An Output Stabilization of Bilinear Distributed Systems

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Abstract

The aim of this paper is to study regional stabilization of the flux of bilinear distributed systems. More precisely it consists in studying the asymptotic behavior of the gradient of such a system not in its whole geometrical evolution domain Ω but only in a subregion ω of Ω . Then we give definitions and under suitable condition we give gradient stabilizing control. We also characterize the control which stabilizes regionally the gradient, and minimizes a given performance cost. Then we develop a numerical approach that is successfully illustrated by simulations.

Keywords : Distributed bilinear systems - Stability - Gradient stability - Regional gradient stability

1 Introduction

The problem of regional stabilization has been the object of various works [8, 6], and it consists in studying the behavior of a distributed system, not in its whole geometrical evolution domain, but just in a subregion which may be inside or in the boundary of this domain. Many approaches were used to characterize different kinds of stabilization, and mainly characterization of control which achieves the stability and minimizing a given cost criterion. Later the notion of regional stabilization was developed for bilinear system [9, 10] where the authors give sufficient conditions to obtain weak, and strong stabilization.

Recently the notion of gradient stabilization was introduced by Zerrik et al [6, 7] in the global and in the regional cases, and it concerns the study of the asymptotic behavior of the gradient for linear distributed system. Necessary and sufficient conditions to stabilize the gradient of the considered system, using the spectrum proprieties for the weak stabilization, the strong stabilization

were obtained under a dissipation hypothesis. The exponential stabilization is obtained by the resolution of a Riccati equation. Examples of unstable systems with a stabilizable gradient are given.

This paper proposes to extend the above results on gradient stabilization to bilinear distributed systems which are close to real application and is organized as follows: In the next section we define different kind of regional gradient stabilization of bilinear distributed systems, and we give sufficient condition to achieve the stabilization of such systems. The third section is focussed on the characterization of control that regionally stabilizes the gradient and minimizes a quadratic performance cost. We also establish an estimation of the stabilization error and the developed approach leads to numerical algorithm. Illustrative simulations are developed for one-dimensional distributed bilinear systems.

2 Regional gradient stabilization

2.1 Preliminaries

Let Ω be an open regular domain of $\mathbb{R}^n$, and ω a nonempty subregion of Ω . We consider a bilinear distributed system given by :

$$\begin{cases} \dot{y}(t) &= Ay(t) + v(t)By(t) & t \geq 0 \\ y(0) &= y_0 \end{cases} \quad (1)$$

where A is a linear operator with domain $\mathcal{D}(A) \subset H^1(\Omega)$, and generates a linear strongly continuous semigroup $(T(t))_{t \geq 0}$ on $H^1(\Omega)$, endowed with its usual complex inner product, denoted by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the associated norm.

We define the operator :

$$\begin{aligned} \nabla_\omega : H^1(\Omega) &\longrightarrow (L^2(\omega))^n \\ y &\longmapsto \left(\chi_\omega \frac{\partial y(x)}{\partial x_1}, \chi_\omega \frac{\partial y(x)}{\partial x_2}, \dots, \chi_\omega \frac{\partial y(x)}{\partial x_n} \right) \end{aligned} \quad (2)$$

where χ_ω is the restriction operator to ω , defined by

$$\begin{aligned} \chi_\omega : L^2(\Omega) &\longrightarrow L^2(\omega) \\ y &\longmapsto y|_\omega \end{aligned} \quad (3)$$

$(L^2(\Omega))^n$ is endowed with its usual complex product $\langle \cdot, \cdot \rangle_n$ and $\|\cdot\|_n$ is the corresponding norm, and $(L^2(\omega))^n$ is endowed with the restriction of $\langle \cdot, \cdot \rangle_n$. B is a linear bounded operator mapping $H^1(\Omega)$ into its self, and let denote $G_\omega = \nabla_\omega^* \nabla_\omega$, where ∇_ω^* is the adjoint operator of ∇_ω .

Definition 2.1 *The system (1) is said to be regionally weakly gradient stabilizable (r.w.g.s) on ω respectively (strongly gradient stabilizable (r.s.g.s), exponentially gradient stabilizable (r.e.g.s)), if for any initial condition $y_0 \in H^1(\Omega)$ the corresponding solution $y(t)$ of (1) is global and $\nabla_\omega y(t)$ converges to 0 weakly respectively (strongly, exponentially) as $t \rightarrow +\infty$.*

2.2 Stabilization control

We are concerned with the problem of regional gradient stabilization which consists in finding an appropriate quadratic feedback control that stabilizes regionally the gradient of the system (1).

We consider controls of the form

$$v(t) = -\langle y(t), Dy(t) \rangle \quad (4)$$

where D is a linear bounded operator mapping $H^1(\Omega)$ into its self.

The following result gives sufficient conditions for regional weak gradient stabilization of system (1) using the control (4) with $D = \widetilde{D}B$ where $\widetilde{D} \in \mathcal{L}(H^1(\Omega))$

Proposition 2.2 *Assume that :*

1. $(T(t))$ is a semigroup of contractions
2. B is compact
3. $\operatorname{Re}(\langle \widetilde{D}Bz, z \rangle \langle z, Bz \rangle) \geq 0 \quad \forall z \in H^1(\Omega)$
4. $\langle \widetilde{D}BT(t)z, T(t)z \rangle \langle T(t)z, BT(t)z \rangle = 0 \quad \forall t \geq 0 \Rightarrow \nabla_\omega z = 0$

then system (1) is regionally weakly gradient stabilizable using the control (4).

Proof : Using 1 and 3 system (1) has a unique global mild solution, using the control (4) (see [9]).

Let $F(z) = -\langle z, \widetilde{D}Bz \rangle Bz$, since F is locally Lipschitz (B maps bounded sets into bounded sets), and B compact, then there exists $z_0 \in H^1(\Omega)$ such that $y(t) \rightarrow z_0$ weakly as $t \rightarrow \infty$ and $\langle F(y(t)), y(t) \rangle = 0, \forall t > 0$ (see ([2])).

So $\langle \widetilde{D}BT(t)z_0, T(t)z_0 \rangle \langle T(t)z_0, BT(t)z_0 \rangle = 0, \forall t > 0$, using 4. we obtain $\nabla_\omega z_0 = 0$.

Let $\tilde{z} \in (\mathcal{L}^2(\Omega))^n$, ∇_ω is a continuous operator then $\langle \nabla_\omega y(t), \tilde{z} \rangle_n \rightarrow \langle \nabla_\omega z_0, \tilde{z} \rangle_n$ as $t \rightarrow \infty$, we deduce that system (1) is regionally weakly gradient stabilizable. We consider the following illustrative example.

Example 2.3 Let $\Omega =]0, 1[^2$, we consider the following system:

$$\begin{cases} \frac{\partial y}{\partial t}(x, z, t) = 0.01\Delta y(x, z, t) + \langle y(t), \varphi_{30} \rangle \varphi_{30} + v(t)By(x, z, t) & \Omega \times]0, +\infty[\\ \frac{\partial y}{\partial \nu}(\xi, \tilde{\xi}, t) = 0 & \partial\Omega \times]0, +\infty[\\ y(0) = y_0 \in H^1(\Omega) \end{cases} \quad (5)$$

where Δ is the Laplace operator.

The operator $A = 0.01\Delta + \langle y(t), \varphi_{30} \rangle \varphi_{30}$, has an orthonormal basis of eigenfunctions $\varphi_{ij}(x, z) = 2a_{ij} \cos(i\pi x) \cos(j\pi z)$, with $a_{ij} = (1 + (i^2 + j^2)\pi^2)^{-\frac{1}{2}}$, and the corresponding eigenvalues are given by :

$$\lambda_{ij} = \begin{cases} -0.01(i^2 + j^2)\pi^2 & \text{if } (i, j) \in \Gamma \\ -0.09\pi^2 + 1 & \text{else} \end{cases}$$

where $\Gamma = \{(i, j) \in \mathbb{N}^2 | (i, j) \neq (3, 0)\}$, A generates a strongly continuous semigroup given by

$$T(t)y = e^{(-0.09\pi^2+1)t} \langle y, \varphi_{30} \rangle \varphi_{30} + \sum_{(i,j) \in \Gamma} e^{\lambda_{ij}t} \langle y, \varphi_{ij} \rangle \varphi_{ij}.$$

Let consider the operator $By = \sum_{(i,j) \in \Gamma} \frac{1}{\lambda_{ij}} \langle y, \varphi_{ij} \rangle \varphi_{ij}$.

For $(i, j) \in \Gamma$, $(\frac{1}{\lambda_{ij}}, \varphi_{ij})$, are the eigencouples of B , and since $\lim_{i,j \rightarrow +\infty} \frac{1}{\lambda_{ij}} = 0$ then B is a compact operator.

Let $\omega = \{\frac{1}{6}\} \times [0, 1]$ be the target subregion.

We have $\langle BT(t)y, T(t)y \rangle = \sum_{(i,j) \in \Gamma} \frac{1}{\lambda_{ij}} e^{2\lambda_{ij}t} \langle y, \varphi_{ij} \rangle^2$. Then

$$\langle BT(t)y, T(t)y \rangle = 0, \forall t \geq 0 \Rightarrow \langle y, \varphi_{ij} \rangle = 0 \quad \forall (i, j) \in \Gamma \quad (6)$$

Since $\nabla_{\omega} \varphi_{30} = 0$ and from (6) we obtain $\nabla_{\omega} y = 0$, then system (5) is r.w.g.s. We remark also that system (5) isn't weakly stabilizable (for $y_0 = \varphi_{30}$, the solution of (5) $y(t) = e^{(-0.09\pi^2+1)t} \varphi_{30} \neq 0$ don't converge to 0 as $t \rightarrow +\infty$).

2.3 Decomposition approach

Along this section we consider system (1) with state space $Z := H^1(\Omega)$, and for $\delta > 0$ we decompose of the spectrum of A in two parts of the complex plan

$$\sigma_{\delta}^{+}(A) = \{\lambda : \mathcal{R}e(\lambda) \geq -\delta\} \quad \text{and} \quad \sigma_{\delta}^{-}(A) = \{\lambda : \mathcal{R}e(\lambda) < -\delta\}$$

If $\sigma_{\delta}^{+}(A)$ is bounded and separated from $\sigma_{\delta}^{-}(A)$ in such a way that a rectifiable, simple, closed curve Γ , enclosing an open set containing $\sigma_{\delta}^{+}(A)$ in its interior

and $\sigma_\delta^+(A)$ in its exterior, then the state space may be decomposed as $Z = Z_u \oplus Z_s$ where $Z_u = PZ$, $Z_s = (I - P)Z$ and P is the projection operator mapping Z into itself given by $P = \frac{1}{2\pi i} \int_\Gamma (\lambda I - A)^{-1} d\lambda$.

Then the operator A satisfies the spectrum decomposition assumption:

$$A = A_u \oplus A_s \text{ where } A_u = PA, \ A_s = (I - P)A. \quad (7)$$

Assume that there exist $B_u \in \mathcal{L}(Z_u)$ and $B_s \in \mathcal{L}(Z_s)$ such that $B = B_u \oplus B_s$, then system (1) can be decomposed as

$$\begin{cases} \dot{y}_u(t) &= A_u y_u(t) + v(t) B_u y_u(t) \\ y_u &= P y \\ y_{0u} &= P y_0 \in Z_u \end{cases} \quad (8)$$

$$\begin{cases} \dot{y}_s(t) &= A_s y_s(t) + v(t) B_s y_s(t) \\ y_s &= (I - P) y \\ y_{0s} &= (I - P) y_0 \in Z_s \end{cases} \quad (9)$$

The solutions of (8) and (9) are given by :

$$y_u(t) = T_u(t) y_{0u} + \int_0^t v(\tau) T_u(t - \tau) B_u y_u(\tau) d\tau \quad (10)$$

and

$$y_s(t) = T_s(t) y_{0s} + \int_0^t v(\tau) T_s(t - \tau) B_s y_s(\tau) d\tau \quad (11)$$

where $T_u(t)$ and $T_s(t)$ denote the restriction of $T(t)$ on Z_u and Z_s which are respectively the strongly continuous semigroups generated by A_u and A_s .

By considering the following control :

$$v(t) = -\langle G_\omega D_u G_\omega y_u(t), y_u(t) \rangle \quad (12)$$

where $D_u \in \mathcal{L}(Z_u)$ and $G_\omega = \nabla_\omega^* \nabla_\omega$, we obtain the following result:

Proposition 2.4 *Assume that A satisfies (7), and A_s satisfies the spectrum determined growth assumption:*

$$\lim_{t \rightarrow +\infty} \frac{\ln \|T_s(t)\|}{t} = \sup \operatorname{Re}(\sigma(A_s)) \quad (13)$$

If system (8) is r.e.g.s on ω using the control (12), then system (1) is r.e.g.s on ω using the same control (12), and if in addition the state of system (8) is bounded, then the state of system (1) remains bounded on Ω .

Proof :

Since system (8) is r.e.g.s, then its solution y_u is global, and we have:

$$\|\nabla_\omega y_u(t)\| \leq M e^{-ct} \|y_{0u}\|, \text{ for some } M, c > 0 \quad (14)$$

Also y_s is a mild solution of (9) defined on $[0, t_{max}[$, $t_{max} > 0$ to conclude that $y_s(t)$ is a global solution, we will show that $y_s(t)$ is bounded for all $t \in [0, t_{max}[$. From (13) we have :

$$\|T_s(t)\| \leq a e^{-bt}, \quad t \geq 0, \quad 0 < b < \delta. \quad (15)$$

With (11) we obtain:

$$\|y_s(t)\| \leq a e^{-bt} \|y_{0s}\| + a \|B_s\| \int_0^t |v_u(\tau)| e^{-b(t-\tau)} \|y_s(\tau)\| d\tau.$$

Applying Gronwall's inequality we have :

$$e^{bt} \|y_s(t)\| \leq a \|y_{0s}\| \exp\left(\int_0^t a \|B_s\| |v_u(\tau)| d\tau\right)$$

From (12) and (14) we obtain:

$$\|y_s(t)\| \leq aL \|y_{0s}\| e^{-bt} \text{ for some } L > 0. \quad (16)$$

Then $y_s(t)$ is bounded which implies that $y_s(t)$ is a global mild solution, thus $y(t)$ is a global mild solution of (1) ($y = y_s + y_u$).

From (16) it follows that $\|y_s(t)\| \rightarrow 0$ exponentially as $t \rightarrow +\infty$, which implies that $\|\nabla_\omega y_s(t)\|_n \rightarrow 0$ exponentially as $t \rightarrow +\infty$. Besides, using (10), (14), and the inequality $\|\nabla_\omega y(t)\|_n \leq \|\nabla_\omega y_u(t)\|_n + \|\nabla_\omega y_s(t)\|_n$ the proof is achieved.

The second point is immediate with (16) ■

If system (9) has a global mild solution $y_s(t)$, then the condition (13) may be relaxed and we obtain the following result.

Proposition 2.5 *Let A satisfies (7), A_s satisfies the following inequality*

$$\lim_{t \rightarrow +\infty} \frac{\ln \|\nabla_\omega T_s(t)\|_n}{t} \leq \sup \operatorname{Re}(\sigma(A_s)) \quad (17)$$

and for some $N > 0$, B_s satisfies the following condition:

$$\|\nabla_\omega B_s y\|_n \leq N \|\nabla_\omega y\|_n \quad \text{for all } y \in Z \quad (18)$$

If system (8) is r.e.g.s on ω , using the control (12) then the system (1) is r.e.g.s on ω using the same control (12). If in addition the state of system (8) is bounded then the state of system (1) remains bounded on Ω .

Proof :

Since system (8) is supposed to be r.e.g.s, then its solution $y_u(t)$ is global, also $y_s(t)$ is a global mild solution of (9) and since $y = y_u + y_s$, then $y(t)$ is a global mild solution of (1).

The decomposition of the spectrum of A gives $\sup \operatorname{Re}(\sigma(A_s)) \leq -\delta$.

From (17) it's follows that $\|\nabla_\omega T_s(t)\|_n \leq a' e^{-b't}$, $t \geq 0$, $0 < b' < \delta$, and $a' > 0$.

By (11) and (18) we have :

$$e^{b't} \|\nabla_\omega y_s(t)\|_n \leq a' \|\nabla_\omega y_{0s}\|_n + a' N \int_0^t |v_u(\tau)| e^{b'\tau} \|\nabla_\omega y_s(\tau)\|_n d\tau$$

Therefore the proof can be achieved similarly to that of proposition 2.4 ■

3 Stabilization problem with decay estimate

Here we characterize the control that stabilizes regionally the gradient of system (1) and minimizes a given performance cost, and we give a decay estimate of the gradient stabilization.

Consider the problem

$$\begin{cases} \min J(v) = \int_0^{+\infty} |\langle P_\omega B y(t), y(t) \rangle|^2 dt + \int_0^{+\infty} |v(t)|^2 dt + \int_0^{+\infty} \langle R y(t), y(t) \rangle dt \\ v \in \mathcal{U}_{ad} = \{v/y(t) \text{ is a global solution and } J(v) < +\infty\} \end{cases} \quad (19)$$

where $P_\omega = G_\omega P G_\omega$ with P and R are positive and self-adjoint operators such that P_ω satisfies the following equation :

$$\langle P_\omega A y, y \rangle + \langle y, P_\omega A y \rangle + \langle R y, y \rangle = 0, \quad \forall y \in \mathcal{D}(A) \quad (20)$$

and

$$\int_0^1 |\langle P_\omega B T(t) y, T(t) y \rangle| dt \geq \alpha \|\nabla_\omega y\|_n^2, \quad y \in H^1(\Omega) \text{ for some } \alpha > 0. \quad (21)$$

Assume that there exist $a, b > 0$, such that $\forall t \geq 0, \forall y \in H^1(\Omega)$

$$\|\nabla_\omega T(t) y\|_n \leq a \|\nabla_\omega y\|_n \text{ and } \|\nabla_\omega B y\|_n \leq b \|\nabla_\omega y\|_n \quad (22)$$

The main result is based on the following lemma.

Lemma 3.1 Denote : $\mu = \int_0^1 \langle P_\omega B y(t), y(t) \rangle dt$ and $\nu = \int_0^1 |v(t)|^2 dt$, then $(\exists \eta, 0 < \eta < 1), (\exists \lambda > 0) : \nu < \eta \Rightarrow \|\nabla_\omega y_0\|_n^2 < \lambda \sqrt{\mu}$

Proof :

Let $\psi(t) = y(t) - T(t) y_0$. Firstly we establish the following estimation

$$\|\nabla_\omega \psi(t)\|_n \leq C_0 \|\nabla_\omega y_0\|_n \sqrt{\nu}, \quad \forall t \in]0, 1[, \text{ for some } C_0 > 0. \quad (23)$$

$\psi(t)$ is written as

$$\psi(t) = \int_0^t v(s)T(t-s)BT(s)y_0 ds + \int_0^t v(s)T(t-s)B(y(s) - T(s)y_0) ds,$$

and by (22) we obtain :

$$\begin{aligned} \|\nabla_\omega \psi(t)\|_n &\leq a^2 b \|\nabla_\omega y_0\|_n \int_0^t |v(s)| ds + ab \int_0^t |v(s)| \|\nabla_\omega \psi(s)\|_n ds \\ &\leq a^2 b \|\nabla_\omega y_0\|_n \sqrt{\nu} + ab \int_0^t |v(s)| \|\nabla_\omega \psi(s)\|_n ds \end{aligned}$$

Using Gronwall's inequality, and for all $t \in]0, 1[$ we have

$$\begin{aligned} \|\nabla_\omega \psi(t)\|_n &\leq a^2 b \|\nabla_\omega y_0\|_n \sqrt{\nu} + a^3 b^2 \sqrt{\nu} \|\nabla_\omega y_0\|_n \int_0^t |v(s)| \exp\left(\int_s^t ab|v(\tau)| d\tau\right) ds \\ &\leq a^2 b \|\nabla_\omega y_0\|_n \sqrt{\nu} + a^3 b^2 \sqrt{\nu} \|\nabla_\omega y_0\|_n \exp(ab) \quad (\nu < 1) \end{aligned}$$

Then (23) is satisfied with $C_0 = a^2 b \|\nabla_\omega y_0\|_n + a^3 b^2 \|\nabla_\omega y_0\|_n \exp(ab)$.

Now the triangle inequality implies that:

$$\begin{aligned} \int_0^1 |\langle P_\omega BT(t)y_0, T(t)y_0 \rangle| dt &\leq \sqrt{\mu} + \int_0^1 |\langle P_\omega BT(t)\psi(t), T(t)y_0 \rangle| dt \\ &\quad + \int_0^1 |\langle P_\omega BT(t)y_0, \psi(t) \rangle| dt + \int_0^1 |\langle P_\omega B\psi(t), \psi(t) \rangle| dt \end{aligned}$$

and from (22), and (23) there exist $\tilde{a}, \tilde{b} > 0$ such that

$$\int_0^1 |\langle P_\omega BT(t)y_0, T(t)y_0 \rangle| dt \leq \sqrt{\mu} + \tilde{a} \sqrt{\nu} \|\nabla_\omega y_0\|_n^2 + \tilde{b} \nu \|\nabla_\omega y_0\|_n^2$$

Since $\nu < 1$ and from (21) we obtain $(\alpha - (\tilde{a} + \tilde{b})\sqrt{\nu}) \|\nabla_\omega y_0\|_n^2 \leq \sqrt{\mu}$, the proof is achieved by choosing $\eta = \frac{1}{2} \inf(1, \frac{\alpha^2}{(\tilde{a} + \tilde{b})^2})$ ■

Proposition 3.2 *If the solution $y^*(t)$ corresponding to the control:*

$$v^*(t) = -\langle P_\omega B y^*(t), y^*(t) \rangle \quad (24)$$

is global, then $v^(t)$ is the unique feedback control solution of (19), and system (1) excited by $v^*(t)$ is regionally strongly gradient stabilizable.*

If in addition the condition

$$|\langle P_\omega B y, y \rangle|^2 + \frac{1}{2} \langle R y, y \rangle \leq d \operatorname{Re}(\langle y, B y \rangle \langle P_\omega y, y \rangle - \langle A y, y \rangle), \quad \forall y \in \mathcal{D}(A) \quad (25)$$

holds for some $d > 0$, then the state remains bounded on ω .

Proof :

Let us define the function $F(z) = \langle P_\omega z, z \rangle$, $\forall z \in H^1(\Omega)$. For $y_0 \in \mathcal{D}(A)$, we have

$$\frac{\partial F(y^*(t))}{\partial t} = -2|\langle P_\omega B y^*(t), y^*(t) \rangle|^2 - \langle R y^*(t), y^*(t) \rangle$$

Integrating, we obtain:

$$\int_0^t |\langle P_\omega B y^*(s), y^*(s) \rangle|^2 ds + \int_0^t \langle R y^*(s), y^*(s) \rangle ds \leq F(y_0) ds, \quad t \geq 0 \quad (26)$$

The solution $y^*(t)$ is continuous with respect to the initial condition (see[4]), and since F and $\langle R y(t), y(t) \rangle$ are continuous then (26) holds for all $y_0 \in H^1(\Omega)$. It follows that $J(v^*)$ is finite for all initial condition y_0 on $H^1(\Omega)$.

Now let us show that system (1) controlled by any control $v \in \mathcal{U}_{ad}$ is regionally strongly gradient stabilizable. Indeed:

Let ϵ be such that $0 < \epsilon < \eta$, since J is finite, Cauchy criterion implies that there exists $T > 0$ such that for all $t > T$

$$\int_t^{t+1} \langle P_\omega B y(s), y(s) \rangle ds < \epsilon \quad \text{and} \quad \int_t^{t+1} |v(s)|^2 ds < \epsilon$$

Applying lemma 3.1 with $y_0 = y(t)$, we obtain $\|\nabla_\omega y(t)\|_n < \lambda\sqrt{\epsilon}$, $t > T$.

Thus $\|\nabla_\omega y(t)\|_n \rightarrow 0$, as $t \rightarrow +\infty$.

Let us show that $v^*(t)$ is the unique solution of (19):

We have $F(y(t)) \leq M \|\nabla_\omega y(t)\|_n^2$, for some $M > 0$, then $\lim_{t \rightarrow +\infty} F(y(t)) = 0$.

For $y_0 \in \mathcal{D}(A)$ integrating the relation :

$$\frac{\partial F(y(t))}{\partial t} = |\langle P_\omega B y(t), y(t) \rangle + v(t)|^2 - |\langle P_\omega B y(t), y(t) \rangle|^2 - |v(t)|^2 - \langle R y(t), y(t) \rangle$$

we obtain $J(v) = F(y_0) + \int_0^{+\infty} |\langle P_\omega B y(t), y(t) \rangle + v(t)|^2 dt$.

It follows that $J(v^*) \leq J(v)$, $\forall v \in \mathcal{U}_{ad}$.

Let $y_0 \in H^1(\Omega)$, there exists $(y_{0n}) \subset \mathcal{D}(A)$ such that $y_{0n} \rightarrow y_0$ as $n \rightarrow +\infty$.

For $v \in \mathcal{U}_{ad}$, $J(v) = F(y_{0n}) + \int_0^{+\infty} |\langle P_\omega B y_n(t), y_n(t) \rangle + v(t)|^2 dt$. Thus $J(v) \geq F(y_{0n})$.

By the continuity of F we deduce that $J(v) \geq F(y_0) = J(v^*)$.

The uniqueness of $v^*(t)$ follows from the uniqueness of $y(t)$ solution of system (1).

Now let us show that the state remains bounded on Ω .

The inequality (25) gives $\frac{\partial}{\partial t} |F(y(t))|^2 \geq d \frac{\partial}{\partial t} |y(t)|^2$ so

$$F(y(t)) \geq d(\|y(t)\|^2 - \|y_0\|^2) \quad \forall y_0 \in \mathcal{D}(A). \quad (27)$$

F and $y(t)$ are continuous with respect to the initial condition, then (27) holds $\forall y_0 \in H^1(\Omega)$.

But $F(y(t)) \rightarrow 0$ as $t \rightarrow +\infty$, which achieves the proof $\blacksquare$.

The following result gives the decay estimate of $\|\nabla_\omega y^*(t)\|_n$.

Proposition 3.3 *Assume that the following assumptions are verified:*

$$\langle P_\omega y, y \rangle \geq d \|\nabla_\omega y\|_n^2 \text{ for some } d > 0 \quad (28)$$

$$\langle Ry, y \rangle \geq e \|\nabla_\omega y\|_n^2 \text{ for some } e > 0 \quad (29)$$

and

$$\mathcal{R}e(\langle RAy, y \rangle) \leq \mathcal{R}e(\langle P_\omega By, y \rangle \langle RBy, y \rangle), \quad \forall y \in \mathcal{D}(A). \quad (30)$$

If in addition the solution $y(t)$ is global then for any initial condition y_0 such that $\nabla_\omega y_0 \neq 0$ we have the estimation $\|\nabla_\omega y(t)\|_n = O(\frac{1}{t})$, as $t \rightarrow +\infty$.

Proof :

Let us consider the function $W(y(t)) = \langle P_\omega y(t), y(t) \rangle + \int_0^t \langle Ry(s), y(s) \rangle ds$, $t \geq 0$, and denote $W_m = \langle P_\omega y^*(t), y^*(t) \rangle + \int_0^m \langle Ry^*(s), y^*(s) \rangle ds$, $t \geq 0$, which is a positive sequence.

Integrating the relation $\frac{\partial W(y(t))}{\partial t} = -2|\langle P_\omega By(t), y(t) \rangle|^2$ between m and $m+1$ we obtain :

$$W_{m+1} - W_m = - \int_m^{m+1} |\langle P_\omega By^*(t), y^*(t) \rangle|^2 dt = -\mu = - \int_m^{m+1} |v^*(t)|^2 dt =: -\nu.$$

Using lemma 3.1 with $y_0 = y^*(t)$ we have

$$W_{m+1} - W_m \leq -\eta \text{ or } W_{m+1} - W_m < \frac{-1}{\lambda^2} \|\nabla_\omega y_0\|_n^4. \quad (31)$$

Let $y_0 \in \mathcal{D}(A)$ be such that $\nabla_\omega y_0 \neq 0$, $W_0 = \frac{1}{2} \langle P_\omega y_0, y_0 \rangle \leq M \|\nabla_\omega y_0\|_n^2$ for some $M > 0$. The inequality (28) implies that $W_0 > 0$ and from (31), we have:

$$W_{m+1} - W_m \leq -f(z_0) W_m^2 \quad (32)$$

where $f(z_0) = \min(\frac{1}{\lambda^2 M^2}, \frac{\eta}{W_0^2})$, then W_m is a nondecreasing sequence.

Let denote $V_m = \frac{1}{W_m}$, then $V_{m+1} - V_m = \frac{W_m - W_{m+1}}{W_{m+1} W_m} \geq \frac{W_m - W_{m+1}}{W_m^2}$ and from (32) we obtain $V_{m+1} - V_m \geq f(y_0)$, then $V_{m+1} \geq V_0 + m f(y_0)$ which implies that $W_m \leq \frac{W_0}{1 + W_0 f(y_0) m}$, $m \geq 0$ and we obtain the following estimate

$$W(y^*(t)) \leq \frac{2W(y_0)}{1 + W(y_0) f(y_0) t}, \quad t \geq 0 \quad (33)$$

Now let $y_0 \in H^1(\Omega)$ such that $\nabla_\omega y_0 \neq 0$, there exists a sequence $y_{0n} \in \mathcal{D}(A)$ and $y_{0n} \rightarrow y_0$ as $n \rightarrow +\infty$, also $\nabla_\omega y_{0n} \neq 0$ (∇_ω is continuous), and using similar above technics we show that (33) is satisfied for y_{0n} .

But both $f(y_0)$ and $y^*(t)$ are continuous with respect to the initial state y_0 , so (33) holds for all $y_0 \in H^1(\Omega)$.

The condition (30) implies that $\frac{\partial}{\partial t} \langle Ry(t), y(t) \rangle \leq 0$, and from (29) we obtain:

$$W(y^*(t)) \geq (d + et) \|\nabla_\omega y^*(t)\|_n^2$$

We conclude that :

$$\|\nabla_\omega y^*(t)\|_n^2 \leq \frac{2U(y_0)}{(d + et)(1 + U(y_0)f(y_0)t)} \quad (34)$$

since $f(y_0)$, $W(y_0)$ and e are not null then the proof is achieved ■

3.1 Numerical approach

Our goal here is to calculate the control v^* solution of problem (19). As shown, this control is given by (24) where P_ω is solution of (20). This turn up to consider the problem

$$\begin{cases} \dot{y}(t) &= Ay(t) + v(t)By(t) \quad t > 0 \\ v(t) &= -\langle P_\omega By(t), y(t) \rangle \\ y(0) &= y_0 \end{cases} \quad (35)$$

To solve (35), let $t_m = mh$ where $m \in \mathbb{N}$, $h > 0$, and for $m \geq 1$, and $t_{m-1} \leq t \leq t_m$. The system (35) is written as

$$\begin{cases} \dot{y}_m(t) &= Ay_m(t) + v_{m-1}(t_{m-1})By_m(t) \\ y_m(t_{m-1}) &= y_{m-1}(t_{m-1}) \\ v_m(t) &= -\langle P_\omega By_m(t), y_m(t) \rangle \\ y(0) &= y_0 \\ v_0(0) &= -\langle P_\omega By_0, y_0 \rangle \end{cases} \quad (36)$$

Now we give a relation between the gradient of system (35) and the one of system (36). For that let suppose that the global mild solution of (35) is bounded, i.e. there exists $\alpha_1 > 0$ such that $\|y(t)\| \leq \alpha_1$ which implies that $v(t) \leq \alpha_2$, where $\alpha_2 = \alpha_1^2 \|P_\omega B\|$, and we have the following result:

Proposition 3.4 Assume that $\|\nabla_\omega T(t)\|_n \leq Me^{-\lambda t}$ for some $\lambda > 0$, and $M \leq 1$. Also B satisfies (22) and $\alpha_2 \leq \frac{\lambda}{M\|B\|}$, then there exists $c > 0$, such that

$$\|\nabla_\omega y_m(t) - \nabla_\omega y(t)\|_n < ch \quad , \quad t_{m-1} \leq t \leq t_m \quad (37)$$

Proof :

We have $\|\nabla_\omega y(t)\|_n \leq M\|\nabla_\omega y_0\|_n e^{-(\lambda - Mb\|B\|\alpha_2)t}$. Indeed :

$$\|\nabla_\omega y(t)\|_n \leq M e^{-\lambda t} \|\nabla_\omega y_0\|_n + Mb\|B\|\alpha_2 \int_0^t e^{-\alpha(t-s)} \|\nabla_\omega y(s)\|_n ds$$

$$\text{Then } e^{\lambda t} \|\nabla_\omega y(t)\|_n \leq M\|\nabla_\omega y_0\|_n + Mb\|B\|\alpha_2 \int_0^t e^{\alpha s} \|\nabla_\omega y(s)\|_n ds$$

Using Gronwall's inequality we obtain $\|\nabla_\omega y(t)\|_n \leq M\|\nabla_\omega y_0\|_n e^{-(\lambda - Mb\|B\|\alpha_2)t}$.
Now let denote $z_m^t = y_m(t) - y(t)$, using Gronwall's inequality we obtain

$$\begin{aligned} e^{\lambda t} \|\nabla_\omega z_m^t\|_n &\leq M e^{\lambda t_{m-1}} \|\nabla_\omega z_{m-1}^{t_{m-1}}\|_n + 2\alpha_2 M^2 b \|\nabla_\omega y_0\|_n \|B\| h e^{Mb\|B\|\alpha_2 t} \\ &\quad + \alpha_2 M^2 b \|B\| e^{\lambda t_{m-1}} \|\nabla_\omega z_{m-1}^{t_{m-1}}\|_n \int_{t_{m-1}}^t e^{Mb\|B\|\alpha_2(t-s)} ds \\ &\quad + 2\alpha_2^2 M^3 b^2 \|\nabla_\omega y_0\|_n \|B\|^2 h^2 e^{Mb\|B\|\alpha_2 t} \end{aligned}$$

We denote $\tilde{\alpha} = -\lambda + bM\|B\|\alpha_2$, it follows that :

$$\begin{aligned} \|\nabla_\omega z_m^t\|_n &\leq M \|\nabla_\omega z_{m-1}^{t_{m-1}}\|_n e^{\tilde{\alpha}(t-t_{m-1})} + 2\lambda M \|\nabla_\omega y_0\|_n h e^{\tilde{\alpha} t} + 2M \|\nabla_\omega y_0\|_n \lambda^2 h^2 e^{\tilde{\alpha} t} \\ &\leq M \|\nabla_\omega z_{m-1}^{t_{m-1}}\|_n + 2\lambda M \|\nabla_\omega y_0\|_n h e^{\tilde{\alpha}(m-1)h} + 2M \|\nabla_\omega y_0\|_n \lambda^2 h^2 e^{\tilde{\alpha}(m-1)h} \end{aligned}$$

Let denote $M_1 = 2\lambda M \|\nabla_\omega y_0\|_n [1 + h\lambda]$, we obtain:

$$\|\nabla_\omega z_m^t\|_n \leq M \|\nabla_\omega z_{m-1}^{t_{m-1}}\|_n + M_1 h e^{\tilde{\alpha}(m-1)h} \quad (38)$$

Finally let show the requested estimation (37): We need to calculate the error $\|\nabla_\omega z_1^t\|_n$, similarly to above we have:

$$\|\nabla_\omega z_1^t\|_n e^{\lambda t} \leq 2\lambda M \|\nabla_\omega y_0\|_n h e^{Mb\|B\|\alpha_2 t} + Mb\|B\|\alpha_2 \int_0^t \|\nabla_\omega z_1^s\|_n e^{\lambda s} ds,$$

Gronwall's inequality implies that :

$$\|\nabla_\omega z_1^t\|_n e^{\lambda t} \leq 2\lambda M \|\nabla_\omega y_0\|_n h e^{Mb\|B\|\alpha_2 t} + 2M h^2 \lambda^2 \|\nabla_\omega y_0\|_n e^{Mb\|B\|\alpha_2 t}$$

then $\|\nabla_\omega z_1^t\|_n \leq M_1 h$.

We have $(m+1)^2 e^{(-\lambda + \alpha_2 Mb\|B\|)mh} \rightarrow 0$ as $m \rightarrow +\infty$ ($-\lambda + \alpha_2 Mb\|B\| < 0$).

So there exists $N_1 > 0$ such that for all $m > N_1$,

$$e^{(-\lambda + \alpha_2 Mb\|B\|)mh} < \frac{1}{(1+m)^2}$$

and from (38) we have for all $m > N_1$

$$\|\nabla_\omega z_m^t\|_n \leq M \|\nabla_\omega z_{m-1}^{t_{m-1}}\|_n + \frac{M_1 h}{m^2}$$

$$\text{then } \|\nabla_{\omega} z_m^t\|_n \leq M^2 \|\nabla_{\omega} z_{m-2}^{t_{m-2}}\|_n + M \frac{M_1 h}{(m-1)^2} + \frac{M_1 h}{m^2}.$$

Thus

$$\|\nabla_{\omega} z_m^t\|_n \leq M^{m-N_1} \|\nabla_{\omega} z_{N_1}^{t_{N_1}}\|_n + M^{m-N_1-1} \frac{M_1 h}{(N_1+1)^2} + M^{m-N_1-2} \frac{M_1 h}{(N_1+2)^2} + \dots + \frac{M_1 h}{m^2} \quad (39)$$

In the other hand, using (38) for $m' \leq N_1$, we have:

$$\begin{aligned} \|\nabla_{\omega} z_{m'}^t\|_n &\leq M \|\nabla_{\omega} z_{m'-1}^{t_{m'-1}}\|_n + M_1 h \\ &\leq \sum_{i=0}^{m'} M^i M_1 h \end{aligned}$$

then $\forall m' \leq N_1$, $\|\nabla_{\omega} z_{m'}^t\|_n \leq h M_1 \sum_{i=0}^{m'} M^i$, and replacing in (39) we obtain

$$\begin{aligned} \|\nabla_{\omega} z_m^t\|_n &\leq M^{m-N_1} \sum_{i=0}^{N_1} M^i M_1 h + \sum_{i=0}^{m-N_1-1} \frac{M^i M_1}{(m-i)^2} h \\ &\leq (M^{m-N_1} \sum_{i=0}^{N_1} M^i + \sum_{j=N_1+1}^m \frac{M^{m-j}}{j^2}) M_1 h, \end{aligned}$$

The above series converges then the proof is achieved ■.

Now let solve the algebraic Riccati equation (20) for that:

Let $H^N = \text{span}\{\phi_i \mid i = 1, \dots, N\}$ a subspace of $H^1(\Omega)$ where $\{\phi_i \mid i \in \mathbb{N}^*\}$ is an orthonormal basis of $H^1(\Omega)$. H^N is endowed with $H^1(\Omega)$ restriction inner product.

We define the projection operator :

$$\begin{aligned} \Pi^N : H^1(\Omega) &\longrightarrow H^N \\ y &\longmapsto \sum_{i=1}^N \langle y, \phi_i \rangle \phi_i(x) \end{aligned} \quad (40)$$

To solve (20) we have to solve the following algebraic Riccati equation :

$$\langle P_{\omega}^N A^N y^N, y^N \rangle + \langle y^N, P_{\omega}^N A^N y^N \rangle + \langle R^N y^N, y^N \rangle = 0 \quad (41)$$

where A^N , P_{ω}^N , and R^N are respectively the projection of A , P_{ω} , and R .

Let $y \in H^1(\Omega)$, $\lim_{N \rightarrow \infty} \|\Pi^N(y) - y\| = 0$ then $\lim_{N \rightarrow \infty} \|P_{\omega}^N \Pi^N y - P y\| = 0$ $\forall y \in H^1(\Omega)$, or equivalently $P_{\omega}^N \Pi^N$ converges to P_{ω} strongly in $H^1(\Omega)$, ([3]). Now for $m \geq 1$ and $t_{m-1} \leq t \leq t_m$, let us consider the system :

$$\begin{cases} \frac{\partial y_m(t)}{\partial t} = A y_m(t) + v_{m-1}^N(t_{m-1}) B y_m(t) \\ y_m(t_{m-1}) = y_{m-1}(t_{m-1}) \\ v_m^N(t) = -\langle P^N B^N y_m^N(t), y_m^N(t) \rangle \\ v(0) = -\langle P^N B^N y_0^N, y_0^N \rangle \end{cases} \quad (42)$$

The linear operator $(A + v_{m-1}^N(t_{m-1})B)$ generates a linear strongly continuous semigroup given by $T^{m,N} = \sum_{i \geq 0} T_i^{m,N}$ where $T_0^{m,N} = T$ and

$$T_i^{m,N}(t)y = T^{m,N}y + v_{m-1}^N(t_{m-1}) \int_0^t T^{m,N}(t-s)BT_{i-1}^{m,N}y ds.$$

Then $\lim_{N \rightarrow +\infty} \|T^{m,N} - T^m\| = 0$, where T^m is the semigroup generated by the operator $(A + v_{m-1}(t_{m-1})B)$.

Thus the mild solution of system (42) converges strongly to the mild solution of system (36) as $N \rightarrow +\infty$ ■.

This leads to the following algorithm:

Step 1 : *Initial data:*

- Threshold accuracy $\varepsilon > 0$, subregion ω , N the dimension of space projection, and y_0^N is the projection of initial state.

Step 2 : *Choose a time sequence $(t_m)_{m \in \mathbb{N}}$, $t_{m+1} = t_m + h$, with $h > 0$ small enough for numerical consideration*

Step 3 : *Solve (41) using the algorithm given in [1] gives P_ω^N*

Step 4 : • $v(0) = -\langle P_\omega^N B^N y_0^N, y_0^N \rangle$
 • While $\|\nabla_\omega y_m(t_m)\| < \epsilon$ repeat
 • $v_m^N(t) = -\langle P_\omega^N B^N y_m^N(t), y_m^N(t) \rangle$
 • solving (42) gives $y_{m+1}(t_{m+1})$
 • $m \leftarrow m + 1$

3.2 Simulation results

Example 1. Let $\Omega =]0, 1[$ and consider the following system

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) &= 0.01\Delta y(x, t) + 0.68y(x, t) + v(t)y(x, t) \\ \frac{\partial y(0, t)}{\partial x} &= \frac{\partial y(1, t)}{\partial x} = 0 \\ y(x, 0) &= x^2(1 - x^3)^2 \end{cases} \quad (43)$$

We consider the subregion $\omega =]0, 0.4[$ and we take $\phi_i(x) = a_i \cos(i\pi x)$, with $a_i = \sqrt{\frac{2}{1+(i\pi)^2}} \forall i \geq 0$, and we consider the problem (19) with $R = 5\nabla_\omega^* \nabla_\omega$.

Let take $N = 6$, the projection of the dynamic A is a diagonal matrix with $\text{diag}(A^6) = (0.5813, 0.2852, -0.2083, -0.8991, -1.7874, -2.8731)$

Applying the previous algorithm the matrix $P_\omega^6 = (p_{ij})_{1 \leq i, j \leq 6}$ is given by:

$$P_\omega^6 = \begin{pmatrix} 1.6725 & 0.6623 & 0.2405 & -0.1397 & -0.1499 & -0.0089 \\ 0.6623 & 1.5564 & 0.5736 & 0.0236 & -0.1218 & -0.0444 \\ 0.2405 & 0.5736 & 0.8415 & 0.5082 & 0.1496 & -0.0458 \\ -0.1397 & 0.0236 & 0.5082 & 0.6683 & 0.4124 & 0.0814 \\ -0.1499 & -0.1218 & 0.1496 & 0.4124 & 0.4252 & 0.2482 \\ -0.0089 & -0.0444 & -0.0458 & 0.0814 & 0.2482 & 0.3048 \end{pmatrix}$$

and we have the following figures:

Figure 1: The flux evolution on $\omega =]0, 0.4[$.

The above figure shows how the gradient is stabilized on ω . The error flux stabilization is $5.9 \cdot 10^{-7}$, and the stabilization cost is $3.1 \cdot 10^{-4}$. The following table shows that there exists a relation between the area of target subregion ω , the cost and the error of gradient stabilization.

ω	$]0, 0.3[$	$]0, 0.4[$	$]0, 0.7[$	$]0, 0.9[$	$]0, 1[$
Error	5.9056×10^{-9}	3.8522×10^{-6}	4.7438×10^{-6}	2.6048×10^{-5}	2.7163×10^{-5}
Cost	5.43×10^{-2}	5.83×10^{-2}	6.65×10^{-2}	6.79×10^{-2}	6.8×10^{-2}

Table 1

More the area of the target region increases, more both the error gradient stabilization and the cost increase. In some situations we are interested by improving the degree of gradient stabilization on $\omega \subset \Omega$, as is showing in the following example.

Example 2. Let $\Omega =]0, 1[$ and consider the following system :

$$\begin{cases} \frac{\partial y}{\partial t}(x, t) &= 0.01\Delta y(x, t) + 0.5y(x, t) + 0.7v(t)y(x, t) \\ y(0, t) &= y(1, t) = 0 \\ y(x, 0) &= 0.5x(1 - x) \end{cases} \quad (44)$$

Let $\omega =]0, 0.4[$ we consider the problem (19) with $R = 3\nabla_{\omega}^* \nabla_{\omega}$.

We take $N = 5$. A^5 the projection of $A = 0.01\Delta + 0.5id$, is diagonal matrix with $diag(A^5) = (0.4013, 0.10521, -0.3882, -1.07913, -1.9674)$ and P_{ω}^5 is a symmetric matrix with

$$P_{\omega}^5 = \begin{pmatrix} 1.8654 & 0.4911 & -0.0599 & -0.1141 & -0.01013 \\ 0.4911 & 1.0613 & 0.63035 & 0.1827 & -0.0805 \\ -0.0599 & 0.6303 & 0.9209 & 0.5089 & 0.0623 \\ -0.1141 & 0.1827 & 0.5089 & 0.5401 & 0.3287 \\ -0.0101 & -0.0805 & 0.0623 & 0.3287 & 0.4292 \end{pmatrix}$$

Applying the above algorithm we obtain the following figures

Figure 2: The flux evolution on $\omega =]0, 0.4[$.

The above figures show that the gradient is regionally stabilizable on ω at $t = 14$, the error is 9.92×10^{-4} and the cost is 1.23×10^{-1} .

Also we remark that as the area of subregion increases as the error and the cost increase.

ω	$]0, 0.1[$	$]0, 0.3[$	$]0, 0.5[$	$]0, 0.8[$	$]0, 1[$
Error	8.2691×10^{-4}	9.5172×10^{-4}	9.9658×10^{-4}	9.9878×10^{-4}	9.9954×10^{-4}
Cost	11.9×10^{-2}	12.22×10^{-2}	13.01×10^{-2}	13.5×10^{-2}	14.29×10^{-2}

Table 2

4 Conclusion

The problem of regional gradient stabilization of bilinear distributed systems is considered, two approaches are used to characterize the gradient stabilization, the first one is based in the decomposition of the state space, and the second one on solving algebraic Riccati equation and leads to an effective algorithm which is successfully implemented.

A natural extension of this work is the study of the gradient stabilization of semilinear distributed systems. The work is under consideration.

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Received: August, 2012