Quasi Superharmonic Functions in Brelot Spaces

M. G. Al-Akhrass

Mathematics Department
Faculty of Science, UOS
Sharjah, United Arab Emirates
malakhrass@sharjah.ac.ae

Abstract

The concept of quasi superharmonic function, in terms of Jensen measures, in Brelot spaces is introduced. A characterization of quasi superharmonic functions in Brelot spaces in terms of harmonic measures is given. We show that a lower semicontinuous regularization of a quasi superharmonic function is superharmonic function. Moreover, we show that a quasi superharmonic function in Brelot spaces equals to its lower semicontinuous regularization out side of a polar set.

Keywords: Brelot space, quasi superharmonic functions, harmonic measures, Jensen measures

1 Introduction

Let Ω be a Hausdorff connected, locally compact, non-compact space satisfying the second axiom of countability. Suppose that there is a system of harmonic functions on Ω satisfying the Brelot axioms. More precisely, the system of the harmonic functions satisfies the sheaf property, there is a base of open set, called regular, where Dirichlet problem is solvable and the limit of an increasing sequence of harmonic function is either harmonic or identically infinity. We assume that there is a potential > 0 on Ω . Moreover, we assume that this system satisfies the domination principle for superharmonic functions (Axiom D). That is, given a locally bounded potential which is harmonic off some closed set $A \subset \Omega$, and given a superharmonic function $u \in SH^+(\Omega)$ such that $u \geq p$ on A , then $u \geq p$ on Ω . We also assume further that constants are harmonic. A Euclidean connected open set $\Omega \subset \mathbb{R}^n$ for $n \geq 3$ with the class of all functions that satisfy Laplace equation on Ω is an example of Brelot space satisfying the above assumptions.

A function $u : \Omega \longrightarrow (-\infty, \infty]$ is defined to be superharmonic if it is not identically ∞ and if

- (a) u is lower semicontinuous on Ω , and
- (b) for each $x \in \Omega$ and each regular open set ω containing x , u satisfies the inequality

$$u(x) \geq \int u \, d\mu_x^\omega.$$

Where μ_x^ω is the harmonic measure at x with respect to ω . The set of all superharmonic (respectively nonnegative superharmonic) functions on Ω is denoted by $SH(\Omega)$ (respectively $SH^+(\Omega)$).

Now it is natural to ask what class of functions we get if either (a) or (b) in the above definition is relaxed or strengthened. Concerning this question, in [2] B. J. Cole and T. J. Ransford proved the following result:

Theorem 1.1. *Let Ω be an open subset of $\mathbb{R}^n$, $n \geq 2$. Let $u : \Omega \longrightarrow (-\infty, \infty]$ be a locally lower bounded function such that $\{x \in \Omega : u(x) > t\}$ is analytic for each $t \in \mathbb{R}$. Then $u(x) \geq \int u \, d\mu$ for each $x \in \Omega$ and every Jensen measure μ for x on Ω if and only if $\hat{u}$ is superharmonic and $\hat{u} = u$ except on a polar set.*

The authors in a later paper [3] proved implicitly that in above theorem, it is enough to consider the harmonic measures at x instead of all Jensen measures.

In this paper the above result of Cole and Ransford is extended to a general set up of axiomatic Brelot spaces. We kept the Cole and Ransford assumptions on the function u , that is: u is never $-\infty$, and u is locally lower bounded such that $\{x \in \Omega : u(x) > t\}$ is analytic for each $t \in \mathbb{R}$. However our starting point is more elementary and we derive the very important Cartan-Brelot convergence theorem as a consequence.

2 Preliminaries

Let E be a subset of Ω and u be any function ≥ 0 on E . The reduced function of u relative to E , R_u^E is defined as the following:

$$R_u^E(x) = \inf\{u(x) : u \in SH^+(\Omega), u \geq u \text{ on } E\}.$$

The lower semicontinuous regularization of R_u^E , denoted by $\hat{R}_u^E$, is called the Balayaged function of u relative to E . For a domain $\omega \subset \Omega$ such that $E \subset \omega$, the restriction of R_u^E to ω is defined as the following:

$$(R_u^E)_\omega = \inf\{u \in SH^+(\omega) : u \geq u \text{ on } E\}.$$

Some important properties of the reduced functions are summarized in the following proposition. For details, see [1].

Proposition 2.1. *Let E be a subset of Ω and u be a nonnegative superharmonic function in Ω . Then*

- (a) R_u^E is harmonic in $\Omega \setminus \overline{E}$, where $\overline{E}$ is the closure of E .
- (b) R_u^E is increasing in E and u , and subadditive in u .
- (c) $0 \leq \hat{R}_u^E \leq R_u^E \leq u$ everywhere, $R_u^E = u$ on E and $\hat{R}_u^E = u$ in E° , the interior of E .
- (d) $\hat{R}_u^E$ is the greatest harmonic minorent of u .
- (e) If ω is a regular open set then $\hat{R}_u^{\omega^c} = R_u^{\omega^c}$ every where and

$$R_u^{\omega^c}(x) = \begin{cases} \int u \, d\rho_x^\omega, & x \in \omega \\ u(x), & x \notin \omega. \end{cases}$$

- (f) Let K be a compact subset of Ω . If ω is a relatively compact regular open set containing K , then $(\hat{R}_u^E)_\omega$ tends to zero at every boundary point of ω .

A set E is said to be a polar set in Ω if there exists superharmonic function $u \geq 0$, called associated to E , such that $u = \infty$ in E . Clearly, any subset of polar set is polar and countable union of polar sets is polar. Polar sets can also be characterized using balayage functions. That is, a set E is a polar set if and only if there exists a superharmonic function $u > 0$ in Ω such that $\hat{R}_u^E(x) = 0$ for some point $x \in \Omega$. The term "quasi-everywhere means except for a polar set. For details, see [1].

A set E is called analytic set if it is a continuous image of a complete separable metric space. In this case E is polar if and only if every compact subset of E is polar. Every Borel set is analytic. Also, although analytic set is not necessary Borel, it is μ -measurable for every finite Borel measure μ on Ω . For details, see [1]. Finally, we recall that the lower semicontinuous regularization of a function u , denoted by $\hat{u}$, is defined at every point $x \in \Omega$ as the following

$$\hat{u}(x) = \liminf_{y \rightarrow x} u(y).$$

2.1 Jensen measures

Definition 2.1. *Let U be an open subset of Ω and $x \in U$. A Jensen measure for x (with respect to U) is a Borel probability Radon measure μ , supported by*

a compact subset of U , such that every superharmonic function u on U satisfies

$$u(x) \geq \int u \, d\mu.$$

The set of all Jensen measures for x (with respect to U) is denoted by $J_x(U)$.

It clear from the definition that if $x \in \Omega$ and U_1, U_2 are two open sets in Ω such that $x \in U_1 \subset U_2$ then $J_x(U_1) \subset J_x(U_2)$ because $SH(U_2) \subset SH(U_1)$.

Remark 2.1. If E is a polar set in Ω , $x \in \Omega$ and $\mu \in J_x(\Omega)$ then $\mu(E \setminus \{x\}) = 0$. To see, this let u be a nonnegative superharmonic function in Ω such that $u = \infty$ on $E \setminus \{x\}$ and $u(x) < \infty$. Then

$$\infty \cdot \mu(E \setminus \{x\}) = \int_{E \setminus \{x\}} u \, d\mu \leq \int u \, d\mu \leq u(x) < \infty.$$

For an open subset U of Ω and $x \in U$, let $H_x(U)$ be the set of all harmonic measures at x with respect to the relatively compact open subsets of U , i.e.

$$H_x(U) = \{\mu_x^V : V \text{ is a relatively compact open set such that } x \in V \subset \bar{V} \subset U\}.$$

It is clear that $H_x(U) \subset J_x(U)$.

3 Main Results

Now, we introduce a definition of quasi-superharmonic functions in Brelot spaces.

Definition 3.1. An extended real valued function u is said to be quasi-superharmonic on Ω if it satisfies the following two conditions:

- (i) u is locally bounded below and the set $\{x : u(x) > \alpha\}$ is analytic $\forall \alpha \in \mathbb{R}$.
- (ii) For each $x \in \Omega$ and each $\mu \in J_x(\Omega)$, u satisfies the inequality

$$u(x) \geq \int u \, d\mu. \tag{1}$$

It can be easily seen from the definition that superharmonic functions are quasi superharmonic functions, and the limit of an increasing sequence of quasi superharmonic function is quasi superharmonic.

The following result justifies the name quasi-superharmonic function.

Theorem 3.1. Let u be an extended real valued function, locally bounded below such that the set $\{x : u(x) > \alpha\}$ is an analytic set $\forall \alpha \in \mathbb{R}$. Then the following are equivalent:

(A) For each $x \in \Omega$, u satisfies the following inequality

$$u(x) \geq \int u \, d\mu \, \forall \mu \in H_x(\Omega). \quad (2)$$

(B) $\hat{u}$ is a superharmonic function and $u = \hat{u}$ quasi everywhere.

Proof. (A) $\Rightarrow$ (B) : Suppose that u satisfies inequality (2). By definition, $\hat{u}$ is lower semicontinuous. Also, $\hat{u}$ can never be $-\infty$ because u is locally bounded below. So, to show that $\hat{u}$ is a superharmonic function on Ω , we only need to show that for every x and a regular domain $\omega \ni x$, the inequality $\hat{u}(x) \geq \int \hat{u} \, d\mu_x^\omega$ holds. Let ω be a regular domain and $x \in \omega$. By inequality (2), we have $u(x) \geq \int u \, d\mu_x^\omega$. Since the function $y \mapsto \int u \, d\mu_y^\omega$ is a continuous function in ω (in fact harmonic), $\hat{u}(x) \geq \int u \, d\mu_x^\omega$. Hence $\hat{u}(x) \geq \int u \, d\mu_x^\omega \geq \int \hat{u} \, d\mu_x^\omega$ and thus $\hat{u}(x) \geq \int \hat{u} \, d\mu_x^\omega$. The first part of (B) is proved.

For the second part, let us first assume that u is bounded above. We must show that the set $\{x : u(x) > \hat{u}(x)\}$ is a polar set. Since this set can be written as a countable union of sets of the form $A = \{x : u(x) - \hat{u}(x) > a\}$, it is enough to prove that A is polar. The function $\hat{u}$ is Borel (actually lower semicontinuous) so it is analytic and the set $\{x : u(x) > a\}$ is analytic (by assumption). Thus the set A is analytic. So, to show that A is polar we only have to show that every compact set $K \subseteq A$ is polar.

Let K be a compact subset of A and let ω be a regular open set such that $K \subset \omega \subset \bar{\omega} \subset \Omega$. Since u is locally lower bounded, there exists a positive number C such that $u + C \geq 0$ in $\bar{\omega}$. So without loss of generality we may assume that $u \geq 0$ on ω (otherwise we consider $u + C$).

Because $\mu_x^{\omega-K} \in H_x(\Omega)$ for $x \in \omega - K$, inequality (2) implies that

$$u(x) \geq \int u \, d\mu_x^{\omega-K} \, \forall x \in \omega - K.$$

Let p be a positive continuous potential such that $p \leq a$ on K . Then $u > p + \hat{u}$ on K . Let

$$f = \begin{cases} p + \hat{u} & \text{on } K \\ 0 & \text{on } \partial\omega. \end{cases}$$

Since $u \geq f$ on $\partial(\omega - K)$, we have

$$u(x) \geq \int u \, d\mu_x^{\omega-K} \geq \int f \, d\mu_x^{\omega-K}, \, \forall x \in \omega - K.$$

Because the function $x \mapsto \int f \, d\mu_x^{\omega-K}$ is continuous (in fact harmonic) in $\omega - K$,

$$\hat{u}(x) \geq \int f \, d\mu_x^{\omega-K} = (\hat{R}_{\hat{u}+p}^K)_\omega(x), \, \forall x \in \omega - K,$$

and hence

$$(\hat{R}_u^K)_\omega \leq (\hat{R}_{u+p}^K)_\omega \leq \hat{u} \text{ in } \omega - K. \quad (3)$$

Axiom D implies that $(\hat{R}_u^K)_\omega$ is the greatest harmonic minorant of $\hat{u}$ in $\omega - K$, so inequalities (3) imply that

$$(\hat{R}_u^K)_\omega = (\hat{R}_{u+p}^K)_\omega = (\hat{R}_u^K)_\omega + (\hat{R}_p^K)_\omega \text{ in } \omega - K.$$

Hence $(\hat{R}_p^K)_\omega = 0$ in $\omega - K$. So K is a polar set in ω and thus a polar set in Ω . For the general case, let $u_n = \min\{u, n\}$. Then for each n , u_n is a bounded above quasi superharmonic function so by the above argument, $\min\{\hat{u}, n\} = \hat{u}_n = u_n$ outside a polar set, say E_n . Let $E = \bigcup_n E_n$. Then E is polar and $\forall n$ we have $\hat{u}_n(x) = u_n(x) \forall x \notin E$, which implies that $\lim_n \hat{u}_n(x) = \lim_n u_n(x) \forall x \notin E$. Hence $\hat{u}(x) = u(x) \forall x \notin E$.

(B) $\Rightarrow$ (A) : Suppose that u is an extended real valued function, locally bounded below such that the set $\{x : u(x) > \alpha\}$ is an analytic set $\forall \alpha \in \mathbb{R}$, and u satisfies (B). Let $x \in \Omega$ and μ be a harmonic measure at x . Since μ does not charge polar sets, $\int u d\mu = \int \hat{u} d\mu$. Furthermore, because $\hat{u}$ is a superharmonic function, $\int u d\mu = \int \hat{u} d\mu \leq \hat{u}(x)$ which is $\leq u(x)$, so $\int u d\mu \leq u(x)$. $\square$

Theorem 3.2. *Let u be an extended real valued function, locally bounded below such that the set $\{x : u(x) > \alpha\}$ is an analytic set $\forall \alpha \in \mathbb{R}$. Then the following are equivalent:*

1. u is quasi superharmonic.
2. For each $x \in \Omega$, u satisfies the following inequality

$$u(x) \geq \int u d\mu \forall \mu \in H_x(\Omega).$$

3. $\hat{u}$ is a superharmonic function and $u = \hat{u}$ quasi everywhere.

Proof. (1) $\Rightarrow$ (2): is obvious because for each $x \in \Omega$, $H_x(\Omega) \subset J_x(\Omega)$.

(2) $\Leftrightarrow$ (3) Theorem 3.1.

] (3) $\Rightarrow$ (1) : Let $x \in \Omega$ and $\mu \in J_x(\Omega)$. By (3), $\hat{u}$ is a superharmonic function and the set $E = \{y : \hat{u}(y) < u(y)\}$ is polar. By remark 2.1, we have $\mu(E \setminus \{x\}) = 0$. Let $\mu(\{x\}) = \alpha$, then

$$\hat{u}(x) \geq \int \hat{u} d\mu = \alpha \hat{u}(x) + \int_{\Omega \setminus \{x\}} \hat{u} d\mu.$$

Hence

$$\int_{\Omega \setminus \{x\}} \hat{u} \, d\mu \leq (1 - \alpha)\hat{u}(x). \quad (4)$$

Now,

$$\begin{aligned} \int u \, d\mu &= \alpha u(x) + \int_{\Omega \setminus \{x\}} u \, d\mu = \alpha u(x) + \int_{\Omega \setminus \{x\}} \hat{u} \, d\mu \\ &\leq \alpha u(x) + (1 - \alpha)\hat{u}(x) \\ &\leq \alpha u(x) + (1 - \alpha)u(x) \\ &= u(x). \end{aligned}$$

This proves (1). The proof is completed. □

We remark that the above theorem implies the well known Cartan-BreLOT convergence theorem.

Corollary 3.1. (*Cartan-BreLOT convergence theorem*) *If u_n is a decreasing sequence of locally uniformly lower bounded superharmonic functions with limit u , then $\hat{u}$ is superharmonic function and $\hat{u} = u$ out side of a polar set.*

References

- [1] BreLOT, M., Lectures on Potential Theory, Tata Institute, No 19, Bombay 1960, re-isside 1967.
- [2] Cole, B.J. and Ransford, T. J., Subharmonicity without upper semicontinuity, J. Funct. Anal. 147(1997), 420-442.
- [3] Cole, B.J. and Ransford, T. J., Jensen measures and harmonic measures, J. Reine Angew. Math. 541(2001), 29-53.

Received: August, 2012