Tube Surfaces with Type-2 Bishop Frame of Weingarten Types in E^3

Sezai KIZILTUĞ, Semra KAYA and Ömer TARAKCI

Department of Mathematics, Faculty of Arts and Sciences
Atatürk University, Erzurum, Turkey
skiziltug24@hotmail.com
semra_kaya85@hotmail.com
tarakci@atauni.edu.tr

Abstract

In this paper, we study tube surfaces with type-2 Bishop frame instead of Frenet frame in Euclidean 3-space E^3 . Besides, we have discussed Weingarten and linear Weingarten conditions for tube surfaces with the Gaussian curvature K , the mean curvature H and the second Gaussian curvature K_{II} .

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1 Introduction

In analysis of surfaces, it is fairly common to determine some surfaces according to their curvatures. For example, it is important the existence of non-trivial functional relationship between the principal curvatures for surfaces in Minkowski 3-space. The resulting surfaces are called Weingarten surfaces or simply W-surfaces. In particular, a surface M in Minkowski 3-space is called *Weingarten surface* if there is some (smooth) relation $U(\kappa_1, \kappa_2) = 0$ between its two principal curvatures κ_1 and κ_2 or equivalently if there exists a non-trivial functional relation $\Phi(K, H) = 0$ with respect to its Gaussian curvature K and its mean curvature H . Functional relation $\Phi(K, H) = 0$ on the surface M with parametrized $X(u, v)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely $\left| \frac{\partial(K, H)}{\partial(u, v)} \right| = 0$ [10]. Also, linear Weingarten surfaces are Weingarten surfaces satisfying a linear equation $aK + bH = c$, $(a, b, c) \neq (0, 0, 0)$, $a, b, c \in \mathbb{R}$ between Gaussian and mean

curvatures. If $b = 0$, linear Weingarten surface M reduces to surface with constant Gaussian curvature. If $a = 0$, linear Weingarten surface M reduces to surface with constant mean curvature. In this sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature. Several geometers have studied W-surfaces and LW-surfaces and obtained many interesting results [5, 7, 8, 9]. If the second fundamental form II of a surface M in a Minkowski 3-space is non-degenerate, then it is regarded as a new (pseudo-) Riemannian metric. Therefore, the Gaussian curvature K_{II} of non-degenerate second fundamental form II can be defined formally on the Riemannian or pseudo-Riemannian manifold (M, II) . We call the curvature K_{II} the second Gaussian curvature on M . If surface M in Minkowski 3-space satisfies $\Phi(X, Y) = 0$ and $aX + bY = c$, then it is respectively said to be a (X, Y) -Weingarten surface and (X, Y) -linear Weingarten surface. Yüksel, Tuncer and Karacan studied tubular surfaces with Bishop frame of Weingarten types in E^3 [14]. Kızıltuğ and Yaylı investigate timelike tube surfaces with Darboux frame of Weingarten types in E_1^3 [18]. Also, Khalifa and Kızıltuğ studied parallel surfaces of Weingarten type in E_1^3 [12]. Kühnel [19] and Stamou [7] investigated ruled (X, Y) -Weingarten surface in Euclidean 3-space. Also, Baikoussis and Koufogiorgos studied helicoidal (H, K_{II}) -Weingarten surfaces [4]. Dillen and Sodsiri gave a classification of ruled (X, Y) -Weingarten surface in Minkowski 3-space [5]. Following Jacobi equation and linear equation with respect to the Gaussian curvature K , the mean curvature H and the second Gaussian curvature K_{II} raise an interesting geometric question: Classifying all surfaces in Euclidean 3-space and Minkowski 3-space satisfying the condition

$$\begin{aligned}\Phi(X, Y) &= 0 \\ aX + bY &= c\end{aligned}$$

where $X, Y \in \{K, H, K_{II}\}$, $X \neq Y$.

In this paper, we would like to contribute the solution of the above question by studying for tube surfaces with type-2 Bishop frame of Weingarten types in E^3 .

2 Preliminaries

Let $\alpha : I \rightarrow E^3$ be an arbitrary curve in E^3 . Recall that the curve α is said to be unit speed, if $\langle \alpha', \alpha' \rangle = 1$, where $\langle, \rangle$ is standard inner product which given by $\langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3$ for each $x = (x_1, x_2, x_3)$, $y = (y_1, y_2, y_3) \in E^3$. The norm of a vector $x \in E^3$ is given by $\|x\| = \sqrt{\langle x, x \rangle}$. Denote by $\{T(s), N(s), B(s)\}$ the moving Frenet frame along the unit speed curve α .

Then, the Frenet formulas are given by

$$\begin{bmatrix} T^i \\ N^i \\ B^i \end{bmatrix} = \begin{bmatrix} 0 & \kappa & 0 \\ -\kappa & 0 & \tau \\ 0 & -\tau & 0 \end{bmatrix} \begin{bmatrix} T \\ N \\ B \end{bmatrix} \quad (1)$$

$$\langle T, T \rangle = \langle N, N \rangle = \langle B, B \rangle = 1,$$

$$\langle T, N \rangle = \langle N, B \rangle = \langle B, T \rangle = 0.$$

Here T , N and B are respectively called the tangent, the principal normal and the binormal vector field of the curves α . $\kappa(s)$ and $\tau(s)$ are called curvature and torsion of the curve α , respectively.

The Bishop frame or parallel transport frame is an alternative approach to define a moving frame that is well defined even when second derivative of the curve has vanished. We can simply parallel transport an orthonormal frame along a curve by parallel transporting each component of the frame.

Let $\alpha = \alpha(s)$ be a unit speed regular curve in E^3 . The type-2 Bishop frame of the $\alpha(s)$ is defined by [15]

$$\begin{bmatrix} N_1^i \\ N_2^i \\ B^i \end{bmatrix} = \begin{bmatrix} 0 & 0 & -k_1 \\ 0 & 0 & -k_2 \\ k_1 & k_2 & 0 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}, \quad (2)$$

The relation matrix between Frenet-Serret and type-2 Bishop frames can be expressed

$$\begin{bmatrix} T \\ N \\ B \end{bmatrix} = \begin{bmatrix} \sin \theta(s) & -\cos \theta(s) & 0 \\ \cos \theta(s) & \sin \theta(s) & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} N_1 \\ N_2 \\ B \end{bmatrix}. \quad (3)$$

Here, the type-2 Bishop curvatures are defined by

$$k_1(s) = -\tau \cos \theta(s), \quad k_2(s) = -\tau \sin \theta(s). \quad (4)$$

It can be also deduced as

$$\theta^i = \kappa = \frac{\left(\frac{k_2}{k_1}\right)^i}{1 + \left(\frac{k_2}{k_1}\right)^2} \quad (5)$$

The frame $\{N_1, N_2, B\}$ is properly oriented, and τ and $\theta(s) = \int_0^s \kappa(s) ds$ are polar coordinates for the curve $\alpha = \alpha(s)$. We shall call the set $\{N_1, N_2, B, k_1, k_2\}$ as type-2 Bishop invariants of the curve $\alpha = \alpha(s)$.

We denote a surface M in E^3 by

$$M(u, v) = (m_1(u, v), m_2(u, v), m_3(u, v))$$

Let U be the standard unit normal vector field on M defined by

$$U = \frac{M_u \times M_v}{\|M_u \times M_v\|}$$

where, $M_u = \frac{\partial M(u, v)}{\partial u}$ and $M_v = \frac{\partial M(u, v)}{\partial v}$. Then, the first fundamental form I and the second fundamental form II of a surface M are respectively defined by

$$\begin{aligned} I &= Edu^2 + 2Fdudv + Gdv^2, \\ II &= edu^2 + 2fdudv + gdv^2 \end{aligned}$$

where

$$\begin{aligned} E &= \langle M_u, M_u \rangle, F = \langle M_u, M_v \rangle, G = \langle M_v, M_v \rangle \\ e &= \langle M_{uu}, U \rangle, f = \langle M_{uv}, U \rangle, g = \langle M_{vv}, U \rangle. \end{aligned}$$

On the other hand, the Gaussian curvature K and the mean curvature H are respectively,

$$\begin{aligned} K &= \frac{eg - f^2}{EG - F^2} \\ H &= \frac{Eg - 2Ff + Ge}{2(EG - F^2)} \end{aligned}$$

If the second fundamental form is non-degenerate; $eg - f^2 \neq 0$, then one define formally the second Gaussian curvature K_{II} a similar one to Brioschi's formula for the Gaussian curvature obtained on M , replacing the components of the first fundamental form E, F, G by those of the second fundamental form e, f, g as [4]

$$K_{II} = \frac{1}{(eg - f^2)^2} \left[\begin{vmatrix} -\frac{1}{2}e_{vv} + f_{uv} - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u - \frac{1}{2}e_v \\ f_v - \frac{1}{2}g_u & e & f \\ \frac{1}{2}g_v & f & g \end{vmatrix} - \begin{vmatrix} 0 & \frac{1}{2}e_v & g_u \\ \frac{1}{2}e_v & e & f \\ \frac{1}{2}g_u & f & g \end{vmatrix} \right].$$

3 Tube Surfaces with Type-2 Bishop Frame of Weingarten Types in E^3

Let $\alpha : (a, b) \rightarrow E^3$ be a smooth unit speed curve of finite length which is topologically imbedded in E^3 . The total space N_α of the normal bundle of

$\alpha((a, b))$ in E^3 is naturally diffeomorphic to the direct product $(a, b) \times E^2$ via the translation along α with respect to the induced normal connection. For a sufficiently small $r > 0$, the tube of radius r about the curve α is the set:

$$T_{r(\alpha)} = \{ \exp_{\alpha(u)} w \mid w \in Z_{\alpha(u)}, \|w\| = r, a < u < b \}.$$

For a sufficiently small $r > 0$, the tube $T_{r(\alpha)}$ is a smooth surface in E^3 . Let T, N, B denote the Frenet frame of the unit speed curve $\alpha = \alpha(u)$. Then the position vector of $T_{r(\alpha)}$ can be expressed as

$$M = M(u, v) = \alpha(u) + r(N(u) \cos v + B(u) \sin v). \quad (6)$$

By taking N_1 and N_2 instead of T and N we can write tube surface with type-2 Bishop frame as follows

$$M = M(u, v) = \alpha(u) + r(N_2(u) \cos v + B(u) \sin v). \quad (7)$$

Let M be tube surface with type-2 Bishop frame in E^3 given in (7). So, from the derivative formulas of type-2 Bishop frame partial differentiation of M with respect to u and v are as follows

$$\begin{aligned} M_u &= (1 + rk_1 \sin v) N_1 + (rk_2 \sin v) N_2 - (rk_2 \cos v) B \\ M_v &= -r \sin v N_2 + r \cos v B. \end{aligned}$$

Therefore, we find the components of the first fundamental form of M as

$$E = (1 + rk_1 \sin v)^2 + r^2 k_2^2, \quad F = -r^2 k_2, \quad G = r^2.$$

On the other hand, the unit surface normal vector field Z is obtained by

$$Z = \frac{M_u \times M_v}{\|M_u \times M_v\|} = -\cos v N_2 - \sin v B. \quad (8)$$

The second order partial differentials of M are found

$$\begin{aligned} M_{uu} &= [rk_1' \sin v - rk_1 k_2 \cos v] N_1 \\ &\quad + [rk_2' \sin v - rk_2^2 \cos v] N_2 \\ &\quad + [-k_1 - rk_1^2 \sin v - rk_2^2 \sin v - rk_2' \cos v] B, \\ M_{uv} &= (rk_1 \cos v) N_1 + (rk_2 \cos v) N_2 + (rk_2 \sin v) B \\ M_{vv} &= -r \cos v N_2 - r \sin v B. \end{aligned}$$

From equation (8) and the last equations we find the second fundamental form coefficients as follows

$$e = rk_2^2 + k_1 \sin v + rk_1^2 \sin^2 v, \quad f = -rk_2, \quad g = r$$

Thus, the Gaussian curvature K , the mean curvature H and the second Gaussian curvature K_{II} are given by, respectively

$$\begin{aligned} K &= \frac{k_1 \sin v}{r(1 + rk_1 \sin v)}, \quad H = \frac{1 + 2rk_1 \sin v}{2r(1 + rk_1 \sin v)}, \\ K_{II} &= \frac{\left\{ \frac{1}{2}rk_1^2 \sin^2 v + \frac{1}{2}rk_1^3 \sin^3 v + \frac{1}{2}rk_1 \cos v \right.}{\left. + \frac{1}{2}r^2k_1^2 \sin 2v - \cos 2v \sin v (k_1^3 r - r^2k_1^4 \sin v) - \frac{r^2k_2^2}{2} \right\}}{(k_1 \sin v + rk_1^2 \sin^2 v)^2}. \end{aligned} \quad (9)$$

Theorem 1 *Let M be tube surface in E^3 defined by (7). Then M is a (K, H) Weingarten surface.*

Proof. Let M be tube surface with type-2 Bishop frame in E^3 . If we take derivative of K and H given by (9) with respect to u and v respectively, then we get

$$K_u = \frac{rk_1^4 \sin v}{r^2(1 + rk_1 \sin v)^2}, \quad (10)$$

$$K_v = \frac{rk_1 \cos v}{r^2(1 + rk_1 \sin v)^2}.$$

$$H_u = \frac{2r^2k_1^4 \sin v}{4r^2(1 + rk_1 \sin v)^2}, \quad (11)$$

$$H_v = \frac{2r^2k_1 \cos v}{4r^2(1 + rk_1 \sin v)^2}.$$

By using (10) and (11), M satisfies identically the Jacobi equation

$$\Phi(K, H) = K_u H_v - K_v H_u = 0.$$

Therefore M is a Weingarten surface. ■

Theorem 2 *Let M be a tube surface in E^3 parametrized by (7) with non-degenerate second fundamental form. If M is a (K_{II}, K) Weingarten surface, then $k_1 = k_2 = 0$. M is an open part of a circular cylinder.*

Proof. Let M be a tube surface with type-2 Bishop frame in E^3 parametrized by (7). We obtained $(K_{II})_u$ and $(K_{II})_v$ by using Mathematica. But the values of this calculations are so long, then we omitted. We consider a tube surface with type-2 Bishop frame in E^3 satisfying the jacobi equation

$$\Phi(K_{II}, K) = (K_{II})_u K_v - (K_{II})_v K_u = 0 \quad (12)$$

with respect to The second Gaussian curvature K_{II} and the mean curvature H . Then by (10) equation (12) becomes

$$\frac{1}{2} (k_1^2 k_2^2 \sin v + k_1 k_2^2 k_1' \cos v) = 0.$$

Since this polynominal is equal to zero for every v , all its coefficients must be zero. Therefore, we conclude that $k_1 = k_2 = 0$ or $\kappa = 0$. ■

Theorem 3 *Let M be a tube surface in E^3 parametrized by (7) with non-degenerate second fundamental form. If M is a (H, K_{II}) Weingarten surface, then $k_1 = k_2 = 0$. M is an open part of a circular cylinder.*

Proof. We suppose that a tube surface parametrized by (7) with non-degenerate second fundamental form in E^3 is (H, K_{II}) Weingarten surface. Then it satisfies the equation

$$\Phi(H, K_{II}) = (K_{II})_v H_u - (K_{II})_u H_v = 0, \quad (13)$$

which implies

$$\frac{1}{4} (k_1^2 k_1' k_2^2 \sin v + k_1^2 k_2^4 k_1' \cos v) = 0. \quad (14)$$

From (14) we can obtain $k_1 = k_2 = 0$ or $\kappa = 0$. Thus M is an open part of a circular cylinder. ■

Now, we consider the following definition to examine the linear Weingarten property of the tube surface with type-2 Bishop frame.

Definition 4 *A surface M is said to be a linear Weingarten surface, if its Gaussian curvature K and mean curvature H satisfy the relation on M*

$$aK + bH = c \quad (15)$$

where, a, b and c are real numbers (not all zero).

Theorem 5 *Suppose that tube surface defined by (7) type-2 Bishop frame is a linear Weingarten surface in E^3 satisfying (15). Then $k_1 = 0$.*

Proof. Consider the parametrization (7) with K and H given by (9) respectively, we rewrite (15) to get

$$\frac{a(k_1 \sin v)}{r(1 + rk_1 \sin v)} + \frac{b(1 + 2rk_1 \sin v)}{2r(1 + rk_1 \sin v)} = c.$$

The above equation can be expressed in a simpler form

$$(2ak_1 + 2brk_1 - 2crk_1) \sin v + 2rc - b = 0.$$

According to the definition of the linear independent of vectors, we obtain $k_1 = 0$. Thus, this completes proof. ■

Theorem 6 *A tube surface defined by (7) type-2 Bishop frame with non-degenerate second fundamental form is a linear Weingarten surface if the curvatures H and K_{II} of the surface are written in a linear form. Then $k_1 = k_2 = 0$ or M is an open part of a circular cylinder.*

Proof. Let us write the relation between H and K_{II} as follows

$$aK_{II} + bH = c. \quad (16)$$

By (9), equation (16) can be written as follows

$$a \left\{ \frac{\frac{1}{2}rk_1^2 \sin^2 v + \frac{1}{2}rk_1^3 \sin^3 v + \frac{1}{2}rk_1 \cos v}{(k_1 \sin v + rk_1^2 \sin^2 v)^2} + \frac{b(1 + 2rk_1 \sin v)}{2r(1 + rk_1 \sin v)} \right\} = c.$$

The above equation can be expressed in a simpler form, we get

$$(2br^2k_1^4 - cr^4k_1^4) \sin^4 v + (3brk_1^3 + ar^2k_1^3 - cr^2k_1^3) \sin^3 v + (2ark_1^2 + bk_1^2 - crk_1^2) \sin^2 v + (ar^2k_1^2) \sin 2v - (ak_1^3r - ar^2k_1^4) \cos 2v \sin^2 v + 2ark_1 \cos v - (ar^2k_2^2 - 2rck_1) = 0.$$

Also, all the coefficients in the above(algebraic) expression must be zero. Therefore, we conclude that $k_1 = k_2 = 0$. Thus, the surface M is an open part of a circular cylinder. ■

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