Boundary Set on Bigeneralized Topological Spaces

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Abstract

The purpose of this paper is to introduce the concept and some fundamental properties of boundary set on bigeneralized topological spaces.

Keywords: generalized topological space, bigeneralized topological space, boundary

1 Introduction

In 1963, J.C. Kelly [1] introduced the notion of bitopological spaces and extended some of the standard results of separation axioms in a topological space to a bitopological space. The concepts of generalized topologies or briefly *GT* was introduced by Á. Császár [2] in 2002. He also introduced two kinds of generalized continuity. In particular, generalized continuity admits a characterization furnishing a known characterization of θ -continuous maps. In 2010, C. Boonpok [4] introduced the notion of bigeneralized topological spaces and the concepts of weakly functions. He also investigated some of their characterizations.

The purpose of this paper is to introduce the concept of boundary set on bigeneralized topological spaces and some of their simple properties.

2 Preliminaries

Let X be a nonempty set and g be a collection of subsets of X . Then g is called a *generalized topology* (briefly *GT*) on X if $\emptyset \in g$ and if $G_i \in g$ for $i \in I \neq \emptyset$ then $G = \bigcup_{i \in I} G_i \in g$.

By (X, g) , we denote a nonempty set X with a generalized topology g on X and it is called a *generalized topological space* (briefly *GTS*) on X . The

elements of G are called g -open sets and the complements are called g -closed sets.

Theorem 2.1. [2] Let (X, g) be a generalized topological space. Then

1. $Cl(A) = X \setminus Int(X \setminus A)$;
2. $Int(A) = X \setminus Cl(X \setminus A)$.

Proposition 2.2. [3] Let (X, g) be a generalized topological space. For subsets A and B of X , the following properties holds;

1. $Cl(X \setminus A) = X \setminus Int(A)$ and $Int(X \setminus A) = X \setminus Cl(A)$;
2. If $X \setminus A \in g$, then $Cl(A) = A$ and if $A \in g$, then $Int(A) = A$;
3. If $A \subseteq B$, then $Cl(A) \subseteq Cl(B)$ and $Int(A) \subseteq Int(B)$;
4. $A \subseteq Cl(A)$ and $Int(A) \subseteq A$;
5. $Cl(Cl(A)) = Cl(A)$ and $Int(Int(A)) = Int(A)$.

Next, we recall some notions and notations of bigeneralized topological spaces.

Definition 2.3. [4] Let X be a nonempty set and g_X^1, g_X^2 be generalized topologies on X . A triple (X, g_X^1, g_X^2) is called a *bigeneralized topological space* (briefly *BGTS*).

Definition 2.4. [4] A subset A of a bigeneralized topological space (X, g_X^1, g_X^2) is called $g_X^i g_X^j$ -closed if $A = g^i Cl(g^j Cl(A))$, where $i, j=1$ or 2 and $i \neq j$. The complement of $g_X^i g_X^j$ -closed set is called $g_X^i g_X^j$ -open.

Proposition 2.5. [4] Let (X, g_X^1, g_X^2) be a bigeneralized topological space. Then A is $g_X^i g_X^j$ -open subset of (X, g_X^1, g_X^2) if and only if $A = g^i Int(g^j Int(A))$.

3 Boundary

In this section, we introduce the concept and study some fundamental properties of boundary set on bigeneralized topological spaces.

Definition 3.1. Let (X, g_X^1, g_X^2) be a bigeneralized topological space, A be a subset of X and $x \in X$. We called x is (i, j) - g_X -boundary point of A if $x \in g^i Cl(g^j Cl(A)) \cap g^i Cl(g^j Cl(X \setminus A))$. We denote the set of all (i, j) - g_X -boundary point of A by $gBdr_{ij}(A)$ where $i, j = 1, 2$ and $i \neq j$.

From definition we have $gBdr_{ij}(A) = g^i Cl(g^j Cl(A)) \cap g^i Cl(g^j Cl(X \setminus A))$.

Example 3.2. Let $X = \{1, 2, 3, 4\}$. Define two generalized topologies g_X^1 and g_X^2 on X as follows: $g_X^1 = \{\emptyset, \{1, 2\}, \{3\}, \{1, 2, 3\}\}$ and $g_X^2 = \{\emptyset, \{1\}, \{2, 3\}, \{1, 2, 3\}\}$. Hence, $gBdr_{12}(\{3\}) = X$, $gBdr_{21}(\{3\}) = \{2, 3, 4\}$ and $gBdr_{12}(\{2\}) = X = gBdr_{21}(\{2\})$.

Lemma 3.3. *Let (X, g_X^1, g_X^2) be a bigeneralized topological space and A be a subset of X . Then $gBdr_{ij}(A) = gBdr_{ij}(X \setminus A)$ where $i, j = 1, 2$ and $i \neq j$.*

$$\begin{aligned} \text{Proof. } gBdr_{ij}(A) &= g^i Cl(g^j Cl(A)) \cap g^i Cl(g^j Cl(X \setminus A)) \\ &= g^i Cl(g^j Cl(X \setminus (X \setminus A))) \cap g^i Cl(g^j Cl(X \setminus A)) \\ &= gBdr_{ij}(X \setminus A) \end{aligned}$$

Thus $gBdr_{ij}(A) = gBdr_{ij}(X \setminus A)$ where $i, j = 1, 2$ and $i \neq j$. $\square$

Theorem 3.4. *Let (X, g_X^1, g_X^2) be a bigeneralized topological space and A, B be a subset of X . Then for any $i, j = 1, 2$ and $i \neq j$, we have the following statements;*

1. $gBdr_{ij}(A) = g^i Cl(g^j Cl(A)) \setminus g^i Int(g^j Int(A))$;
2. $gBdr_{ij}(A) \cap g^i Int(g^j Int(A)) = \emptyset$;
3. $gBdr_{ij}(A) \cap g^i Int(g^j Int(X \setminus A)) = \emptyset$;
4. $g^i Cl(g^j Cl(A)) = gBdr_{ij}(A) \cup g^i Int(g^j Int(A))$;
5. $X = g^i Int(g^j Int(X \setminus A)) \cup gBdr_{ij}(A) \cup g^i Int(g^j Int(A))$
is a pairwise disjoint union.

Proof. Assume that (X, g_X^1, g_X^2) is a bigeneralized topological space and A, B are subsets of X .

1.
$$\begin{aligned} gBdr_{ij}(A) &= g^i Cl(g^j Cl(A)) \cap g^i Cl(g^j Cl(X \setminus A)) \\ &= g^i Cl(g^j Cl(A)) \cap [X \setminus g^i Int(g^j Int(A))] \\ &= g^i Cl(g^j Cl(A)) \setminus g^i Int(g^j Int(A)). \end{aligned}$$
2. From (1), it's obvious that, $gBdr_{ij}(A) \cap g^i Int(g^j Int(A)) = \emptyset$.
3. By Lemma 3.3 and (2), we have $gBdr_{ij}(A) \cap g^i Int(g^j Int(X \setminus A)) = \emptyset$.
4.
$$\begin{aligned} g^i Cl(g^j Cl(A)) &= [g^i Cl(g^j Cl(A)) \setminus g^i Int(g^j Int(A))] \cup g^i Int(g^j Int(A)) \\ &= gBdr_{ij}(A) \cup g^i Int(g^j Int(A)). \end{aligned}$$
5.
$$\begin{aligned} g^i Int(g^j Int(X \setminus A)) \cup gBdr_{ij}(A) \cup g^i Int(g^j Int(A)) \\ &= [X \setminus g^i Cl(g^j Cl(A))] \cup g^i Cl(g^j Cl(A)) \\ &= X. \end{aligned}$$

By (2) and (3) we have $gBdr_{ij}(A) \cap g^i Int(g^j Int(A)) = \emptyset$
and $gBdr_{ij}(A) \cap g^i Int(g^j Int(X \setminus A)) = \emptyset$.

Now, we will show that $g^i Int(g^j Int(A)) \cap g^i Int(g^j Int(X \setminus A)) = \emptyset$.

Since $g^i Int(g^j Int(A)) \cap g^i Int(g^j Int(X \setminus A)) \subseteq A \cap (X \setminus A) = \emptyset$. we also have $g^i Int(g^j Int(A)) \cap g^i Int(g^j Int(X \setminus A)) = \emptyset$.

Therefore, $X = g^i Int(g^j Int(A)) \cup gBdr_{ij}(A) \cup g^i Int(g^j Int(X \setminus A))$
is a pairwise disjoint union.

$\square$

Theorem 3.5. *Let (X, g_X^1, g_X^2) be a bigeneralized topological space and A be a subset of X . Then for any $i, j = 1, 2$ and $i \neq j$, we have;*

1. *A is $g_X^i g_X^j$ - closed if and only if $gBdr_{ij}(A) \subseteq A$.*
2. *A is $g_X^i g_X^j$ - open if and only if $gBdr_{ij}(A) \subseteq (X \setminus A)$.*

Proof. Let A be a subset of X .

1. ($\implies$) Assume that A is $g_X^i g_X^j$ - closed. Thus $g^i Cl(g^j Cl(A)) = A$.

$$gBdr_{ij}(A) \cap (X \setminus A) = g^i Cl(g^j Cl(A)) \cap g^i Cl(g^j Cl(X \setminus A)) \cap (X \setminus A)$$

$$= A \cap g^i Cl(g^j Cl(X \setminus A)) \cap (X \setminus A) = \emptyset.$$

Therefore, $gBdr_{ij}(A) \subseteq A$.

- ($\impliedby$) Assume that $gBdr_{ij}(A) \subseteq A$. Thus $gBdr_{ij}(A) \cap (X \setminus A) = \emptyset$, and also

$$g^i Cl(g^j Cl(A)) \cap (X \setminus A) = g^i Cl(g^j Cl(A)) \cap [g^i Cl(g^j Cl(X \setminus A)) \cap (X \setminus A)]$$

$$= gBdr_{ij}(A) \cap (X \setminus A)$$

$$= \emptyset.$$

Therefore, $g^i Cl(g^j Cl(A)) \subseteq A$. But $A \subseteq g^i Cl(g^j Cl(A))$.

Consequently, $A = g^i Cl(g^j Cl(A))$. Hence A is $g_X^i g_X^j$ - closed.

2. ($\implies$) Assume that A is $g_X^i g_X^j$ - open. Thus $g^i Int(g^j Int(A)) = A$.

$$gBdr_{ij}(A) \cap A = [g^i Cl(g^j Cl(A)) \setminus g^i Int(g^j Int(A))] \cap A$$

$$= [g^i Cl(g^j Cl(A)) \setminus A] \cap A$$

$$= \emptyset.$$

Therefore, $gBdr_{ij}(A) \subseteq (X \setminus A)$.

- ($\impliedby$) Assume that $gBdr_{ij}(A) \subseteq (X \setminus A)$. Thus $gBdr_{ij}(A) \cap A = \emptyset$, and also $[g^i Cl(g^j Cl(A)) \setminus g^i Int(g^j Int(A))] \cap A = \emptyset$. Since $A \subseteq g^i Cl(g^j Cl(A))$, we have $A \setminus g^i Int(g^j Int(A)) = \emptyset$. But $g^i Int(g^j Int(A)) \subseteq A$.
 Consequently, $A = g^i Int(g^j Int(A))$. Hence A is $g_X^i g_X^j$ - open.

□

Theorem 3.6. *Let (X, g_X^1, g_X^2) be a bigeneralized topological space and A be a subset of X . Then $gBdr_{ij}(A) = \emptyset$ if and only if A is $g_X^i g_X^j$ - closed and $g_X^i g_X^j$ - open where $i, j = 1, 2$ and $i \neq j$.*

Proof. ($\implies$) Assume that $gBdr_{ij}(A) = \emptyset$.

Thus by Theorem 3.5, we have A is $g_X^i g_X^j$ - closed and $g_X^i g_X^j$ - open.

- ($\impliedby$) Assume that A is $g_X^i g_X^j$ - closed and $g_X^i g_X^j$ - open.
 By Theorem 3.5, we have $gBdr_{ij}(A) \subseteq A \cap (X \setminus A) = \emptyset$.
 Hence, $gBdr_{ij}(A) = \emptyset$.

□

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