
Tube Surfaces with Type-2 Bishop Frame of Weingarten Types in $E^3$

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Abstract

In this paper, we study tube surfaces with type-2 Bishop frame instead of Frenet frame in Euclidean 3-space $E^3$. Besides, we have discussed Weingarten and linear Weingarten conditions for tube surfaces with the Gaussian curvature $K$, the mean curvature $H$ and the second Gaussian curvature $K_{II}$.

Mathematics Subject Classification: 53A05

Keywords: Tube surfaces, Weingarten property, Type-2 Bishop frame, Mean and Gaussian curvatures, Second Gaussian curvature

1 Introduction

In analysis of surfaces, it is fairly common to determine some surfaces according to their curvatures. For example, it is important the existence of non-trivial functional relationship between the principal curvatures for surfaces in Minkowski 3-space. The resulting surfaces are called Weingarten surfaces or simply W-surfaces. In particular, a surface $M$ in Minkowski 3-space is called Weingarten surface if there is some (smooth) relation $U(\kappa_1, \kappa_2) = 0$ between its two principal curvatures $\kappa_1$ and $\kappa_2$ or equivalently if there exists a non-trivial functional relation $\Phi(K, H) = 0$ with respect to its Gaussian curvature $K$ and its mean curvature $H$. Functional relation $\Phi(K, H) = 0$ on the surface $M$ with parametrized $X(u, v)$ is equivalent to the vanishing of the corresponding Jacobian determinant, namely $\left| \frac{\partial(K, H)}{\partial(u, v)} \right| = 0$ [10]. Also, linear Weingarten surfaces are Weingarten surfaces satisfying a linear equation $aK + bH = c$, $(a, b, c) \neq (0, 0, 0)$, $a, b, c \in \mathbb{R}$ between Gaussian and mean...
curvatures. If \( b = 0 \), linear Weingarten surface \( M \) reduces to surface with constant Gaussian curvature. If \( a = 0 \), linear Weingarten surface \( M \) reduces to surface with constant mean curvature. In this sense, the linear Weingarten surfaces can be regarded as a natural generalization of surfaces with constant Gaussian curvature or with constant mean curvature. Several geometers have studied W-surfaces and LW-surfaces and obtained many interesting results [5, 7, 8, 9]. If the second fundamental form \( II \) of a surface \( M \) in a Minkowski 3-space is non-degenerate, then it is regarded as a new (pseudo-) Riemannian metric. Therefore, the Gaussian curvature \( K_{II} \) of non-degenerate second fundamental form \( II \) can be defined formally on the Riemannian or pseudo-Riemannian manifold \((M, II)\). We call the curvature \( K_{II} \) the second Gaussian curvature on \( M \).

If surface \( M \) in Minkowski 3-space satisfies \( \Phi(X, Y) = 0 \) and \( aX + bY = c \), then it is respectively said to be a \((X, Y)\)-Weingarten surface and \((X, Y)\)–linear Weingarten surface. Yüksel, Tuncer and Karacan studied tubular surfaces with Bishop frame of Weingarten types in \( E^3 \)[14]. Kızıltuğ and Yaylı investigate timelike tube surfaces with Darboux frame of Weingarten types in \( E^3_1 \)[18]. Also, Khalifa and Kızıltuğ studied parallel surfaces of Weingarten type in \( E^3_1 \)[12]. Kühnel [19] and Stamou [7] investigated ruled \((X,Y)\)-Weingarten surface in Euclidean 3-space. Also, Baikoussis and Koufogiorgos studied helicoidal \((H,K_{II})\)-Weingarten surfaces [4]. Dillen and Sodsiri gave a classification of ruled \((X,Y)\)-Weingarten surface in Minkowski 3-space [5]. Following Jacobi equation and linear equation with respect to the Gaussian curvature \( K \), the mean curvature \( H \) and the second Gaussian curvature \( K_{II} \) raise an interesting geometric question: Classifying all surfaces in Euclidean 3-space and Minkowski 3-space satisfying the condition

\[
\Phi(X, Y) = 0 \\
aX + bY = c
\]

where \( X, Y \in \{K, H, K_{II}\}, X \neq Y \).

In this paper, we would like to contribute the solution of the above question by studying for tube surfaces with type-2 Bishop frame of Weingarten types in \( E^3 \).

2 Preliminaries

Let \( \alpha : \mathbb{R} \rightarrow \mathbb{R}^3 \) be an arbitrary curve in \( \mathbb{R}^3 \). Recall that the curve \( \alpha \) is said to be unit speed, if \( \langle \alpha', \alpha'' \rangle = 1 \), where \( \langle \cdot, \cdot \rangle \) is standard inner product which given by \( \langle x, y \rangle = x_1y_1 + x_2y_2 + x_3y_3 \) for each \( x = (x_1, x_2, x_3), y = (y_1, y_2, y_3) \in \mathbb{R}^3 \). The norm of a vector \( x \in \mathbb{R}^3 \) is given by \( ||x|| = \sqrt{\langle x, x \rangle} \). Denote by \( \{T(s), N(s), B(s)\} \) the moving Frenet frame along the unit speed curve \( \alpha \).
Then, the Frenet formulas are given by

\[
\begin{bmatrix}
T' \\
N' \\
B'
\end{bmatrix} = \begin{bmatrix}
0 & \kappa & 0 \\
-\kappa & 0 & \tau \\
0 & -\tau & 0
\end{bmatrix} \begin{bmatrix}
T \\
N \\
B
\end{bmatrix}
\tag{1}
\]

\[\langle T, T \rangle = \langle N, N \rangle = \langle B, B \rangle = 1,\]
\[\langle T, N \rangle = \langle N, B \rangle = \langle B, T \rangle = 0.\]

Here \(T, N\) and \(B\) are respectively called the tangent, the principal normal, and the binormal vector field of the curves \(\alpha\). \(\kappa(s)\) and \(\tau(s)\) are called curvature and torsion of the curve \(\alpha\), respectively.

The Bishop frame or parallel transport frame is an alternative approach to define a moving frame that is well defined even when second derivative of the curve has vanished. We can simply parallel transport an orthonormal frame along a curve by parallel transporting each component of the frame.

Let \(\alpha = \alpha(s)\) be a unit speed regular curve in \(E^3\). The type-2 Bishop frame of the \(\alpha(s)\) is defined by [15]

\[
\begin{bmatrix}
N_1' \\
N_2' \\
B'
\end{bmatrix} = \begin{bmatrix}
0 & 0 & -k_1 \\
0 & 0 & -k_2 \\
k_1 & k_2 & 0
\end{bmatrix} \begin{bmatrix}
N_1 \\
N_2 \\
B
\end{bmatrix},
\tag{2}
\]

The relation matrix between Frenet-Serret and type-2 Bishop frames can be expressed

\[
\begin{bmatrix}
T \\
N \\
B
\end{bmatrix} = \begin{bmatrix}
\sin \theta(s) & -\cos \theta(s) & 0 \\
\cos \theta(s) & \sin \theta(s) & 0 \\
0 & 0 & 1
\end{bmatrix} \begin{bmatrix}
N_1 \\
N_2 \\
B
\end{bmatrix}.
\tag{3}
\]

Here, the type-2 Bishop curvatures are defined by

\[k_1(s) = -\tau \cos \theta(s), \quad k_2(s) = -\tau \sin \theta(s).\]

It can be also deduced as

\[\theta' = \kappa = \frac{\left(\frac{k_2}{k_1}\right)'}{1 + \left(\frac{k_2}{k_1}\right)^2}.\]

The frame \(\{N_1, N_2, B\}\) is properly oriented, and \(\tau\) and \(\theta(s) = \int_0^s \kappa(s) \, ds\) are polar coordinates for the curve \(\alpha = \alpha(s)\). We shall call the set \(\{N_1, N_2, B, k_1, k_2\}\) as type-2 Bishop invariants of the curve \(\alpha = \alpha(s)\).
We denote a surface \( M \) in \( E^3 \) by
\[
M(u, v) = (m_1(u, v), m_2(u, v), m_3(u, v))
\]
Let \( U \) be the standard unit normal vector field on \( M \) defined by
\[
U = \frac{M_u \times M_v}{\| M_u \times M_v \|}
\]
where, \( M_u = \frac{\partial M(u, v)}{\partial u} \) and \( M_v = \frac{\partial M(u, v)}{\partial v} \). Then, the first fundamental form \( I \) and the second fundamental form \( II \) of a surface \( M \) are respectively defined by
\[
I = Edu^2 + 2Fdudv + Gdv^2, \quad II = edu^2 + 2fdudv + gdv^2
\]
where
\[
E = \langle M_u, M_u \rangle, \quad F = \langle M_u, M_v \rangle, \quad G = \langle M_v, M_v \rangle \quad e = \langle M_{uu}, U \rangle, \quad f = \langle M_{uv}, U \rangle, \quad g = \langle M_{vv}, U \rangle.
\]
On the other hand, the Gaussian curvature \( K \) and the mean curvature \( H \) are respectively,
\[
K = \frac{eg - f^2}{EG - F^2}, \quad H = \frac{Eg - 2Ff + Ge}{2(EG - F^2)}
\]
If the second fundamental form is non-degenerate; \( eg - f^2 \neq 0 \), then one define formally the second Gaussian curvature \( K_{II} \) a similar one to Brioschi’s formula for the Gaussian curvature obtained on \( M \), replacing the components of the first fundamental form \( E, F, G \) by those of the second fundamental form \( e, f, g \) as \([4]\)
\[
K_{II} = \frac{1}{(eg - f^2)^2} \left[ \begin{array}{c}
-\frac{1}{2}e_{vv} + f_uv - \frac{1}{2}g_{uu} & \frac{1}{2}e_u & f_u - \frac{1}{2}e_v \\
f_v - \frac{1}{2}g_u & e & f \\
\frac{1}{2}g_v & f & g
\end{array} \right] - \left[ \begin{array}{c}
0 & \frac{1}{2}e_v & g_u \\
\frac{1}{2}e_v & e & f \\
\frac{1}{2}g_u & f & g
\end{array} \right].
\]

3 Tube Surfaces with Type-2 Bishop Frame of Weingarten Types in \( E^3 \)

Let \( \alpha : (a, b) \to E^3 \) be a smooth unit speed curve of finite length which is topologically imbedded in \( E^3 \). The total space \( N_\alpha \) of the normal bundle of
Tube surfaces with type-2 Bishop frame

$\alpha ((a, b))$ in $E^3$ is naturally diffeomorphic to the direct product $(a, b) \times E^2$ via the translation along $\alpha$ with respect to the induced normal connection. For a sufficiently small $r > 0$, the tube of radius $r$ about the curve $\alpha$ is the set:

$$T_{r(\alpha)} = \{ \exp_{\alpha(u)} w | w \in Z_{\alpha(u)}, \|w\| = r, a < u < b \}.$$

For a sufficiently small $r > 0$, the tube $T_{r(\alpha)}$ is a smooth surface in $E^3$. Let $T, N, B$ denote the Frenet frame of the unit speed curve $\alpha = \alpha (u)$. Then the position vector of $T_{r(\alpha)}$ can be expressed as

$$M = M (u, v) = \alpha (u) + r (N (u) \cos v + B (u) \sin v).$$  \hfill (6)

By taking $N_1$ and $N_2$ instead of $T$ and $N$ we can write tube surface with type-2 Bishop frame as follows

$$M = M (u, v) = \alpha (u) + r (N_2 (u) \cos v + B (u) \sin v).$$  \hfill (7)

Let $M$ be tube surface with type-2 Bishop frame in $E^3$ given in (7). So, from the derivative formulas of type-2 Bishop frame partial differentiation of $M$ with respect to $u$ and $v$ are as follows

$$M_u = (1 + rk_1 \sin v) N_1 + (rk_2 \sin v) N_2 - (rk_2 \cos v) B$$
$$M_v = -r \sin v N_2 + r \cos v B.$$

Therefore, we find the components of the first fundamental form of $M$ as

$$E = (1 + rk_1 \sin v)^2 + r^2 k_2^2, \quad F = -r^2 k_2, \quad G = r^2.$$

On the other hand, the unit surface normal vector field $Z$ is obtained by

$$Z = \frac{M_u \times M_v}{\|M_u \times M_v\|} = - \cos v N_2 - \sin v B.$$  \hfill (8)

The second order partial differentials of $M$ are found

$$M_{uu} = 
\begin{bmatrix}
  rk_1' \sin v & -rk_1 k_2 \cos v
\end{bmatrix} N_1
+ 
\begin{bmatrix}
  rk_2' \sin v & -rk_2^2 \cos v
\end{bmatrix} N_2
+ 
\begin{bmatrix}
  -k_1 - rk_1^2 \sin v & -rk_1^2 \sin v - rk_2 \cos v
\end{bmatrix} B,$$

$$M_{uv} = (rk_1 \cos v) N_1 + (rk_2 \cos v) N_2 + (rk_2 \sin v) B$$

$$M_{vv} = -r \cos v N_2 - r \sin v B.$$
From equation (8) and the last equations we find the second fundamental form coefficients as follows

\[ e = rk_2^2 + k_1 \sin v + rk_1^2 \sin^2 v, \quad f = -rk_2, \quad g = r \]

Thus, the Gaussian curvature \( K \), the mean curvature \( H \) and the second Gaussian curvature \( K_{II} \) are given by, respectively

\[ K = \frac{k_1 \sin v}{r (1 + rk_1 \sin v)}, \quad H = \frac{1 + 2rk_1 \sin v}{2r (1 + rk_1 \sin v)}, \quad K_{II} = \frac{1 \cdot \frac{1}{2} rk_2^2 \sin^2 v + \frac{1}{2} rk_3^2 \sin^3 v + \frac{1}{2} rk_4 \cos v + \frac{1}{2} r^2 k_1^2 \sin 2v - \cos 2v \sin (k_3 r - r^2 k_1 \sin v) - r^2 k_2^2}{(k_1 \sin v + rk_1^2 \sin^2 v)^2}. \]

**Theorem 1** Let \( M \) be tube surface in \( E^3 \) defined by (7). Then \( M \) is a \((K, H)\) Weingarten surface.

**Proof.** Let \( M \) be tube surface with type-2 Bishop frame in \( E^3 \). If we take derivative of \( K \) and \( H \) given by (9) with respect to \( u \) and \( v \) respectively, then we get

\[ K_u = \frac{rk_1 \sin v}{r^2 (1 + rk_1 \sin v)^2}, \quad H_u = \frac{2r^2 k_1 \sin v}{4r^2 (1 + rk_1 \sin v)^2}, \]

\[ K_v = \frac{rk_1 \cos v}{r^2 (1 + rk_1 \sin v)^2}, \quad H_v = \frac{2r^2 k_1 \cos v}{4r^2 (1 + rk_1 \sin v)^2}. \]

By using (10) and (11), \( M \) satisfies identically the Jacobi equation

\[ \Phi(K, H) = K_u H_v - K_v H_u = 0. \]

Therefore \( M \) is a Weingarten surface. ■

**Theorem 2** Let \( M \) be a tube surface in \( E^3 \) parametrized by (7) with non-degenerate second fundamental form. If \( M \) is a \((K_{II}, K)\) Weingarten surface, then \( k_1 = k_2 = 0 \). \( M \) is an open part of a circular cylinder.
Proof. Let \( M \) be a tube surface with type-2 Bishop frame in \( E^3 \) parametrized by (7). We obtained \((K_{II})_u\) and \((K_{II})_v\) by using Mathematica. But the values of this calculations are so long, then we omitted. We consider a tube surface with type-2 Bishop frame in \( E^3 \) satisfying the Jacobi equation
\[
\Phi(K_{II}, K) = (K_{II})_u K_v - (K_{II})_v K_u = 0 \quad (12)
\]
with respect to The second Gaussian curvature \( K_{II} \) and the mean curvature \( H \). Then by (10) equation (12) becomes
\[
\frac{1}{2} (k_1^2 k_2^2 \sin v + k_1 k_2^2 k_1^2 \cos v) = 0. \quad (14)
\]
Since this polynomial is equal to zero for every \( v \), all its coefficients must be zero. Therefore, we conclude that \( k_1 = k_2 = 0 \) or \( \kappa = 0 \).

Theorem 3 Let \( M \) be a tube surface in \( E^3 \) parametrized by (7) with non-degenerate second fundamental form. If \( M \) is a \((H, K_{II})\) Weingarten surface, then \( k_1 = k_2 = 0 \). \( M \) is an open part of a circular cylinder.

Proof. We suppose that a tube surface parametrized by (7) with non-degenerate second fundamental form in \( E^3 \) is \((H, K_{II})\) Weingarten surface. Then it satisfies the equation
\[
\Phi(H, K_{II}) = (K_{II})_v H_u - (K_{II})_u H_v = 0, \quad (13)
\]
which implies
\[
\frac{1}{4} (k_1^2 k_1^2 \sin v + k_1 k_2^2 k_1^2 \cos v) = 0. \quad (14)
\]
From (14) we can obtain \( k_1 = k_2 = 0 \) or \( \kappa = 0 \). Thus \( M \) is an open part of a circular cylinder.

Now, we consider the following definition to examine the linear Weingarten property of the tube surface with type-2 Bishop frame.

Definition 4 A surface \( M \) is said to be a linear Weingarten surface, if its Gaussian curvature \( K \) and mean curvature \( H \) satisfy the relation on \( M \)
\[
aK + bH = c \quad (15)
\]
where, \( a, b \) and \( c \) are real numbers (not all zero).

Theorem 5 Suppose that tube surface defined by (7) type-2 Bishop frame is a linear Weingarten surface in \( E^3 \) satisfying (15). Then \( k_1 = 0 \).
Proof. Consider the parametrization (7) with $K$ and $H$ given by (9) respectively, we rewrite (15) to get

$$\frac{a (k_1 \sin v)}{r (1 + rk_1 \sin v)} + \frac{b (1 + 2rk_1 \sin v)}{2r (1 + rk_1 \sin v)} = c.$$ 

The above equation can be expressed in a simpler form

$$(2ak_1 + 2brk_1 - 2crk_1) \sin v + 2rc - b = 0.$$ 

According to the definition of the linear independent of vectors, we obtain $k_1 = 0$. Thus, this completes proof. 

Theorem 6 A tube surface defined by (7) type-2 Bishop frame with non-degenerate second fundamental form is a linear Weingarten surface if the curvatures $H$ and $K_{II}$ of the surface are written in a linear form. Then $k_1 = k_2 = 0$ or $M$ is an open part of a circular cylinder.

Proof. Let us write the relation between $H$ and $K_{II}$ as follows

$$aK_{II} + bH = c. \quad (16)$$

By (9), equation (16) can be written as follows

$$a \left\{ \frac{\frac{1}{2}rk_1^2 \sin^2 v + \frac{1}{2}rk_1^3 \sin^3 v + \frac{1}{2}rk_1 \cos v}{(k_1 \sin v + rk_1^2 \sin^2 v)^2} + \frac{b (1 + 2rk_1 \sin v)}{2r (1 + rk_1 \sin v)} \right\} = c.$$ 

The above equation can be expressed in a simpler form, we get

$$(2brk_1^4 - crk_1^4) \sin^4 v + (3brk_1^3 + ar^2k_1^3 - crk_1^3) \sin^3 v + (2ark_1^2 + bk_1^2 - crk_1^2) \sin^2 v \\ (ar^2k_1^2) \sin 2v - (ak_1^2 - ar^2k_1^4) \cos 2v \sin^2 v + 2ark_1 \cos v - (ar^2k_1^2 - 2rck_1) = 0.$$ 

Also, all the coefficients in the above(algebraic) expression must be zero. Therefore, we conclude that $k_1 = k_2 = 0$. Thus, the surface $M$ is an open part of a circular cylinder. 

References


[18] S. Kiziltug and Y. Yayli, Timelike tubes surfaces with Darboux frame of Weingarten types in $E^3_1$, Annales Polonici Mathematici (Submitted).


Received: August, 2012