On the Category of Pointed G-Sets

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Abstract
In this paper, we obtain that the category of pointed G-sets has kernels and cokernels. Further, we prove that the category of pointed G-sets is normal and factorizable.

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1. Introduction

Pointed sets may be regarded as a rather simple algebraic structure. In the sense of universal algebra, these are structures with a single nullary operation which picks out the base point. Pointed sets are mainly used as illustrative examples in the study of universal algebra as algebra with a single constant operator. This operator takes every element in the algebra to a unique constant, which is known as a base point. Any homomorphism between two algebras
preserves base points (taking the base point of the domain algebra to the base point of the codomain algebra). Motivated by the idea of pointed sets and pointed mappings, pointed G-sets and pointed G-morphisms have been studied in [2,3].

Henceforth the category $G$-Sets*, of pointed G-sets has been constructed by taking into account pointed G-sets as the objects of the category and pointed G-morphisms as the morphisms of the category. The above mentioned paper deals with the behaviour of some special morphisms such as monomorphisms, epimorphisms, sections and retractions in the category $G$-Sets*. As a follow up, in the present analysis we obtain some interesting results in this category by proving that the category $G$-Sets* possesses kernels and cokernels. The existence of these notions is used in achieving that the category $G$-Sets* is normal and factorizable.

2. Preliminaries

We begin with the following definitions and results that will be needed in the sequel [2,3,6]:

**Definition 2.1.** Let $G$ be a group and $X$ be a set. Then $X$ is said to be a G-set if there exists a mapping $\phi : G \times X \rightarrow X$ such that for all $a, b \in G$ and $x \in X$ the following conditions are satisfied: (i) $\phi(ab, x) = \phi(a, \phi(b, x))$ (ii) $\phi(e, x) = x$, where $e$ is the identity of $G$. The G-set $X$ defined above will be denoted by the pair $(X, \phi)$.

For the sake of convenience, one can denote $\phi(a, x)$ as $ax$. Under this notation, above conditions become (i) $(ab)x = a(bx)$ (ii) $ex = x$.

**Definition 2.2.** Let $X$ and $Y$ be two G-sets. Then a mapping $f : X \rightarrow Y$ is called a G-morphism from $X$ to $Y$ if $f(ax) = af(x)$ for all $a \in G, x \in X$.

**Definition 2.3.** Let $(X, \phi)$ be a G-set. Then a subset $A$ of $X$ is called a G-subset of $X$ if $(A, \phi)$ is also a G-set.

**Definition 2.4.** A pointed set $(X, x')$ is said to be a pointed G-set if there exists a mapping $\phi : G \times X \rightarrow X$ such that

(i) $(X, \phi)$ is a G-set,

(ii) $\phi(g, x') = x'$ for all $g \in G$. 
Definition 2.5. Let \((X, x')\) be a pointed G-set. Then a pointed G-subset of \((X, x')\) is an ordered pair \((A, x')\), where \(A\) is a G-subset of \(X\).

Definition 2.6. Let \((X, x')\) and \((Y, y')\) be two pointed G-sets. Then a mapping \(f : (X, x') \rightarrow (Y, y')\) is called a pointed G-morphism if

(i) \(f\) is a G-morphism i.e., \(f(ax) = af(x)\) for all \(a \in G, \ x \in X\),

(ii) \(f(x') = y'\).

Definition 2.7. Let \(\alpha : (X, x') \rightarrow (Y, y')\) be a morphism in \(\mathcal{G}\)-Sets*. Then the subset \(Ker(\alpha) = \{x \mid \alpha(x) = y'\}\) of \(X\) is called the kernel of the morphism \(\alpha\).

Lemma 2.1[6]. Let \((X, \phi)\) be a G-set. Then

(i) for any \(x, y \in X\), a relation \(\sim_G\) on \(X\) defined by \(x \sim_G y \Leftrightarrow y = \phi(g, x)\) for some \(g \in G\), is an equivalence relation,

(ii) the set \(X/\sim_G\) of all G-equivalence classes is a G-set.

Definition 2.8. Let \(\alpha : (X, x') \rightarrow (Y, y')\) be a morphism in \(\mathcal{G}\)-Sets* and \(\sim_G\) be a G-equivalence relation on \(Y\). Then the pointed G-set \((Y/\sim_G, [y'])\) is called the co-kernel of \(\alpha\) and is denoted by \(Coker(\alpha)\).

3. Main Results

Proposition 3.1. Let \(\alpha : (X, x') \rightarrow (Y, y')\) be a pointed G-morphism. Then \(Ker(\alpha)\) is a pointed G-subset of \(X\) in \(\mathcal{G}\)-Sets*.

Proof. In order to show that \(Ker(\alpha)\) is a pointed G-subset of \(X\), define a mapping \(\phi : G \times Ker(\alpha) \rightarrow Ker(\alpha)\) by \(\phi(a, x) = ax\) for all \(a \in G, x \in Ker(\alpha)\).

Let \(x \in Ker(\alpha)\). Then we have \(\alpha(x) = y'\) implying thereby \(a\alpha(x) = ay'\) for all \(a \in G\). Therefore, \(\alpha(ax) = y'\) yielding \(ax \in Ker(\alpha)\) and so the mapping \(\phi\) is well defined.

For any \(a, b \in G\) and \(x \in X\), we get \(\phi(ab, x) = (ab)x = a(\phi(b, x)) = \phi(a, \phi(b, x))\). Also for the identity element \(e \in G\), one gets \(\phi(e, x) = x\). Therefore \(Ker(\alpha)\) is a G-subset of \(X\).

Further, since \(f\) is a pointed G-morphism, then \(f(x') = y'\) implies \(x' \in Ker(\alpha)\) and also for any \(a \in G\), we get \(\phi(a, x') = ax' = x'\) which amounts to say that \((Ker(\alpha), x')\) is a pointed G-subset of \((X, x')\).
Theorem 3.1. The category \( \mathcal{G}\text{-Sets}^* \) has kernels.

**Proof.** Let \( f : (X, x') \to (Y, y') \) be a morphism in \( \mathcal{G}\text{-Sets}^* \). Consider the set \( K = \{ x \mid f(x) = y' \} \subseteq X \), then in view of Proposition 3.1, \( K \) is a pointed \( \mathcal{G} \)-subset. Let \( i : (K, x') \to (X, x') \) be an inclusion morphism in \( \mathcal{G}\text{-Sets}^* \). We claim that \( i : (K, x') \to (X, x') \) is the kernel of \( f : (X, x') \to (Y, y') \) in \( \mathcal{G}\text{-Sets}^* \).

For any \( x \in K \), one gets \((f \circ i)(x) = f(i(x)) = f(x) = y' = O_{KY}(x)\) which implies \( f \circ i = O_{KY} \).

Now, for any object \((Z, z') \in \mathcal{G}\text{-Sets}^* \), let \( \alpha : (Z, z') \to (X, x') \) be a morphism in \( \mathcal{G}\text{-Sets}^* \) such that \( f \circ \alpha = O_{ZY} \).

For any \( z \in Z \), one gets \((f \circ \alpha)(z) = O_{ZY}(z)\) implying thereby \( f(\alpha(z)) = y' \). Henceforth \( \alpha(z) \in K \) which yields \( \text{Im}(\alpha) \subseteq K \). Therefore we can define a mapping \( \eta : (Z, z') \to (K, x') \) by \( \eta(z) = \alpha(z) \) for all \( z \in Z \).

For any \( z \in Z \) and \( a \in G \), we get \( \eta(az) = \alpha(az) = a\alpha(z) = a(\eta(z)) \) showing that \( \eta \) is a \( G \)-morphism and also \( \eta(z') = \alpha(z') = x' \) which amounts to say that \( \eta : (Z, z') \to (K, x') \) is a morphism in \( \mathcal{G}\text{-Sets}^* \).

Moreover, for any \( z \in Z \), we have \((i \circ \eta)(z) = i(\eta(z)) = i(\alpha(z)) = \alpha(z)\) which implies \( i \circ \eta = \alpha \).

Finally, in order to show that \( \eta \) is unique, suppose there is another morphism \( \xi : (Z, z') \to (K, x') \) in \( \mathcal{G}\text{-Sets}^* \) such that \( i \circ \xi = \alpha \). Then for any \( z \in Z \), we have \((i \circ \xi)(z) = \alpha(z)\) implying \( i(\xi(z)) = \eta(z) \) which gives \( \xi(z) = \eta(z) \). Therefore \( \xi = \eta \) and hence the result follows.

Theorem 3.2. The category \( \mathcal{G}\text{-Sets}^* \) has co-kernels.

**Proof.** Let \( f : (X, x') \to (Y, y') \) be a morphism in \( \mathcal{G}\text{-Sets}^* \). Let \( R \) be a relation on \( Y \) such that for any \( y_1, y_2 \in Y \), \( y_1 \bar{R}y_2 \Leftrightarrow y_1, y_2 \in \text{Im}(f) \). Consider a smallest equivalence relation \( \bar{R} \) on \( Y \) containing \( R \) such that for any \( y_1, y_2 \in Y \), \( y_1 \bar{R}y_2 \Leftrightarrow y_1 = y_2 \) or \( y_1, y_2 \in \text{Im}(f) \). The equivalence classes \([y]\) under \( \bar{R} \) are given by

\[
[y] = \begin{cases} 
[f(x)] = \text{Im}f & \text{if } y = f(x) \in \text{Im}f \\
\{y\} & \text{if } y \notin \text{Im}f.
\end{cases}
\]

Clearly, \( Y/\bar{R} \) is a pointed \( G \)-set with the base point \([y']\).
Since \( y' = f(x') \), it follows that \( y' \in \text{Im}(f) \) yielding thereby \( [y'] = \text{Im}(f) \). Define a natural mapping \( p : (Y, y') \rightarrow (Y/\bar{R}, [y']) \) by \( p(y) = [y] \) for all \( y \in Y \) in \( \text{G-Sets}^* \). We claim that \( p : (Y, y') \rightarrow (Y/\bar{R}, [y']) \) is the co-kernel of \( f : (X, x') \rightarrow (Y, y') \) in \( \text{G-Sets}^* \).

For any \( x \in X \), one gets \( (p \circ f)(x) = p(f(x)) = [f(x)] = \text{Im}(f) = [y'] = O_{XY/R}(x) \) which implies \( p \circ f = O_{XY/R} \).

For any object \((Z, z') \in \text{G-Sets}^* \), let \( q : (Y, y') \rightarrow (Z, z') \) be a morphism in \( \text{G-Sets}^* \) such that \( q \circ f = O_{XZ} \).

For any \( x \in X \), one gets \( (q \circ f)(x) = O_{XZ}(x) \) implying thereby \( q(f(x)) = z' \). Thus \( f(x) \in \text{Ker}(q) \) which gives \( \text{Im}(f) \subseteq \text{Ker}(q) \). Therefore we can define a mapping \( \eta : Y/\bar{R} \rightarrow Z \) by \( \eta([y]) = q(y) \) for all \( y \in Y \).

Let \([y_1] = [y_2] \) which gives \( y_1 \sim \bar{R} y_2 \). Therefore \( y_1 = y_2 \) or \( y_1, y_2 \in \text{Im}(f) \).

Now, if \( y_1 = y_2 \), then \( q(y_1) = q(y_2) \) which implies \( \eta([y_1]) = \eta([y_2]) \). Again, if \( y_1, y_2 \in \text{Im}(f) \subseteq \text{Ker}(q) \), then \( q(y_1) = z' = q(y_2) \) which implies \( \eta([y_1]) = \eta([y_2]) \). Consequently \( \eta \) is well defined.

For any \([y] \in Y/\bar{R} \) and \( a \in G \), one gets \( \eta(a[y]) = \eta([ay]) = q(ay) = a(q(y)) = a(\eta([y])) \) which shows that \( \eta \) is a G-morphism and also \( \eta([y']) = q(y') = z' \) which in turn yields that \( \eta : (Y/\bar{R}, [y']) \rightarrow (Z, z') \) is a morphism in \( \text{G-Sets}^* \).

Moreover, for any \( y \in Y \), we have \( (\eta \circ p)(y) = \eta(p(y)) = \eta([y]) = q(y) \) which implies \( \eta \circ p = q \).

Finally, in order to prove that \( \eta \) is unique, suppose there is another morphism \( \xi : (Y/\bar{R}, [y']) \rightarrow (Z, z') \) in \( \text{G-Sets}^* \) such that \( \xi \circ p = q \). Then for any \( y \in Y \), we have \( \xi([y]) = \eta([y]) \) yielding thereby \( \xi(p(y)) = \eta([y]) \). Therefore \( \xi([y]) = \eta([y]) \) which implies \( \xi = \eta \) and hence the result follows.

**Theorem 3.3.** In the category \( \text{G-Sets}^* \) every monomorphism is a kernel.

**Proof.** Let \( f : (X, x') \rightarrow (Y, y') \) be a monomorphism in \( \text{G-Sets}^* \), then \( f \) is injective in \( \text{G-Sets}^* \) [2, Corollary 3.1]. Thus \( X \simeq \text{Im}(f) \subseteq Y \). In view of Theorem 3.2, \( p : (Y, y') \rightarrow (Y/\bar{R}, [y']) \) is the co-kernel of \( f : (X, x') \rightarrow (Y, y') \) in the category \( \text{G-Sets}^* \) and we have \( p \circ f = O_{XY/R} \). We claim that \( f : (X, x') \rightarrow (Y, y') \) is the kernel of \( p : (Y, y') \rightarrow (Y/\bar{R}, [y']) \).
Let $\alpha : (Z, z') \rightarrow (Y, y')$ be a morphism in $\mathcal{G}$-Sets* such that $p \circ \alpha = O_{ZY/R}$. Then for any $z \in Z$, we have $(p \circ \alpha)(z) = O_{ZY/R}(z)$ yielding thereby $p(\alpha(z)) = [y'] = \text{Im}(f)$. Thus $[\alpha(z)] = \text{Im}(f)$ which gives $\alpha(z) \in \text{Im}(f)$ implying thereby $\text{Im}(\alpha) \subseteq \text{Im}(f)$. Therefore we can define a mapping $\eta : (Z, z') \rightarrow (X, x')$ by $\eta(z) = x$ where $f(x) = \alpha(z)$.

Here, it may be noted that $x$ is unique in view of the injectivity of $f$, therefore the mapping $\eta$ is well defined.

For any $a \in G$, we have $\alpha(az) = a(\alpha(z)) = a(f(x)) = f(ax)$ which amounts to say that $\eta(az) = ax = a(\eta(z))$ implying thereby that $\eta$ is a $G$-morphism and also $\alpha(z') = y' = f(x')$ yields that $\eta(z') = x'$. Consequently, $\eta$ is a morphism in $\mathcal{G}$-Sets*.

Moreover, for any $z \in Z$, we get $(f \circ \eta)(z) = f(\eta(z)) = f(x) = \alpha(z)$ implying thereby $f \circ \eta = \alpha$.

Finally, in order to prove that $\eta$ is unique, suppose there is another morphism $\xi : (Z, z') \rightarrow (X, x')$ in $\mathcal{G}$-Sets* such that $f \circ \xi = \alpha$. Then $f \circ \xi = f \circ \eta$ which implies $\xi = \eta$ as $f$ is left cancellable. This completes the proof.

**Theorem 3.4.** The category $\mathcal{G}$-Sets* is normal.

**Proof.** Since, the category $\mathcal{G}$-Sets* has zero object, therefore the proof follows from Theorem 3.1, Theorem 3.2 and Theorem 3.3.

**Theorem 3.5.** The category $\mathcal{G}$-Sets* is factorizable.

**Proof.** In view of the above Theorem 3.4, the category $\mathcal{G}$-Sets* is normal, therefore it is factorizable [1, Theorem 4.8].

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**References**


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