Fixed Point Theorems without Continuity of any Mappings in Dislocated Quasi Metric Space

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Abstract. The aim of this paper is to prove some common fixed point theorems for dislocated quasi-metric spaces. We point out that for the existence of fixed point, continuity of any mapping is not needed.

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1. Introduction

In 1922, a famous and well appreciated result in the field of fixed point theory for contracting mapping in complete metric space commonly known as a Banach fixed point theorem was proved by Banach [2]. The proven result becomes an asset for an applied mathematician due to its enormous applications in various branches of mathematics which includes differential equations, integral equation etc. and other areas of science involving mathematics especially in logic programming and electronic engineering. Banach fixed point theorem attracted many researcher involved in the said area to study and explore new results in contraction mapping using varying conditions. Further, Kannan [6] came out with a new type of contraction mapping which may not be continuous where as Das and Gupta [4] and Ciric [3] gave a generalization of Banach contraction principle in metric space. Also Rohades [8] made a successful attempt to establish a partial ordering for various

definitions of contractive mapping. The study of Abramsky and Jung [9] emphasized on the concept of dislocated metrics under the name of metric domains in the context of domain theory by considering that the self distance for any point need to be equal to zero for a metric. Hitzler and Seda [7] came out with a notion of dislocated metric spaces thereby extended the Banach Contraction Principle for such spaces.

Zeyada et al. [10] generalized the work of Hitzler and Seda [7] in dislocated quasi-metric spaces. Furthermore Aage and Salunke [1] studied the mappings refereed by Kannan [6] and Ćirić [3] and obtained fixed point theorems in dislocated quasi-metric space. Recently, Isufati [5] proved some fixed point theorems for continuous contraction mapping defined by Das and Gupta [4] and Rhoades [8] in dislocated quasi-metric spaces. The objective of this paper is to improve results of Isufati [5] by striking out the condition of continuity.

2. Preliminaries

Definition 2.1 [10] Let X be a non-empty set and let $d : X \times X \rightarrow [0, \infty)$ be a function satisfying following conditions :

- (i) $d(x, y) = d(y, x) = 0$, implies $x = y$,
- (ii) $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$.

Then d is called a dislocated quasi-metric on X . If d satisfies $d(x, y) = d(y, x)$, then it is called dislocated metric.

Definition 2.2 [10] A sequence $\{x_n\}$ is dislocated quasi-metric space (X, d) is called Cauchy sequence if for given $\varepsilon > 0$, there exists $n_0 \in \mathbb{N}$, such that for all $m, n \geq n_0$, implies $d(x_m, x_n) < \varepsilon$ or $d(x_n, x_m) < \varepsilon$
i.e. $\min\{d(x_m, x_n), d(x_n, x_m)\} < \varepsilon$.

Definition 2.3 [10] A sequence $\{x_n\}$ dislocated quasi-convergence to x if

$$\lim_{n \rightarrow \infty} d(x_n, x) = \lim_{n \rightarrow \infty} d(x, x_n) = 0$$

In this case x is called a dislocated quasi limit of $\{x_n\}$ and we write $x_n \rightarrow x$.

Lemma 2.1 [10] Dislocated quasi-limits in a dislocated quasi metric space are unique.

Definition 2.4 [10] A dislocated quasi-metric space (X, d) is called complete if every Cauchy sequence in it is a dislocated quasi-convergent.

Definition 2.5 [10] Let (X, d_1) and (Y, d_2) be dislocated quasi-metric spaces and let $f : X \rightarrow Y$ be a function. Then f is continuous to $x_0 \in X$, if for each sequence

$\{x_n\}$ which is d_1 – quasi convergent to x_0 , the sequence $\{f(x_n)\}$ is d_2 – quasi convergent to $f(x_0)$ in Y .

Definition 2.6 [10] let (X, d) be a dq – metric space. A map $T : X \rightarrow X$ is called contraction if there exists $0 \leq \lambda < 1$, such that

$$d(Tx, Ty) \leq \lambda d(x, y) , \text{ for all } x, y \in X.$$

3. Main Results

Theorem 3.1 Let (X, d) be a complete dq – metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition.

$$(3.1) \quad d(Tx, Ty) \leq \alpha \frac{d(y, Ty) [1 + d(x, Tx)]}{(1 + d(x, y))} + \beta d(x, y) ,$$

for all $x, y \in X$, $\alpha > 0$, $\beta > 0$, $\alpha + \beta < 1$.

Then T has a unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in X , defined as follows

Let $x_0 \in X$, $T(x_0) = x_1$, $T(x_1) = x_2$, $\dots$, $T(x_n) = x_{n+1}$.

Consider

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, x_n) \leq \frac{\alpha d(x_n, Tx_n) [1 + d(x_{n-1}, Tx_{n-1})]}{(1 + d(x_{n-1}, x_n))} + \beta d(x_{n-1}, x_n) ,$$

$$\leq \frac{\alpha d(x_n, x_{n+1}) [1 + d(x_{n-1}, x_n)]}{(1 + d(x_{n-1}, x_n))} + \beta d(x_{n-1}, x_n) ,$$

Therefore

$$d(x_n, x_{n+1}) \leq \frac{\beta d(x_{n-1}, x_n)}{(1 - \alpha)} = \lambda d(x_{n-1}, x_n) , \text{ where } \lambda = \frac{\beta}{(1 - \alpha)} \text{ with } 0 \leq \lambda < 1 .$$

Similarly we can show that $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$,
and $d(x_n, x_{n+1}) \leq \lambda^2 d(x_{n-2}, x_{n-1})$

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Thus $d(x_n, x_{n+1}) \leq \lambda^n d(x_1, x_0)$.

Since $0 \leq \lambda < 1$ as $n \rightarrow \infty$, $\lambda^n \rightarrow 0$. Hence $\{x_n\}$ is a dq -sequence in Complete dislocated quasi-metric space X . Thus $\{x_n\}$ dislocated quasi converges to some t_0 .

By (3.1), we have

$$d(Tz, Tx_n) \leq \frac{\alpha d(x_n, Tx_n) [1 + d(z, Tx_n)]}{(1 + d(z, x_n))} + \beta d(z, x_n) .$$

Taking the limit $n \rightarrow \infty$, we have

$$d(Tz, z) \leq \frac{\alpha d(z, z) [1 + d(z, Tz)]}{(1 + d(z, z))} + \beta d(z, z)$$

$$d(Tz, z) \leq 0,$$

which is a contradiction. Therefore $Tz = z$ i.e. z is a fixed point of Tz .

For uniqueness let w ($z \neq w$) be another fixed point of T . Then by (3.1), we have

$$d(z, w) = d(Tz, Tw) \leq \frac{\alpha d(w, Tw) [1 + d(z, Tz)]}{(1 + d(z, w))} + \beta d(z, w),$$

$$d(z, w) \leq \frac{\alpha d(w, w) [1 + d(z, z)]}{(1 + d(z, w))} + \beta d(z, w),$$

$$d(z, w) \leq \beta d(z, w),$$

which is a contradiction, since $0 \leq \beta < 1$ and therefore $z = w$. Thus the fixed point is unique.

Theorem 3.2 Let (X, d) be a complete dq-metric space and let $T : X \rightarrow X$ be a mapping satisfying the following condition.

$$(3.2) \quad d(Tx, Ty) \leq \alpha d(x, Ty) + \beta d(y, Tx) + \gamma d(x, y),$$

where α, β, γ are non negative, which may depends on both x and y , such that $\sup\{2\alpha + 2\beta + \gamma; x, y \in X\} < 1$.

Then T has a unique fixed point.

Proof: Let $\{x_n\}$ be a sequence in X , defined as follows.

$$\text{Let } x_0 \in X, \quad T(x_0) = x_1, T(x_1) = x_2, \dots, T(x_n) = x_{n+1}.$$

Consider

$$\begin{aligned} d(x_n, x_{n+1}) &= d(Tx_{n-1}, Tx_n) \leq \alpha d(x_{n-1}, Tx_n) + \beta d(x_n, Tx_{n-1}) + \gamma d(x_{n-1}, x_n), \\ &\leq \alpha d(x_{n-1}, x_{n+1}) + \beta d(x_n, x_n) + \gamma d(x_{n-1}, x_n), \\ &\leq \alpha d(x_{n-1}, x_n) + \alpha d(x_n, x_{n+1}) + \gamma d(x_{n-1}, x_n), \\ d(x_n, x_{n+1}) &\leq \frac{(\alpha + \gamma)}{(1 - \alpha)} d(x_{n-1}, x_n) = \lambda d(x_{n-1}, x_n), \text{ where } \lambda = \frac{(\alpha + \gamma)}{(1 - \alpha)}. \end{aligned}$$

Similarly, we have $d(x_{n-1}, x_n) \leq \lambda d(x_{n-2}, x_{n-1})$.

In this way, we get $d(x_n, x_{n+1}) \leq \lambda^n d(x_0, x_1)$.

Since $0 \leq \lambda < 1$, so for $n \rightarrow \infty$, we have $d(x_n, x_{n+1}) \rightarrow 0$.

Similarly we can show that $d(x_{n-1}, x_n) \rightarrow 0$. Hence $\{x_n\}$ is a Cauchy sequence in the complete dislocated quasi-metric space, so there is a point $z \in X$ such that $x_n \rightarrow z$.

By (3.2), we have

$$\begin{aligned} d(Tz, Tx_n) &\leq \alpha d(z, Tx_n) + \beta d(x_n, Tz) + \gamma d(z, x_n), \\ d(Tz, x_{n+1}) &\leq \alpha d(z, x_{n+1}) + \beta d(x_n, Tz) + \gamma d(z, x_n). \end{aligned}$$

Taking limit $\rightarrow \infty$, we have

$$\begin{aligned} d(Tz, z) &\leq \alpha d(z, z) + \beta d(z, Tz) + \gamma d(z, z), \\ d(Tz, z) &\leq \beta d(z, Tz). \end{aligned}$$

a contradiction. Thus $d(z, Tz) = 0$ implies $Tz = z$.

For uniqueness let w ($z \neq w$) be another fixed point of T . Then by (3.2), we have

$$\begin{aligned} d(z, w) &= d(Tz, Tw) \leq \alpha d(z, Tw) + \beta d(w, Tz) + \gamma d(z, w), \\ d(z, w) &\leq (\alpha + \beta + \gamma) d(z, w), \end{aligned}$$

a contradiction. Therefore $z = w$. i.e. z is a unique fixed point of T .

Theorem 3.3 Let (X, d) be a complete dislocated metric space and let $A, B : X \rightarrow X$ be mappings such that

$$(3.3) \quad d(Ax, By) \leq h \max \left\{ d(x, y), d(x, Ax), d(y, By), \frac{(d(x, By) + d(y, Ax))}{2} \right\},$$

for all $x, y \in X$ and $0 < h < 1$.

Then A and B have unique common fixed point.

Proof: Let $x_0 \in X$. Define the sequence $\{x_n\}$ by $x_1 = Ax_0$, $x_2 = Bx_1$, ..., $x_{2n} = Bx_{2n-1}$, $x_{2n+1} = Ax_{2n}$,

$$\begin{aligned} \text{Consider } d(x_{2n+1}, x_{2n+2}) &= d(Ax_{2n}, Bx_{2n+1}) \\ &\leq h \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, Ax_{2n}), d(x_{2n+1}, Bx_{2n+1}), \frac{(d(x_{2n}, Bx_{2n+1}) + d(x_{2n+1}, Ax_{2n}))}{2} \right\}, \end{aligned}$$

$$\begin{aligned} &= h \max \left\{ d(x_{2n}, x_{2n+1}), d(x_{2n}, x_{2n+1}), d(x_{2n+1}, x_{2n+2}), \frac{(d(x_{2n}, x_{2n+2}) + d(x_{2n+1}, x_{2n+1}))}{2} \right\}, \\ &= h d(x_{2n}, x_{2n+1}). \end{aligned}$$

$$\text{Therefore } d(x_{2n+1}, x_{2n+2}) \leq h d(x_{2n}, x_{2n+1}),$$

$$\text{Similarly } d(x_{2n}, x_{2n+1}) \leq h d(x_{2n-1}, x_{2n}),$$

and so on

$$d(x_{2n+1}, x_{2n+2}) \leq h^2 d(x_{2n-1}, x_{2n}).$$

Proceeding in the same way, we have

$$d(x_{2n+1}, x_{2n+2}) \leq h^n d(x_0, x_1).$$

Since $0 < h < 1$, $h^{2n} \rightarrow 0$ as $n \rightarrow \infty$. Thus $\{x_n\}$ is a Cauchy sequence in a complete dislocated metric space X . There exists a point $z \in X$ such that $x_n \rightarrow z$.

$$d(Az, Bx_{2n}) \leq h \max \left\{ d(z, x_{2n}), d(z, Az), d(x_{2n}, Bx_{2n}), \frac{(d(z, Bx_{2n}) + d(x_{2n}, Az))}{2} \right\},$$

$$d(Az, x_{2n+1}) \leq h \max \left\{ d(z, x_{2n}), d(z, Az), d(x_{2n}, x_{2n+1}), \frac{(d(z, x_{2n+1}) + d(x_{2n}, Az))}{2} \right\}$$

Taking the limit $n \rightarrow \infty$, we have

$$\begin{aligned} d(Az, z) &\leq h \max \left\{ d(z, z), d(z, Az), d(z, z), \frac{(d(z, z) + d(z, Az))}{2} \right\}, \\ &\leq h \max \left\{ 0, d(z, Az), 0, \frac{(0 + d(z, Az))}{2} \right\}. \end{aligned}$$

This yields $d(Az, z) \leq h d(Az, z)$.

This gives $Az = z$. Similarly we can prove that $Bz = z$.

For uniqueness let w ($z \neq w$) be another fixed point of T . Then by (3.3), we have

$$d(z, w) = d(Az, Bw) \\ \leq h \max \left\{ d(z, w), d(z, Az), d(w, Bw), \frac{(d(z, Bw) + d(w, Az))}{2} \right\},$$

$$d(z, w) \leq h \max \left\{ d(z, w), d(z, z), d(w, w), \frac{(d(z, w) + d(w, z))}{2} \right\},$$

This gives $d(z, w) \leq h d(z, w)$.

Therefore $z = w$. i.e. common fixed point of A and B is unique.

Remark 3.1 In Theorem 3.1 if we take T continuous and $\alpha = 0$, we obtain Theorem of Zeyada, Hassan and Ahmed [10].

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