Int. Journal of Math. Analysis, Vol. 7, 2013, no. 2, 91 - 103

Positive Solutions of a Second-Order Difference Equation with Summation Boundary Value Problem

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Abstract

In this paper, we study the existence of positive solutions to the summation boundary value problem

$$\Delta^2 u(t-1) + a(t)f(u) = 0, \qquad t \in \{1, 2, ..., T\},$$

$$u(0) = 0, \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s),$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, ..., T-1\}$, $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$ and $\Delta u(t-1) = u(t) - u(t-1)$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

Mathematics Subject Classification: 39A10

Keywords: Positive solution; Boundary value problem; Fixed point theorem; Cone

1 Introduction

The study of the existence of solutions of multipoint boundary value problems for linear second-order ordinary differential and difference equations was initiated by Ilin and Moiseev [1]. Then Gupta [2] studied three-point boundary value problems for nonlinear second-order ordinary differential equations.

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Since then, nonlinear second-order three-point boundary value problems have also been studied by many authors, one may see the text books [3-4] and the papers [6-11]. However, all these papers are concerned with problems with three-point boundary condition restrictions on the difference of the solutions and the solutions themselves, for example,

$$\begin{aligned} u(0) &= 0, & u(T+1) &= 0 \\ u(0) &= 0, & au(s) &= u(T+1), \\ u(0) &= 0, & u(T+1) - au(s) &= b. \\ u(0) - \alpha\Delta u(0) &= 0, & u(T+1) &= \beta u(s). \\ u(0) - \alpha\Delta u(0) &= 0, & \Delta u(T+1) &= 0 \end{aligned}$$

and so forth.

In [6], Leggett-Williams developed a fixed point theorem to prove the existence of three positive solutions for Hammerstein integral equations. Since then, this theorem has been reported to be a successful technique for dealing with the existence of three solutions for the two-point boundary value problems of differential and difference equations; see [7,8]. In [9], X. Lin and W. Liu using the properties of the associate Green's function and Leggett-Williams fixed point theorem, studied the existence of positive solutions of the problem.

In [10], G. Zhang and R. Medina studied the existence of positive solutions for second order boundary value problems of difference equations by applying the Krasnoselskii's fixed point theorem. In [11], J. Henderson and H.B. Thompson used lower and upper solution methods.

In this paper, we consider the existence of positive solutions to the equation

$$\Delta^2 u(t-1) + a(t)f(u) = 0, \qquad t \in \{1, 2, ..., T\}, \tag{1}$$

with the three-point summation boundary condition

$$u(0) = 0, \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{2}$$

where f is continuous, $T \geq 3$ is a fixed positive integer, $\eta \in \{1, 2, ..., T-1\}$.

The aim of this paper is to give some results for existence of positive solutions to (1)-(2), assuming that $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$ and f is either superlinear or sublinear. Set

$$f_0 = \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

Then $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.

Let $\mathbb{N}$ be the nonnegative integer, we let $\mathbb{N}_{i,j} = \{k \in \mathbb{N} | \ \ i \leq k \leq j\}$ and $\mathbb{N}_p = \mathbb{N}_{0,p}$. By the positive solution of (1)-(2) we mean that a function $u(t) : \mathbb{N}_{T+1} \to [0, \infty)$ and satisfies the problem (1)-(2).

Throughout this paper, we suppose the following conditions hold:
$(H1)$ $f \in C([0,\infty),[0,\infty))$;
$(H2)$ $a \in C(\mathbb{N}_{T+1},[0,\infty))$ and there exists $t_0 \in \mathbb{N}_{\eta,T+1}$ such that $a(t_0) > 0$.

The proof of the main theorem is based upon an application of the following Krasnoselskii's fixed point theorem in a cone.

Theorem 1.1. ([5]). *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_2$; *or*
(ii) $\|Au\| \geqslant \|u\|, \quad u \in K \cap \partial\Omega_1$, *and* $\|Au\| \leqslant \|u\|, \quad u \in K \cap \partial\Omega_2$.
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminary

We now state and prove several lemmas before stating our main results.

Lemma 2.1. *Let $\alpha \neq \frac{2T+2}{\eta(\eta+1)}$. Then for $y \in C(\mathbb{N}_{T+1},[0,\infty))$, the problem*

$$\Delta^2 u(t-1) + y(t) = 0, \qquad t \in \mathbb{N}_{1,T}, \tag{3}$$

$$u(0) = 0, \qquad u(T+1) = \alpha \sum_{s=1}^{\eta} u(s), \tag{4}$$

has a unique solution

$$\begin{aligned} u(t) =& \frac{2t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{T} (T-s+1)y(s) \\ &- \frac{\alpha t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s) \\ &- \sum_{s=1}^{t-1} (t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \end{aligned}$$

Proof. From $\Delta^2 u(t-1) = \Delta u(t) - \Delta u(t-1)$ and the first equation of (3), we get

$$\begin{gathered} \Delta u(t) - \Delta u(t-1) = -y(t), \\ \Delta u(t-1) - \Delta u(t-2) = -y(t-1), \\ \vdots \\ \Delta u(1) - \Delta u(0) = -y(1). \end{gathered}$$

We sum the above equations to obtain

$$\Delta u(t) = \Delta u(0) - \sum_{s=1}^{t} y(s), t \in \mathbb{N}_T. \tag{5}$$

We define $\sum_{s=p}^{q} y(s) = 0$; if $p < q$. Similarly, we sum (5) from $t = 0$ to $t = h$, and by using the boundary condition $u(0) = 0$ in (4), we obtain

$$u(h+1) = (h+1)\Delta u(0) - \sum_{s=1}^{h} (h+1-s)y(s), h \in \mathbb{N}_T,$$

by changing the variable from $h+1$ to t, we have

$$u(t) = t\Delta u(0) - \sum_{s=1}^{t-1} (t-s)y(s), t \in \mathbb{N}_{T+1}. \tag{6}$$

From (6),

$$\begin{aligned}\sum_{s=1}^{\eta} u(s) &= \frac{\eta(\eta+1)}{2}\Delta u(0) - \sum_{s=1}^{\eta-1}\sum_{l=1}^{\eta-s} ly(s)\\ &= \frac{\eta(\eta+1)}{2}\Delta u(0) - \frac{1}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s)\end{aligned}$$

Again using the boundary condition $u(T+1) = \alpha \sum_{s=1}^{\eta} u(s)$ in (4), we obtain

$$(T+1)\Delta u(0) - \sum_{s=1}^{T}(T-s+1)y(s) = \frac{\alpha\eta(\eta+1)}{2}\Delta u(0) - \frac{\alpha}{2}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s)$$

Thus,

$$\begin{aligned}\Delta u(0) =& \frac{2}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)y(s)\\ &- \frac{\alpha}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)y(s).\end{aligned}$$

Therefore, (3)-(4) has a unique solution

$$\begin{aligned} u(t) =& \frac{2t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{T} (T-s+1)y(s) \\ & - \frac{\alpha t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{\eta-1} (\eta-s)(\eta-s+1)y(s) \\ & - \sum_{s=1}^{t-1} (t-s)y(s), \quad t \in \mathbb{N}_{T+1}. \end{aligned}$$

□

Lemma 2.2. *Let* $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$. *If* $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ *and* $y(t) \geqslant 0$ *for* $t \in \mathbb{N}_{1,T}$, *then the unique solution* u *of* (3)-(4) *satisfies* $u \geqslant 0$ *for* $t \in \mathbb{N}_{T+1}$.

Proof. From the fact that $\Delta^2 u(t-1) = u(t+1) - 2u(t) + u(t-1) = -y(t) \le 0$, we get $u(t) \ge \frac{u(t+1)+u(t-1)}{2}$, imply that $\frac{u(t+1)}{t+1} < \frac{u(t)}{t}$.
Hence,

$$\frac{u(T+1)}{T+1} < \frac{u(\eta)}{\eta}, \quad \eta \in \mathbb{N}_{1,T-1}. \tag{7}$$

Moreover, we know that

$$u(i) > \frac{i}{\eta} u(\eta) \quad \text{for} \quad i < \eta, \tag{8}$$

we get,

$$\begin{aligned} \sum_{s=1}^{\eta} u(s) >& \frac{1}{\eta}u(\eta) + \frac{2}{\eta}u(\eta) + \dots + \frac{\eta}{\eta}u(\eta) \\ =& \frac{1}{\eta}u(\eta)[1+2+\dots+\eta] = \frac{1}{2}(\eta+1)u(\eta) \end{aligned}$$

$$\therefore \quad \sum_{s=1}^{\eta} u(s) > \frac{1}{2}(\eta+1)u(\eta). \tag{9}$$

If $u(T+1) \geqslant 0$, then, by (7) and the boundary condition $u(0) = 0$ imply that $u(t) \geqslant 0$ for $t \in \mathbb{N}_{T+1}$.

Assume that $u(T+1) < 0$. From (4), we have

$$\sum_{s=1}^{\eta} u(s) < 0. \tag{10}$$

By (7),(9) and (10), implies that $u(\eta) < 0$. Hence,

$$\frac{u(T+1)}{T+1} = \frac{\alpha}{T+1}\sum_{s=1}^{\eta} u(s) > \frac{\alpha(\eta+1)}{2T+2}u(\eta) = \frac{\alpha\eta(\eta+1)}{2T+2}\frac{u(\eta)}{\eta} > \frac{u(\eta)}{\eta},$$

which contradicts with (7). □

Lemma 2.3. *Let $\alpha\eta(\eta+1) > 2T+2$. If $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ and $y(t) \geqslant 0$ for $t \in \mathbb{N}_{1,T}$, then* (3)-(4) *has no positive solution.*

Proof. Assume (3)-(4) has a positive solution u.

If $u(T+1) > 0$, then $\sum_{s=1}^{\eta} u(s) > 0$ and by (7) and (9), imply that $u(\eta) > 0$ and

$$\frac{u(T+1)}{T+1} = \frac{\alpha}{T+1}\sum_{s=1}^{\eta} u(s) > \frac{\alpha(\eta+1)}{2T+2}u(\eta) = \frac{\alpha\eta(\eta+1)}{2T+2}\frac{u(\eta)}{\eta} > \frac{u(\eta)}{\eta},$$

which contradicts with (7).

If $u(T+1) = 0$, then $\sum_{s=1}^{\eta} u(s) = 0$ and by (9), imply that $u(t) \equiv 0$ for all $t \in \mathbb{N}_{1,\eta}$. If there exists $\tau \in \mathbb{N}_{\eta+1,T}$ such that $u(\tau) > 0$, then $u(0) = \frac{u(\eta)}{\eta} < \frac{u(\tau)}{\tau}$, which contradicts with (8).

Therefore, no positive solutions exist. □

In the rest of the paper, we assume that $0 < \alpha\eta(\eta+1) < 2T+2$. Moreover, we will work in the Banach space $C(\mathbb{N}_{T+1}, [0,\infty))$, and only the sup norm is used.

Lemma 2.4. *Let $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$. If $y \in C(\mathbb{N}_{T+1}, [0,\infty))$ and $y \geqslant 0$, then the unique solution u of the problem* (3)-(4) *satisfies*

$$\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) \geqslant \gamma\|u\|,$$

where

$$\gamma := \min\left\{\frac{\eta}{T+1}, \frac{\alpha\eta(\eta+1)}{2T+2}, \frac{\alpha(\eta+1)(T+1-\eta)}{2T+2-\alpha\eta(\eta+1)}\right\} \tag{11}$$

Proof. Let $u(t)$ be maximal at $t = \tau$ and $\|u\| = u(\tau)$. We divide the proof into three cases.

Case (i). If $\eta \leqslant \tau \leqslant T+1$ and $\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) = u(\eta)$, then

$$\frac{u(\eta)}{\eta} \geqslant \frac{u(\tau)}{\tau} \geqslant \frac{u(\tau)}{T+1}.$$

Thus,

$$\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) \geqslant \frac{\eta}{T+1}\|u\|.$$

Case (ii). If $\eta \leqslant \tau \leqslant T+1$ and $\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) = u(T+1)$, then (4), (7) and (9) implies

$$u(T+1) = \alpha\sum_{s=1}^{\eta} u(s) > \frac{\alpha\eta(\eta+1)}{2}\Big[\frac{u(\eta)}{\eta+1}\Big] \geqslant \frac{\alpha\eta(\eta+1)}{2}\frac{u(\tau)}{\tau+1} \geqslant \frac{\alpha\eta(\eta+1)}{2T+2}u(\tau).$$

Therefore,

$$\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) \geqslant \frac{\alpha\eta(\eta+1)}{2T+2}\|u\|.$$

Case (iii). If $\tau \leqslant \eta < T+1$, then $\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) = u(T+1)$. Using (4) and (7), we have

$$\begin{aligned} u(\tau) &\leqslant u(T+1) + \frac{u(T+1)-u(\eta)}{T+1-\eta}(\tau-(T+1)) \\ &< u(T+1) + \frac{u(T+1)-u(\eta)}{T+1-\eta}(0-(T+1)) \\ &\leqslant u(T+1)\Big[1 - \frac{T+1-\frac{2(T+1)}{\alpha(\eta+1)}}{T+1-\eta}\Big] \\ &= u(T+1)\frac{2T+2-\alpha\eta(\eta+1)}{\alpha(\eta+1)(T+1-\eta)}. \end{aligned}$$

This implies

$$\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) \geqslant \frac{\alpha(\eta+1)(T+1-\eta)}{2T+2-\alpha\eta(\eta+1)}\|u\|.$$

This completes the proof. □

3 Main Results

Now we are in the position to establish the main result.

Theorem 3.1. *Assume* $(H1)$ *and* $(H2)$ *hold. Then the problem* (1)-(2) *has at least one positive solution in the case*
(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear) or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. It is known that $0 < \alpha < \frac{2T+2}{\eta(\eta+1)}$. From Lemma 2.1, u is a solution to the boundary value problem (1)-(2) if and only if u is a fixed point of operator A, where A is defined by

$$\begin{aligned} u(t) =& \frac{2t}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ & - \frac{\alpha t}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s)) - \sum_{s=1}^{t-1}(t-s)a(s)f(u(s)) \\ :=& (Au)(t). \end{aligned}$$

Denote

$$K = \{u \mid u \in C(\mathbb{N}_{T+1}, [0,\infty)), u \geqslant 0, \inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) \geqslant \gamma\|u\|\}.$$

where γ is defined in (11).

It is obvious that K is a cone in $C(\mathbb{N}_{T+1}, [0,\infty))$. by Lemma 2.4, $A(K) \subseteq K$. It is also easy to check that $A : K \longrightarrow K$ is completely continuous.

Superlinear case. $f_0 = 0$ and $f_\infty = \infty$. Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leqslant \epsilon u$, for $0 < u \leqslant H_1$, where $\epsilon > 0$ satisfies

$$\frac{2\epsilon(T+1)}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \leqslant 1.$$

Thus, if we let

$$\Omega_1 = \{u \in C(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < H_1\},$$

then for $u \in K \cap \partial\Omega_1$, we get

$$\begin{aligned} Au(t) \leqslant& \frac{2t}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ \leqslant& \frac{2t\epsilon}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)u(s) \\ \leqslant& \frac{2\epsilon(T+1)}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)\|u\| \leqslant \|u\|. \end{aligned}$$

Thus $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geqslant \rho u$, for $u \geqslant \widehat{H}_2$, where $\rho > 0$ is chosen so that

$$\frac{2\eta\rho\gamma}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s) \geqslant 1.$$

Let $H_2 = \max\{2H_1, \frac{\widehat{H}_2}{\gamma}\}$ and $\Omega_2 = \{u \in C(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < H_2\}$. Then $u \in K \cap \partial\Omega_2$ implies

$$\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) \geqslant \gamma\|u\| = \gamma H_2 \geqslant \widehat{H}_2,$$

and so

$$\begin{aligned}
Au(\eta) =&\frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))\\
&-\frac{\alpha\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s))-\sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s))\\
=&\frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))\\
&-\frac{\alpha\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}(\eta^2-2\eta s+s^2+\eta-s)a(s)f(u(s))\\
&-\frac{1}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}(2T+2-\alpha\eta(\eta+1))(\eta-s)a(s)f(u(s))\\
=&\frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)f(u(s))\\
&+\frac{2\eta(T+1)}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}a(s)f(u(s))-\frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}sa(s)f(u(s))\\
&-\frac{\alpha\eta(\eta^2+\eta)}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}a(s)f(u(s))+\frac{\alpha\eta(2\eta+1)}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}sa(s)f(u(s))\\
&-\frac{\alpha\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}s^2a(s)f(u(s))\\
&-\frac{2T\eta+2\eta-\alpha\eta^3-\alpha\eta^2}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}a(s)f(u(s))+\frac{2T+2-\alpha\eta^2-\alpha\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}sa(s)f(u(s))\\
=&\frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)f(u(s))\\
&+\frac{2(T+1-\eta)}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}sa(s)f(u(s))\\
&+\frac{\alpha\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}(\eta s-s^2)a(s)f(u(s))
\end{aligned}$$

$$\begin{aligned}Au(\eta) \geqslant & \frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)f(u(s))\\ \geqslant & \frac{2\eta\rho}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)u(s)\\ \geqslant & \frac{2\eta\rho\gamma}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)\|u\| \geq \|u\|.\end{aligned}$$

Hence, $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_2$. By the first part of Theorem **??**, A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$ such that $H_1 \leqslant \|u\| \leqslant H_2$.

Sublinear case. $f_0 = \infty$ and $f_\infty = 0$. Since $f_0 = \infty$, choose $H_3 > 0$ such that $f(u) \geqslant Mu$ for $0 < u \leqslant H_3$, where $M > 0$ satisfies

$$\frac{2\eta\gamma M}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)a(s) \geqslant 1.$$

Let

$$\Omega_3 = \{u \in C(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < H_3\},$$

then for $u \in K \cap \partial\Omega_3$, we get

$$\begin{aligned}Au(\eta) =& \frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s)f(u(s))\\ &- \frac{\alpha\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s))\\ &- \sum_{s=1}^{\eta-1}(\eta-s)a(s)f(u(s))\\ \geqslant & \frac{2\eta}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)f(u(s))\\ \geqslant & \frac{2\eta M}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)u(s)\\ \geqslant & \frac{2\eta\gamma M}{2T+2-\alpha\eta(\eta+1)}\sum_{s=\eta}^{T}(T-s+1)a(s)\|u\| \geqslant \|u\|.\end{aligned}$$

Thus, $\|Au\| \geqslant \|u\|$, $u \in K \cap \partial\Omega_3$. Now, since $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ so that $f(u) \leqslant \lambda u$ for $u \geqslant \widehat{H}_4$, where $\lambda > 0$ satisfies

$$\frac{2\lambda(T+1)}{2T+2-\alpha\eta(\eta+1)}\sum_{s=1}^{T}(T-s+1)a(s) \leqslant 1.$$

Choose $H_4 = \max\{2H_3, \frac{\widehat{H}_4}{\gamma}\}$. Let

$$\Omega_4 = \{u \in C(\mathbb{N}_{T+1}, [0,\infty)) \mid \|u\| < H_4\},$$

then $u \in K \cap \partial\Omega_4$ implies

$$\inf_{t\in\mathbb{N}_{\eta,T+1}} u(t) \geqslant \gamma\|u\| = \gamma H_4 \geqslant \widehat{H}_4.$$

Therefore,

$$\begin{aligned} Au(t) =& \frac{2t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ & - \frac{\alpha t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{\eta-1}(\eta-s)(\eta-s+1)a(s)f(u(s)) \\ & - \sum_{s=1}^{t-1}(t-s)a(s)f(u(s)) \end{aligned}$$

$$\begin{aligned} Au(t) \leqslant& \frac{2t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)f(u(s)) \\ \leqslant& \frac{2\lambda t}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)u(s) \\ \leqslant& \frac{2\lambda(T+1)}{2T+2-\alpha\eta(\eta+1)} \sum_{s=1}^{T}(T-s+1)a(s)\|u\| \leqslant \|u\|. \end{aligned}$$

Thus $\|Au\| \leqslant \|u\|$, $u \in K \cap \partial\Omega_4$. By the second part of Theorem **??**, A has a fixed point u in $K \cap (\overline{\Omega}_4 \setminus \Omega_3)$, such that $H_3 \leqslant \|u\| \leqslant H_4$. This completes the sublinear part of the theorem. Therefore, the problem (1)-(2) has at least one positive solution. □

4 Some examples

In this section, in order to illustrate our result, we consider some examples.
Example 4.1 Consider the BVP

$$\Delta^2 u(t-1) + t^2 u^k = 0, \qquad t \in N_{1,4}, \tag{12}$$

$$u(0) = 0, \qquad u(5) = \frac{2}{3}\sum_{s=1}^{2} u(s). \tag{13}$$

Set $\alpha = \frac{2}{3},\ \eta = 2,\ T = 4,\ a(t) = t^2,\ f(u) = u^k$.
We can show that

$$0 < \alpha = \frac{2}{3} < \frac{5}{3} = \frac{2T+2}{\eta(\eta+1)},$$

Case I : $k \in (1, \infty)$. In this case, $f_0 = 0,\ f_\infty = \infty$ and (i) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.
Case II : $k \in (0, 1)$. In this case, $f_0 = \infty,\ f_\infty = 0$ and (ii) of theorem 3.1 holds. Then BVP (4.1)-(4.2) has at least one positive solution.

Example 4.2 Consider the BVP

$$\Delta^2 u(t-1) + e^t t^e (\frac{\pi \sin u + 2\cos u}{u^2}) = 0, \qquad t \in N_{1,4}, \tag{14}$$

$$u(0) = 0, \qquad u(5) = \frac{1}{3}\sum_{s=1}^{3} u(s), \tag{15}$$

Set $\alpha = \frac{1}{3},\ \eta = 3,\ T = 4,\ a(t) = e^t t^e,\ f(u) = \frac{\pi \sin u + 2\cos u}{u^2}$.
We can show that

$$0 < \alpha = \frac{1}{3} < \frac{5}{6} = \frac{2T+2}{\eta(\eta+1)},$$

Through a simple calculation we can get $f_0 = \infty,\ f_\infty = 0$. Thus, by (ii) of theorem 3.1, we can get BVP (4.3)-(4.4) has at least one positive solution.

ACKNOWLEDGEMENTS. This research is supported by the Centre of Excellence in Mathematics, Commission on Higher Education, Thailand.

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Received: August, 2012