, $f(x, y), g(x, y)$ are polynomials of x and y , and $\max\{\deg f(x, y), \deg g(x, y)\} = n$. Assume $I(h)$ not vanish identically, $c = \gcd(a, b)$, $\lambda = \max\{a/c, b/c\}$, then $B(n) = \frac{1}{2}[\frac{n-1}{2}][\frac{n-1}{2}] + 3$ for $n \leq 2\lambda$, $B(n) = \lambda[\frac{n-1}{2}] - \frac{1}{2}(\lambda-1)(\lambda-2)$ for $n \geq 2\lambda + 1$.

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1 Introduction and statement of the main results

Consider a real polynomial $H(x, y)$ of degree $d + 1$ in the plane. A closed connected component of a level curve $H = h$ is denoted by $\Gamma(h)$ and called an oval of H . Let $\Sigma = (h_1, h_2) \subset \mathbb{R}^1$ be a maximal interval of existence of $\Gamma(h)$.

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Let $\omega = -f(x, y) dy + g(x, y) dx$ be a real 1-form with polynomial coefficients of degree at most n . Define the complete Abelian integral

$$I(h) = \oint_{\Gamma(h)} \omega, \quad h \in \Sigma. \quad (1)$$

Any small perturbation corresponds to a polynomial vector field of degree n reads

$$dH + \epsilon\omega = 0. \quad (2)$$

Here, ϵ is a small parameter. The general problem of estimating the number of limit cycles in a polynomial unfolding (2) of a polynomial system having a center, is known as the weakened Hilbert's 16th problem, initially posed by Arnold[1]. The problem reduces to the study of the first order approximation term $I(h)$ of the corresponding displacement map

$$d(h, \epsilon) = \epsilon I(h) + O(\epsilon^2). \quad (3)$$

Namely, the zeros of $I(h)$ in the interval Σ where the level sets $H = h$ form a continuous family of ovals (period annuli), correspond to the limit cycles emerging from this period annulus for ϵ small (*i.e.* generated by Poincare bifurcation).

The weakened Hilbert's problem has been solved only in its original form and in the quadratic case $n = 2$. It is known in this case that the function $I(h)$ can have at most two zeros. The general problem is still open even in the case $n = 2$. The general results about this problem was achieved in [6,11], where the existence of an upper bound $Z(d, n)$, the number of zeros of the integral (1), was proved, but its explicit expression is not solved except for some special cases. Petrov[10] achieved $Z(2, n) \leq n - 1$ for $H(x, y) = y^2 + x^3 - x$. Li & Zhang[7] obtained $M_2(h)$ has at most $2(n - 1)$ zeros for n even and at most $2n - 3$ zeros for n odd when $M_1(h) \equiv 0$ for $H(x, y) = y^2 + x^3 - x$. Gavrilov[2] gave $Z(2, n) \leq [(2/3)(n - 1)]$ for $H(x, y) = (1/2)(x^2 + y^2) - (1/3)x^3 + xy^2$ ($[p]$ denotes the entire part of p). Horozov & Iliev[3] concluded $Z(2, n) \leq 5n + 15$ for the Abelian integrals corresponding to cubic Hamiltonians. Zhao & Zhang[12] stated $Z(3, n) \leq 7n + 5$ for $H(x, y) = (1/2)y^2 + U(x)$, where $\deg U(x) = 4$. Iliev[4] achieved $M_k(h)$ has at most $k(n - 1)$ zeros, counting the multiplicity, where $M_k(h)$ is the first Melnikov function in (3) not vanishing identically for $H(x, y) = (1/2)(x^2 + y^2) - (1/3)x^3$. And further references therein[5,8,9,13].

In this paper, we consider one kind of Hamiltonian with symmetry under polynomial perturbation

$$\begin{cases} \dot{x} = \frac{\partial H(x, y)}{\partial y} + \epsilon f(x, y) \\ \dot{y} = -\frac{\partial H(x, y)}{\partial x} + \epsilon g(x, y) \end{cases}, \quad (4)$$

where $H(x, y) = x^{2a}/A + y^{2b}/B$, A, B are positive real numbers, and a, b are positive integers, $f(x, y)$ and $g(x, y)$ are polynomials in x and y with real coefficients, and $\max\{\deg f(x, y), \deg g(x, y)\} = n$, ϵ as given above in (2). In the present paper, we investigate the limit cycles generated from the Poincare bifurcation of the above system and obtain the following main result. Additionally, in the following gcd (*resp.* lcm) is short for greatest common divisor (*resp.* least common multiple).

Theorem 1.1 *Assume $I(h)$ is not vanish identically, $c = \gcd(a, b)$, $\lambda = \max\{a/c, b/c\}$. Then for (4), $B(n) = \frac{1}{2}[\frac{n-1}{2}](\frac{n-1}{2} + 3)$ for $n \leq 2\lambda$, $B(n) = \lambda[\frac{n-1}{2}] - \frac{1}{2}(\lambda - 1)(\lambda - 2)$ for $n \geq 2\lambda + 1$. $B(n)$ is the sharp upper bound of the number of zeros of Abelian integrals $I(h)$ and the number of limit cycles bifurcated from Poincare bifurcation of (4).*

The rest of this paper is organized as follows. In Sec. 2, $I(h)$ can be expressed as a polynomial of new variable u , which is

$$A(u) = \sum_{i+j \leq [\frac{n-1}{2}]} a_{ij} u^{i\alpha + j\beta}.$$

In Sec. 3, we can derive the number of terms of $A(u)$ and obtain the number of zeros of $A(u)$, using these results, we prove our main Theorem 1.1.

2 Preliminary

We derive the following preliminary results which will be used later on. The Abelian integral for system (4) is the following

$$\begin{aligned} I(h) &= \oint_{\Gamma_h} g(x, y) dx - f(x, y) dy \\ &= \iint_{\text{Int } \Gamma_h} \sum_{i+j \leq n-1} c_{ij} x^i y^j dx dy \\ &= \iint_{\text{Int } \Gamma_h} \sum_{i+j \leq [\frac{n-1}{2}]} c_{2i, 2j} x^{2i} y^{2j} dx dy \quad (\text{by symmetry}) \\ &= \oint_{\Gamma_h} \sum_{i+j \leq [\frac{n-1}{2}]} \frac{c_{2i, 2j}}{2j+1} x^{2i} y^{2j+1} dx \\ &= 4 \int_{\substack{\Gamma_h \\ x \geq 0 \\ y \geq 0}} \sum_{i+j \leq [\frac{n-1}{2}]} \frac{c_{2i, 2j}}{2j+1} x^{2i} y^{2j+1} dx. \end{aligned}$$

Let

$$\begin{cases} x = x(h, \theta) = A^{\frac{1}{2a}} h^{\frac{1}{2a}} (\cos \theta)^{\frac{1}{a}} \\ y = y(h, \theta) = B^{\frac{1}{2b}} h^{\frac{1}{2b}} (\sin \theta)^{\frac{1}{b}} \end{cases}, \quad \theta \in [0, \frac{\pi}{2}],$$

then

$$\begin{aligned} I(h) &= 4 \int_{\frac{\pi}{2}}^0 \sum_{i+j \leq [\frac{n-1}{2}]} \frac{c_{2i,2j}}{a(2j+1)} \left(A^{\frac{1}{2a}} h^{\frac{1}{2a}} (\cos \theta)^{\frac{1}{a}} \right)^{2i} \left(B^{\frac{1}{2b}} h^{\frac{1}{2b}} (\sin \theta)^{\frac{1}{b}} \right)^{2j+1} \\ &\quad \times A^{\frac{1}{2a}} h^{\frac{1}{2a}} (\cos \theta)^{\frac{1}{a}-1} (-\sin \theta) d\theta \\ &= \sum_{i+j \leq [\frac{n-1}{2}]} a_{ij} h^{\frac{2i+1}{2a} + \frac{2j+1}{2b}}, \end{aligned}$$

where

$$a_{ij} = \frac{2c_{2i,2j}}{a(2j+1)} A^{\frac{2i+1}{2a}} B^{\frac{2j+1}{2b}} \mathcal{B}\left(\frac{2i+1}{2a}, \frac{2j+2b+1}{2b}\right),$$

and $\mathcal{B}(p, q)$ is a Beta function, $\mathcal{B}(p, q) = 2 \int_0^{\frac{\pi}{2}} (\cos \theta)^{2p-1} (\sin \theta)^{2q-1} d\theta \neq 0$. Assume $\gcd(a, b) = c$, then there exist positive integers α, β suffice to $a = \alpha c$, $b = \beta c$, and $\gcd(\alpha, \beta) = 1$, also $\lambda = \max\{a/c, b/c\} = \max\{\alpha, \beta\}$, $\text{lcm}(a, b) = \alpha\beta c$. Take $u = h^{\frac{1}{\alpha\beta c}}$, then $u \in (0, \infty)$ and

$$\begin{aligned} I(h) &= \sum_{i+j \leq [\frac{n-1}{2}]} a_{ij} h^{\frac{(2i+1)\alpha + (2j+1)\beta}{2\alpha\beta c}} \\ &= u^{\frac{\alpha+\beta}{2}} \sum_{i+j \leq [\frac{n-1}{2}]} a_{ij} u^{i\alpha+j\beta}. \end{aligned}$$

Denote $A(u) = \sum_{i+j \leq [\frac{n-1}{2}]} a_{ij} u^{i\alpha+j\beta}$, $u \in (0, \infty)$. Next section we study the property of polynomial $A(u)$ in $(0, \infty)$.

3 Proof of Theorem 1.1

Lemma 3.1 Denote the set $X := \{p\gamma + q\delta \mid p, q \in \mathbb{N}, p+q \leq K, \gamma, \delta \in \mathbb{N}^+, \gcd(\gamma, \delta) = 1, \gamma \geq \delta\}$, then $|X| = \frac{1}{2}(K+1)(K+2)$ for $K \leq \gamma-1$, $|X| = \gamma K + 1 - \frac{1}{2}(\gamma-1)(\gamma-2)$ for $K \geq \gamma$, where $|X|$ denote the number of elements in the set X .

Proof Obviously it holds for $\gamma = 1$. When $\gamma > 1$, denote $A_m = \{l\gamma + m \mid l \in \mathbb{Z}\}$, $B_m = \{l\gamma + m\delta \mid l \in \mathbb{Z}\}$. Since $\gcd(\gamma, \delta) = 1$, all the divisor from γ by $0, \delta, 2\delta, \dots, (\gamma-1)\delta$ is different each other. Hence, $B_0, B_1, \dots, B_{\gamma-1}$ is a rearrangement of $A_0, A_1, \dots, A_{\gamma-1}$. By $A_i \cap A_j = \emptyset (i \neq j)$ and $\bigcup_{m=0}^{\gamma-1} A_m = \mathbb{Z}$,

we obtain $B_i \cap B_j = \emptyset (i \neq j)$, $\bigcup_{m=0}^{\gamma-1} B_m = \mathbb{Z}$, $X = \bigcup_{m=0}^{\gamma-1} (B_m \cap X)$, and $|X| = \sum_{m=0}^{\gamma-1} |B_m \cap X|$.

Furthermore,

$$B_0 \cap X = \{p\gamma \mid 0 \leq p \leq K\} \Rightarrow |B_0 \cap X| = K + 1,$$

$$B_1 \cap X = \{p\gamma + \delta \mid 0 \leq p \leq K - 1\} \Rightarrow |B_1 \cap X| = K,$$

$$B_2 \cap X = \{p\gamma + 2\delta \mid 0 \leq p \leq K - 2\} \Rightarrow |B_2 \cap X| = \max\{0, K - 1\},$$

$$\dots,$$

$$B_{\gamma-1} \cap X = \{p\gamma + (\gamma-1)\delta \mid 0 \leq p \leq K - (\gamma-1)\} \Rightarrow |B_{\gamma-1} \cap X| = \max\{0, K - \gamma + 2\}.$$

Hence,

$$|X| = \sum_{m=0}^{\gamma-1} \max\{0, K - m + 1\},$$

then,

$$|X| = (K + 1) + K + \dots + 2 + 1 = \frac{1}{2}(K + 1)(K + 2)$$

for $K \leq \gamma - 1$, and

$$|X| = (K + 1) + K + \dots + (K - \gamma + 2) = \gamma K + 1 - \frac{1}{2}(\gamma - 1)(\gamma - 2)$$

for $K \geq \gamma$.

Lemma 3.2 Denote $F(x) = \sum_{k=0}^m C_k x^{a_k}$, $0 = a_0 < a_1 < \dots < a_m$, then the isolated zeros of $F(x)$ in $(0, \infty)$ is less than m , and there exist the real numbers C_i , $i = 0, \dots, m$, let $F(x)$ has exactly m simple zeros in $(0, \infty)$.

Proof Without any loss of generality, assume $\sum_{i=0}^m C_i^2 \neq 0$.

Firstly, we get the isolated zeros $F(x)$ in $(0, \infty)$ is less than m by the induction. Suppose it holds for $m = l - 1$, then for $m = l$,

$$F'(x) = \sum_{k=0}^l C_k a_k x^{a_k-1} = x^{a_1-1} (C_1 a_1 + C_2 a_2 x^{a_2-a_1} + \dots + C_l a_l x^{a_l-a_1}),$$

where $C_1 a_1 + C_2 a_2 x^{a_2-a_1} + \dots + C_l a_l x^{a_l-a_1}$ has $l - 1$ zeros at most in $(0, \infty)$. From Rolle Theorem, $F(x)$ has l zeros at most in $(0, \infty)$.

Secondly, one elaborate there exist real numbers C_i , $i = 0, \dots, m$, let $F(x)$ has exactly m simple zeros in $(0, \infty)$. One can put $C_0 = 1$, suppose $F(x) = \sum_{k=0}^m C_k x^{a_k} = 0$ has m positive roots $1, 2, \dots, 2^{m-1}$, then $\sum_{k=0}^m C_k =$

$0, \sum_{k=0}^m C_k 2^{a_k} = 0, \dots, \sum_{k=0}^m C_k (2^{m-1})^{a_k} = 0$. Consider a set of m linear equations in m unknowns $C_1, C_2, \dots, C_m$, and

$$\sum_{k=0}^m C_k = -1, \sum_{k=0}^m C_k 2^{a_k} = -1, \dots, \sum_{k=0}^m C_k (2^{m-1})^{a_k} = -1.$$

The determinant of coefficient matrix D of above linear system is,

$$\begin{aligned} \det D &= \begin{vmatrix} 1 & 1 & \dots & 1 \\ 2^{a_0} & 2^{a_1} & \dots & 2^{a_m} \\ \dots & \dots & \dots & \dots \\ (2^{m-1})^{a_0} & (2^{m-1})^{a_1} & \dots & (2^{m-1})^{a_m} \end{vmatrix} \\ &= \begin{vmatrix} 1 & 1 & \dots & 1 \\ 2^{a_0} & 2^{a_1} & \dots & 2^{a_m} \\ (2^{a_0})^2 & (2^{a_1})^2 & \dots & (2^{a_m})^2 \\ \dots & \dots & \dots & \dots \\ (2^{a_0})^{m-1} & (2^{a_1})^{m-1} & \dots & (2^{a_m})^{m-1} \end{vmatrix} \\ &= \prod_{0 \leq i < j \leq m} (2^{a_j} - 2^{a_i}) \neq 0, \end{aligned}$$

Hence, the m th linear equation exist unique real solution $C_1, C_2, \dots, C_m$, so as to $F(x)$ at least has m zeros in $(0, \infty)$.

Combine the two parts, $F(x)$ has exactly m simple zeros in $(0, \infty)$.

Proof[Proof of Theorem 1.1] From Lemma 3.1, for $n \leq 2\lambda$, $K = \lfloor \frac{n-1}{2} \rfloor \leq \lambda - 1$, polynomial $A(u)$ has $\frac{1}{2}(\lfloor \frac{n-1}{2} \rfloor + 1)(\lfloor \frac{n-2}{2} \rfloor + 2)$ terms, by Lemma 2, the zeros of $A(u)$ in $(0, \infty)$ is less than $B(n) = \frac{1}{2}(\lfloor \frac{n-1}{2} \rfloor + 1)(\lfloor \frac{n-2}{2} \rfloor + 2) - 1 = \frac{1}{2}\lfloor \frac{n-1}{2} \rfloor(\lfloor \frac{n-2}{2} \rfloor + 3)$. Similarly, for $n \geq 2\lambda + 1$, $K = \lfloor \frac{n-1}{2} \rfloor \geq \lambda$, polynomial $A(u)$ has $\lambda\lfloor \frac{n-1}{2} \rfloor + 1 - \frac{1}{2}(\lambda - 1)(\lambda - 2)$ terms, the positive zeros of $A(u)$ is less than $B(n) = \lambda\lfloor \frac{n-1}{2} \rfloor + 1 - \frac{1}{2}(\lambda - 1)(\lambda - 2) - 1 = \lambda\lfloor \frac{n-1}{2} \rfloor - \frac{1}{2}(\lambda - 1)(\lambda - 2)$.

From Lemma 3.2, we can get the sharp upper bound $B(n)$ of positive zeros by construct the real numbers C_i , $i = 0, \dots, m$ (i.e. construct polynomials $f(x, y)$ and $g(x, y)$). The number of zeros of Abelian integrals $I(h)$ equals the positive zeros of polynomial $A(u)$, which the sharp upper bound both is $B(n)$.

From above discussion, $I(h)$ has exactly $B(n)$ simple zeros in $(0, \infty)$, that is, system (4) have $B(n)$ hyperbolic limit cycles in a neighborhood of the periodic orbits Γ_h . Therefore, the sharp bound of the limit cycles generated by Poincare bifurcation of system (4) also is $B(n)$.

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