Strong and Semi Strong Outer Mod Sum Graphs

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Abstract

A semi strong outer mod sum labeling of a graph G is an injective mapping $f : V(G) \rightarrow Z^+$ with an additional property that for each vertex v of G , there exist vertices w_1, w_2 in $V(G)$ such that $f(w_1) = \sum_{u \in N(v)} f(u)$ and $f(v) = \sum_{u \in N(w_2)} f(u)$, where both the sums are taken under addition modulo m for some positive integer m . A graph G which admits a semi strong outer mod sum labeling is called a semi strong outer mod sum graph. In this paper we show that the Paths P_n for all $n \neq 3$, the Cycle C_6 , and the Complete graphs K_n for all $n \neq 3$ are semi strong outer mod sum graphs.

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1 Introduction

All the graphs considered here are non trivial, undirected, finite, connected and simple. We use the standard terminology, the terms not defined here may be found in [2] and [3]

A sum labeling λ of a graph is a mapping of the vertices of G into distinct positive integers such that for $u, v \in V(G)$, $uv \in E(G)$ if and only if $\lambda(u) + \lambda(v) = \lambda(w)$ for some vertex w of G . A graph which admits a sum labeling is called a *sum graph*. Sum graph were originally proposed by F. Harary [8] and later he extended to include all integers in [9]. Similarly, Bolan, Laskar, Turner and Domke [1] introduced a *mod sum labeling* of a graph in which the sum $\lambda(u) + \lambda(v)$ is taken under addition modulo m for some positive integer m .

An outer sum labeling f of a non-trivial graph is a mapping of the vertices of G into distinct positive integers such that for each vertex $v \in V(G)$, there exists

a vertex $w \in V(G)$ with $f(w) = \sum_{u \in N(v)} f(u)$, where $N(v) = \{x : vx \in E(G)\}$. A graph G which admits an outer sum labeling is called an *outer sum graph*. If G is not an outer sum graph, then by adding certain number of isolated vertices to G , we can make the resultant graph an outer sum graph. The minimum of such isolated vertices required for a graph G , to make the resultant graph an outer sum graph, is called the outer sum number of G and is denoted by $on(G)$. Outer sum graph were originally proposed by B. Sooryanarayana, Manjula K and Vishu Kumar M [14]. Further Jayalakshmi M, B. Sooryanarayana, P Devadas Rao [10] introduced a *outer mod sum labeling* of a graph. An *outer mod sum labeling* of a graph G is an injective mapping $f : V(G) \rightarrow Z^+$ with an additional property that for each vertex v of G , there is a vertex w in G such that $f(w) = \sum_{u \in N(v)} f(u)$, where the sum is taken under addition modulo m for some fixed positive integer m . The vertex w is then called a *working vertex* for v and in particular if $\sum_{u \in N(v)} f(u) = f(v)$, then vertex v is called a *self working vertex*. A graph G which admits an outer mod sum labeling is called an *outer mod sum graph* (OMSG). For the entire survey on sum labeling and mod sum labeling we refer the latest survey article by Joseph A. Gallian [6].

The purpose of this paper is to introduce the special types of outer mod sum graphs and their properties.

2 Strong and Semi Strong Outer Mod Sum Graphs

A *semi-strong outer mod sum labeling* (SSOMSL) f of a graph G is an outer mod sum labeling of G with an additional property that every vertex of G is a working vertex of some vertex of G . Further a semi strong outer mod sum labeling f of a graph G , under which every vertex of G is a self working vertex is called a *strong outer mod sum labeling* (SOMSL). A graph G which admits a semi-strong outer mod sum labeling is called a *semi-strong outer mod sum graph* (SSOMSG). Similarly, a *strong outer mod sum graph* (SOMSG) is defined. For notational convenience, the sum, $\sum_{u \in N(v)} f(u)$ is denoted by $N_f(v)$ and call it as f -neighborhood sum of the vertex v and that of under modulo m is denoted by $N_{f_m}(v)$ and call it as f -neighborhood sum of the vertex v under modulo m .

Example 1 Consider the labeling $f : V(P_5) \rightarrow Z^+$ defined by $f(v_1) = 4$, $f(v_2) = 7$, $f(v_3) = 5$, $f(v_4) = 8$, $f(v_5) = 9$, as shown in the figure 1.

Under the addition modulo 10, we see that the working vertex of v_1 is v_2 and v_1 is the working vertex of v_4 ; the working vertex of v_2 is v_5 and v_2 is the working vertex of v_1 ; v_3 is the self working vertex; the working vertex of v_4 is v_1 and v_4 is the working vertex of v_5 ; the working vertex of v_5 is v_4 and v_5 is

Figure 1: A SSOMSL of the graph P_5 .

the working vertex of v_2 . So P_5 is a semi-strong outer mod sum graph.

Example 2 Consider the labeling $f : V(C_6) \rightarrow \mathbb{Z}^+$ defined by $f(v_1) = 1$, $f(v_2) = 3$, $f(v_3) = 2$, $f(v_4) = 9$, $f(v_5) = 7$, $f(v_6) = 8$, as shown in the figure 2.

Figure 2: A SOMSL of the graph C_6 .

Under addition modulo 10, $N_{f_{10}}(v_1) = 1 = f(v_1)$, $N_{f_{10}}(v_2) = 3 = f(v_2)$, $N_{f_{10}}(v_3) = 2 = f(v_3)$, $N_{f_{10}}(v_4) = 9 = f(v_4)$, $N_{f_{10}}(v_5) = 7 = f(v_5)$, $N_{f_{10}}(v_6) = 8 = f(v_6)$. Therefore, v_i is a self working vertex for each i , $1 \leq i \leq 6$ and hence C_6 is a strong outer mod sum graph.

By the definition it follows that every SOMSL is SSOMSL and every SSOMSL is OMSL. Thus we conclude;

Lemma 2.1. *Every strong outer mod sum graph is a semi strong outer mod sum graph.*

The converse of the lemma 2.1 need not be true in general. As a counter example we see that from example 1, the path P_5 is semi strong outer mod sum graph. But path P_5 is not strong outer mod sum graph. Because, the working vertex of $v_1 \in P_5$ is always v_2 with respect to any labeling f .

Lemma 2.2. *Every semi strong outer mod sum graph is an outer mod sum graph.*

The converse of the lemma 2.2 need not be true in general. As a counter example the path P_3 is an outer mod sum graph. But the path P_3 is not semi strong outer mod sum graph. Because

In figure 3, v_2 is the working vertex of both v_1 and v_3 . Therefore at the most either v_1 or v_3 will be the working vertex of v_2 . So that the vertex v_1 or v_3 will not be the working vertex of any of the vertices of P_3 , under any labeling f and any addition modulo m . Hence P_3 is not a semi strong outer mod sum graph.

Figure 3: An example for a graph which is not a SSOMSG.

Lemma 2.3. *A graph G is a semi strong outer mod sum graph if and only if each vertex of G*

- (i) *has a unique working vertex in G and*
- (ii) *is the working vertex of exactly one of the vertex of G .*

Proof. Let G be a semi-strong outer mod sum graph with n vertices $v_1, v_2, v_3, \dots, v_n$. Then, by the definition of SSOMSG, there exists a SSOMSL f under which ,

- i) each vertex v_i of G has a working vertex in G and is unique (since f is injective), and
- ii) each vertex of G is a working vertex of some vertex of G . Let $w_i \in V(G)$ be the working vertex of v_i for each $i = 1, 2, \dots, n$. We now see that w_i 's are all distinct. In fact, if $w_i = w_j$ for some $i \neq j$ then, as f is injective, at least one vertex in $V(G)$ is not a working vertex of any vertex of G , which is a contradiction.

Converse part follows directly by the definition. $\square$

Remark 2.4. *The Lemma 2.3 says that a graph G is a semi-strong outer mod sum graph if and only if $N_{f_m}(v_i) \neq N_{f_m}(v_j)$ for all $i \neq j$.*

By Remark 2.4, we immediately conclude the following;

Theorem 2.5. *The cycle C_4 is not a SSOMSG and hence not a SOMSG.*

Theorem 2.6. *Let G be a r -regular graph and f be a SSOMSL of G , under modulo m . Then $(r-1) \sum_{v_i \in V(G)} f(v_i) \equiv 0 \pmod{m}$ and hence $\sum_{v_i \in V(G)} f(v_i) \equiv 0 \pmod{m}$ if $\gcd(r-1, m) = 1$.*

Proof. Let G be a r - regular graph having the vertices $v_1, v_2, v_3, \dots, v_n$ and let f be any SSOMSL of G under modulo m . Then each vertex v_i of G is the working vertex of exactly one of the vertices of G . Also $\deg v_i = r$ which means that each vertex v_i has r neighborhoods and hence $f(v_i)$ contribute for f -neighborhood sum of exactly r vertices. Therefore, $\sum_{v_i \in V(G)} f(v_i) = \sum_{v_j \in V(G)} N_{f_m}(v_j) \equiv \left[\sum_{v_i \in V(G)} r \times f(v_i) \right] \pmod{m} \Rightarrow (r-1) \sum_{v_i \in V(G)} f(v_i) \equiv 0 \pmod{m} \Rightarrow \sum_{v_i \in V(G)} f(v_i) \equiv 0 \pmod{m}$ if $(r-1, m) = 1$. $\square$

Since the cycle C_n , ($n \geq 3$) is a 2-regular graph, it follows immediately, as a consequence of the above Theorem 2.6, that:

Corollary 2.7. *If f is a SSOMSL of a Cycle C_n ($n \geq 3$) under modulo m , then*

$$\sum_{v_i \in V(G)} f(v_i) \equiv 0 \pmod{m}.$$

3 SSOMSL of a Path

Throughout this section, $v_1, v_2, \dots, v_n$ denote the vertices of the path P_n such that v_i is adjacent to v_j if and only if $|i - j| = 1$ for each i, j , $1 \leq i, j \leq n$.

Theorem 3.1. *For $n = 2$ or any integer $n \geq 4$, the path P_n is a SSOMSG under modulo $m(> n)$.*

Proof. For $n = 2, 4, 6$, it is easy to verify that the function $f : V(G) \rightarrow Z^+$ defined by $f(v_i) = i$ serves as a SSOMSL under modulo $n + 1$ and the case $n = 5$ follows by the Figure 1 under modulo 10.

For $n \geq 7$ and any $l \in Z^+$, define $f : V(P_n) \rightarrow Z^+$ as follows.

$$f(v_i) = \begin{cases} l, & \text{if } i = 1 \\ l + 1, & \text{if } i = 3 \\ f(v_1) + f(v_3), & \text{if } i = 4 \\ f(v_4) + 1, & \text{if } i = 2 \\ f(v_{k-1}) + f(v_{k-3}), & \text{if } i = k \text{ for any } k \in Z^+ \text{ with } 4 < k < n - 1 \\ f(v_{n-2}) + f(v_{n-4}), & \text{if } i = n \\ m + f(v_3) - f(v_{n-3}), & \text{if } i = n - 1 \text{ where } m = f(v_n) + f(v_{n-2}) - f(v_1) \end{cases}$$

The function f defined above is clearly injective. We now show that it is a SSOMSL of P_n . In fact,

- i) $N_f(v_1) = f(v_2)$ and $N_f(v_{n-1}) = f(v_n) + f(v_{n-2}) = m + f(v_1) \equiv f(v_1) \pmod{m} \Rightarrow$ the working vertex of v_1 is v_2 and v_1 is the working vertex of v_{n-1} .
- ii) $N_f(v_2) = f(v_1) + f(v_3) = f(v_4)$ and $N_f(v_1) = f(v_2) \Rightarrow$ the working vertex of v_2 is v_4 and v_2 is the working vertex of v_1 .
- iii) $N_f(v_3) = f(v_2) + f(v_4) = f(v_5)$ and $N_f(v_{n-2}) = f(v_{n-1}) + f(v_{n-3}) = m + f(v_3) \equiv f(v_3) \pmod{m} \Rightarrow$ the working vertex of v_3 is v_5 and v_3 is the working vertex of v_{n-2} .
- iv) For each $k \in Z^+$ with $4 \leq k \leq n - 4$, $N_f(v_k) = f(v_{k-1}) + f(v_{k+1}) = f(v_{k+2})$ and $N_f(v_{k-2}) = f(v_{k-1}) + f(v_{k-3}) = f(v_k) \Rightarrow$ the working vertex of v_k is v_{k+2} and v_k is the working vertex of v_{k-2} .

- v) $N_f(v_{n-3}) = f(v_{n-2}) + f(v_{n-4}) = f(v_n)$ and $N_f(v_{n-5}) = f(v_{n-6}) + f(v_{n-4}) = f(v_{n-3}) \Rightarrow$ the working vertex of v_{n-3} is v_n and v_{n-3} is the working vertex of v_{n-5} .
- vi) $N_f(v_{n-2}) = f(v_{n-1}) + f(v_{n-3}) = m + f(v_3) \equiv f(v_3) \pmod{m}$ and $N_f(v_{n-4}) = f(v_{n-5}) + f(v_{n-3}) = f(v_{n-2}) \Rightarrow$ the working vertex of v_{n-2} is v_3 and v_{n-2} is the working vertex of v_{n-4} .
- vii) $N_f(v_{n-1}) = f(v_{n-2}) + f(v_n) = m + f(v_1) \equiv f(v_1) \pmod{m}$ and $N_f(v_n) = f(v_{n-1}) \Rightarrow$ the working vertex of v_{n-1} is v_1 and v_{n-1} is the working vertex of v_n .
- viii) $N_f(v_n) = f(v_{n-1})$ and $N_f(v_{n-3}) = f(v_{n-2}) + f(v_{n-4}) = f(v_n) \Rightarrow$ the working vertex of v_n is v_{n-1} and v_n is the working vertex of v_{n-3} .

Hence the result. $\square$

Open Problem 3.2. *Certain values of m considered in the above theorem are not optimal. It is exponentially increasing with n as shown in the table 1. The next Theorem 3.3 gives an optimal value for m , whenever n is even. Determine the optimal value of m for the case n is odd.*

Value of n	Value of m	Value of n	Value of m
7	15	20	2117
8	20	21	3103
9	30	22	4548
10	46	23	6666
11	67	24	9770
12	98	25	14319
13	145	26	20986
14	213	27	30757
15	312	28	45077
16	458	29	66064
17	672	30	96822
18	985	31	141900
19	1444	32	207965

Table 1: The values of m and n (when $l = 1$) for a SSOMSL of the graph P_n as in the proof of Theorem 3.1.

Theorem 3.3. *For any integer $k \geq 1$, the path P_{2k} is a SSOMSG under modulo $2k + 1$.*

Figure 4: An SSOMS-labeling of P_2 .

Proof. Let $n = 2k$. Now, the proof when $k = 1$ follows from the Figure 4 as v_2 is the working vertex of v_1 and vice versa.

For $k \geq 2$, define $f : V(P_n) \rightarrow \{1, 2, \dots, n\}$ by $f(v_i) = i$ for all $i, 1 \leq i \leq n$. Then $N_f(v_1) = f(v_2) = 2$ and $N_f(v_n) = f(v_{n-1}) = n - 1$, implies that the

Figure 5: An SSOMS-labeling of even path.

working vertex of v_1 is v_2 and the working vertex of v_n is v_{n-1} . Also for all $j \in \mathbb{Z}^+$, $2 \leq j \leq n - 1$, $N_f(v_j) = f(v_{j-1}) + f(v_{j+1}) = j - 1 + j + 1 = 2j$. Therefore, if $2 \leq j \leq \frac{n}{2}$, then $4 \leq N_{f_{n+1}}(v_j) = 2j = f(v_{2j}) \leq n$, which implies that the working vertex of $v_2, v_3, \dots, v_{\frac{n}{2}}$ are respectively $v_4, v_6, \dots, v_n$ and if $\frac{n}{2} + 1 \leq j \leq n - 1$, then $1 \leq N_{f_{n+1}}(v_j) = 2j - n - 1 = f(v_{2j-n-1}) \leq n - 3$, which implies that the working vertex of $v_{\frac{n}{2}+1}, v_{\frac{n}{2}+2}, \dots, v_{n-1}$ are respectively $v_1, v_3, \dots, v_{n-3}$. Thus each vertex has a unique working vertex and each vertex is the working vertex of exactly one of the vertex. So f is a SSOMSL and hence the theorem. $\square$

Remark 3.4. The labeling $f : V(P_n) \rightarrow \{1, 2, \dots, n\}$, defined by $f(v_k) = k$ as in the theorem 3.3, is not a semi strong outer mod sum labeling under addition modulo $n + 1$ if n is odd (since v_k and $v_{k+\frac{n+1}{2}}$ have the common working vertex).

In every path P_n and any injective labeling $f : V(G) \rightarrow \mathbb{Z}^+$, working vertex of v_1 is always v_2 and working vertex of v_n is always v_{n-1} . So, v_1 and v_n can not be the self working vertices under any modulo m . Thus;

Theorem 3.5. For $n = 2$ and any integer $n \geq 4$, the path P_n is a SSOMSG and is not a SOMSG.

4 SOMSL and SSOMSL of a Cycle

Let $v_1, v_2, \dots, v_n$ be the vertices of the cycle C_n with v_i adjacent to v_{i+1} for all $i, 1 \leq i \leq n - 1$ and v_n adjacent to v_1 .

Theorem 4.1. For any integer $n \geq 3$, the Cycle C_n is a SOMSG if and only if $n = 6$.

Proof. In view of Figure 2, it suffices to show that the cycle C_n is not a SOMSG whenever $n \neq 6$. We first consider the case $n \geq 7$. If possible suppose that $f : V(C_n) \rightarrow Z^+$ be an injective function such that $f(v_i) = N_{f_m}(v_i)$ for some $m \in Z^+$ with $m > n$ and for all $i, 1 \leq i \leq n$. Without loss of generality we assume that $f(v_1) = l < f(v_3) = k$, Clearly $l, k < m$

Case 1: $l + k < m$

In this case,

- $N_f(v_2) = f(v_1) + f(v_3) = l + k \Rightarrow f(v_2) = N_{f_m}(v_2) = l + k$ ($\because l + k < m$).
- $N_f(v_3) = f(v_2) + f(v_4) = l + k + f(v_4) \Rightarrow k = f(v_3) = N_{f_m}(v_3) \equiv l + k + f(v_4) \pmod{m} \Rightarrow m = l + f(v_4) \Rightarrow f(v_4) = m - l$.
- $N_f(v_4) = f(v_3) + f(v_5) = k + f(v_5) \Rightarrow f(v_4) = N_{f_m}(v_4) \Rightarrow m - l \equiv k + f(v_5) \pmod{m} \Rightarrow f(v_5) = m - l - k$ ($\because m > m - (l + k) > 0$).
- $N_f(v_5) = f(v_4) + f(v_6) = m - l + f(v_6) \Rightarrow f(v_5) = N_{f_m}(v_5) \Rightarrow m - l - k \equiv m - l + f(v_6) \pmod{m} \Rightarrow f(v_6) = m - k$.
- $N_f(v_6) = f(v_5) + f(v_7) = m - l - k + f(v_7) \Rightarrow m - k = f(v_6) = N_{f_m}(v_6) \equiv m - l - k + f(v_7) \pmod{m} \Rightarrow f(v_7) = l$.

Case 2: $l + k > m$

- $N_f(v_2) = f(v_1) + f(v_3) = l + k \Rightarrow f(v_2) = N_{f_m}(v_2) = l + k - m$ ($\because l + k > m$).
- $N_f(v_3) = f(v_2) + f(v_4) = l + k - m + f(v_4) \Rightarrow k = f(v_3) = N_{f_m}(v_3) \equiv l + k - m + f(v_4) \pmod{m} \Rightarrow f(v_4) = m - l$ ($\because m > m - l > 0$).
- $N_f(v_4) = f(v_3) + f(v_5) = k + f(v_5) \Rightarrow m - l = f(v_4) = N_{f_m}(v_4) \equiv k + f(v_5) \pmod{m} \Rightarrow f(v_5) = 2m - l - k$ ($\because m > 2m - (l + k) > 0$).
- $N_f(v_5) = f(v_4) + f(v_6) = m - l + f(v_6) \Rightarrow 2m - l - k = f(v_5) = N_{f_m}(v_5) \equiv m - l + f(v_6) \pmod{m} \Rightarrow f(v_6) = m - k$.
- $N_f(v_6) = f(v_5) + f(v_7) = 2m - l - k + f(v_7) \Rightarrow m - k = f(v_6) = N_{f_m}(v_6) \equiv 2m - l - k + f(v_7) \pmod{m} \Rightarrow f(v_7) = l$.

Thus in either of the cases, $f(v_7) = l = f(v_1)$ and hence $v_7 = v_1$ (since f is injective), a contradiction.

We now consider the small cases where $n \leq 5$. When $n = 3$, it is shown in [10] that the cycle C_3 is not an outer mod sum graph and hence, by Lemma 2.1 and Lemma 2.2, C_3 is not a SSOMSG and hence not a SOMSG. When $n = 4$, it follows by Theorem 2.5 that the graph C_4 is not a SSOMSG. Finally, when $n = 5$, we see by labeling $f(v_1) = l$ and $f(v_3) = k$ as in the above two cases respectively that $m - l - k = f(v_5) = f(v_1) + f(v_4) = l + m - l \pmod{m}$

or $2m - l - k = f(v_5) = f(v_1) + f(v_4) = l + m - l \pmod{m}$. In either of these cases, we get $l \equiv k \pmod{m} \Rightarrow l = k$ (as $l, k < m$), which is a contradiction to the fact that f is injective. $\square$

Conjecture 4.2. *For every integer $n \geq 3$ ($n \neq 6$), the cycle C_n is not a SSOMSG.*

5 SOMSL and SSOMSL of a Complete graph

Theorem 5.1. *For every integer $n \geq 2$ ($n \neq 3$), the complete graph K_n is a SSOMSG under addition modulo m , where $m = n + 1$ if n is even, and $m = \frac{(n+1)(n-2)}{2}$ if n is odd.*

Proof. When $n = 3$, $K_3 \equiv C_3$, is not an outer mod sum graph. Consider a complete graph K_n on $n \neq 3$ vertices $v_1, v_2, \dots, v_n$. Define $f : V(K_n) \rightarrow \{1, 2, \dots, n\}$ as $f(v_i) = i$ for all i , $1 \leq i \leq n$. It is shown in [10] that the function f so defined is an outer mod sum labeling under addition modulo $n + 1$ if n is even and under addition modulo $(\frac{(n+1)(n-2)}{2})$ if n is odd. Therefore each vertex v_i has a working vertex. Further, for all i , $1 \leq i \leq n$, $N_f(v_i) = \sum_{k=1, k \neq i}^n f(v_k) = \sum_{k=1}^n f(v_k) - f(v_i) = \sum_{k=1}^n k - i = \frac{n(n+1)}{2} - i$. Now when n is even, $N_f(v_i) = \frac{n(n+1)}{2} - i \equiv (-i) \pmod{n+1} \Rightarrow N_f(v_i) \equiv n+1-i \pmod{n+1} \Rightarrow N_{f_{n+1}}(v_i) = n+1-i = f(v_{n+1-i})$ or equivalently, $N_{f_{n+1}}(v_{n+1-i}) = f(v_i)$. When n is odd, $N_f(v_i) = \frac{n(n+1)}{2} - i = \frac{(n-2+2)(n+1)}{2} - i = \frac{(n-2)(n+1)}{2} + n+1-i \equiv n+1-i \pmod{\frac{(n-2)(n+1)}{2}} \Rightarrow N_{f_{\frac{(n+1)(n-2)}{2}}}(v_i) = n+1-i = f(v_{n+1-i})$ or equivalently, $N_{f_{\frac{(n+1)(n-2)}{2}}}(v_{n+1-i}) = f(v_i)$ which imply in both the cases, each vertex v_i is the working vertex of the unique vertex v_{n+1-i} . Thus the function f defined above is a semi strong outer mod sum labeling under addition modulo $n + 1$ if n is even and under addition modulo $(\frac{(n+1)(n-2)}{2})$ when n is odd. Hence the theorem. $\square$

Theorem 5.2. *For every integer $n \geq 3$, the complete graph K_n is not a strong outer mod sum graph.*

Proof. In view of Theorem 3.5 and Theorem 4.1, it suffices to prove the result for $n > 3$. If possible, suppose K_n ($n > 3$) is a SOMSG and f be a SOMSL of K_n under modulo m . Then, by Theorem 2.6, it follows that $(n-2) \sum_{k=1}^n f(v_i) \equiv 0 \pmod{m}$. Now, we have the following cases.

Case 1: $\gcd(n-2, m) = 1$

In this case $\sum_{k=1}^n f(v_i) \equiv 0 \pmod{m} \Rightarrow f(v_i) \equiv -(f(v_1) + f(v_2) + \dots + f(v_{i-1}) + f(v_{i+1}) + \dots + f(v_n)) \pmod{m} \Rightarrow f(v_i) \equiv -(f(v_i)) \pmod{m} \Rightarrow 2f(v_i) \equiv 0 \pmod{m} \Rightarrow f(v_i) \equiv 0 \pmod{\frac{m}{2}}$ ($\because 0 < f(v_i) < m$ and

hence $2|m$, for every i , $1 \leq i \leq n \Rightarrow f(v_i) = \frac{m}{2}$ for each i , which is a contradiction (since f is injective).

Case 2: $\gcd(n-2, m) \neq 1$

Let $\gcd(n-2, m) = k > 1$. Then $\gcd(\frac{n-2}{k}, \frac{m}{k}) = 1$, so $\sum_{k=1}^n f(v_i) \equiv 0 \pmod{\frac{m}{k}} \Rightarrow f(v_i) \equiv -(f(v_1) + f(v_2) + \dots + f(v_{i-1}) + f(v_{i+1}) \dots + f(v_n)) \pmod{\frac{m}{k}} \Rightarrow f(v_i) \equiv -(f(v_i)) \pmod{\frac{m}{k}} \Rightarrow 2f(v_i) \equiv 0 \pmod{\frac{m}{k}}$. Now if $(\frac{m}{k}, 2) = 1$, then $f(v_i) \equiv 0 \pmod{\frac{m}{k}}$ for every i , $1 \leq i \leq n$, which is a contradiction, otherwise $(\frac{m}{k}, 2) = 2 \Rightarrow (\frac{m}{2k}, 1) = 1$, then $f(v_i) \equiv 0 \pmod{\frac{m}{2k}}$ for every i , $1 \leq i \leq n$, which is again a contradiction (since in either of the cases $1 < k \leq n-2 < m$ and f is injective).

Hence the theorem. □

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References

- [1] J. Boland, R. Laskar, C. Turner, and G. Domke, *On mod sum graphs*, Congr. Numer., 70 (1990) 131-135.
- [2] F. Buckley and Frank Harary, *Distance in Graphs*, Addison-Wesley, (1990).
- [3] Gary Chartrand and Ping Zhang, *Introduction to Graph theory*, Tata McGraw-Hell Edition (2006).
- [4] Z. Chen, *Harary's conjectures on integral sum graphs*, Disc. Math., 160(1996) 241-244.
- [5] Wenqing Dou and Jingzhen Gao, *The (mod, integral) sum number of fans and $K_{n,n} - E(nK_2)$* , Discrete Mathematics, 306 (2006) 2655-2669.
- [6] Joseph A. Gallian, *A dynamic survey of graph labeling*, The Electronic Journal of Combinatorics, # DS6, (2009), 1-219.
- [7] Hartsfield Gerhard and Ringel, *Pearls in Graph Theory*, Academic Press, USA, 1994.
- [8] Frank Harary, *Sum graph and difference graphs*, Congr. Numer., 72 (1990) 101-108.

- [9] Frank Harary, *Sum graph over all the integers*, Disc. Math., 124(1994) 99-105.
- [10] Jayalakshmi M, B. Sooryanarayana, P Devadas Rao, *Outer Mod Sum Labelings of a Graph*, Internatinal Journal of Information Science and Computer Mathematics, Volume 2, Number 2, 2010, pp87-102.
- [11] K. N. Meera and B. Sooryanarayana, *Optimal Outer Sum number of a Graphs*, International Journal of Combinatorial graph theory and applications, Vol 4, No. 1, (Jan-june 2011) pp 23-35.
- [12] A. V. Pyatkin, *New formula for the sum number for the complete bipartite graphs*, Disc. Math., 239 (2001) 155-160.
- [13] Joe Ryan, *Exclusive sum labeling of graphs: A survey*, AKEC International Journal of Graphs. Combinatorics., Vol 6, No. 1, (2009) 113-126.
- [14] B. Sooryanarayana, Manjula K and Vishu Kumar M, *Outer Sum Labeling of a Graph*, International Journal of Combinatorial graph theory and applications ., Vol 4, No. 1, (2011) 37-60.
- [15] Martin Sutton and Anna Draganova and Mirka Miller, *Mod Sum Number of Wheels*, Ars Combinatoria, 63(2002) 273-287.
- [16] Haiying Wang, *The sum numbers and integral sum number of the graph $K_{1,n} \setminus E(K_{1,n})$* , Disc. Math. 309(2009) 4137-4143.

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