Related Fixed Point Theorems for Three Set-Valued Mappings on Three Uniform Spaces

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Abstract

In this paper we prove some related fixed point theorems for three set valued mappings on three uniform spaces. Our results generalize the results of Jain and Fisher [3], Fisher and Turkoglu [1], Jain, Sahu and Fisher [2].

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1. Introduction

The following related fixed point theorem was proved by Jain, Sahu and Fisher [2].
Theorem 1.1[2]: Let \((X,d)\), \((Y,\rho)\) and \((Z,\sigma)\) be complete metric spaces and suppose \(T\) is a continuous mapping of \(X\) into \(Y\), \(S\) is a continuous mapping of \(Y\) into \(Z\) and \(R\) is a mapping of \(Z\) into \(X\), satisfying the inequalities
\[
d(RSTx, RSTx') \leq c \max \{d(x, x'), d(x, RSTx), d(x', RSTx'), \rho(Tx, Tx'), \sigma(STx, STx')\}
\]
\[
\rho(TRSy, TRSy') \leq c \max \{\rho(y, y'), \rho(y, TRSy), \rho(y', TRSy'), \sigma(Sy, Sy'), d(RSy, RSy')\}
\]
\[
\sigma(STRz, STRz') \leq c \max \{\sigma(z, z'), \sigma(z, STRz), \sigma(z', STRz'), d(Rz, Rz'), \rho(TRz, TRz')\}
\]
for all \(x,x'\) in \(X\), \(y,y'\) in \(Y\) and \(z,z'\) in \(Z\) where \(0 \leq c < 1\). Then \(RST\) has a unique fixed point \(u\) in \(X\), \(TRS\) has a unique fixed point \(v\) in \(Y\) and \(STR\) has a unique fixed point \(w\) in \(Z\). Further, \(Tu = v\), \(Sv = w\), \(Rw = u\).

The following theorem was proved by Jain and Fisher [3].

Theorem 1.2[3]: Let \((X,d_1)\), \((Y,d_2)\) and \((Z,d_3)\) be complete metric spaces. If \(F\) is a continuous mapping of \(X\) in \(B(Y)\), \(G\) is a continuous mapping of \(Y\) into \(B(Z)\) and \(H\) is a mapping of \(Z\) into \(B(X)\) satisfying the inequalities
\[
\delta_1 (HGFx, HGFx') \leq c \max \{d_1(x, x'), \delta_1(x, HGFx), \delta_1(x', HGFx')\}
\]
\[
\delta_2 (FHGY, FHGY') \leq c \max \{d_2(y, y'), \delta_2(y, FHGY), \delta_2(y', FHGY')\}
\]
\[
\delta_3 (GFHz, GFHz') \leq c \max \{d_3(z, z'), \delta_3(z, GFHz), \delta_3(z', GFHz')\}
\]
for all \(x, x'\) in \(X\), \(y, y'\) in \(Y\) and \(z, z'\) in \(Z\), where \(0 \leq c < 1\), then \(HGF\) has a unique fixed point \(u\) in \(X\), \(FHG\) has a unique fixed point \(v\) in \(Y\) and \(GFH\) has a unique fixed point \(w\) in \(Z\). Further, \(Fu = \{v\}\), \(Gv = \{w\}\), \(Hw = \{u\}\).

The following theorem was proved by Turkoglu and Fisher [1].

Theorem 1.3[1]: Let \((X,U_1)\) and \((Y,U_2)\) be complete Hausdorff uniform spaces defined by \(\{d_1^i, i \in I\} = P_1^\ast\), \(\{d_2^i, i \in I\} = P_2^\ast\) and \((2^X, U_1^\ast)\), \((2^Y, U_2^\ast)\) hyperspaces, let \(F\) is a mapping of \(X\) into \(2^Y\) and \(G\) is a mapping of \(Y\) into \(2^X\) satisfying the inequalities
\[
\delta_1^i(GFx, GFx') \leq c_i \max \{d_1^i(x, x'), \delta_1^i(x, GFx), \delta_1^i(x', GFx'), \delta_1^i(Fx, Fx')\}
\]
\[
\delta_2^i(FGy, Fgy') \leq c_i \max \{d_2^i(y, y'), \delta_2^i(y, FGy), \delta_2^i(y', FGy'), \delta_2^i(Gy, Gy')\}
\]
for all \(x, x'\) in \(X\) and \(y, y'\) in \(Y\), \(i \in I\) where \(0 \leq c_i < 1\). If \(F\) is continuous, then \(GF\) has a unique fixed point \(z\) in \(X\) and \(FG\) has a unique fixed point \(w\) in \(Y\). Further, \(Fz = \{w\}\) and \(Gw = \{z\}\).

The aim of the present work is to generalize theorem 1.3 by considering three pairs of set valued mappings on three complete Hausdorff uniform spaces. Before proceeding to our main result we recall the following from Turkoglu and Fisher [1] with necessary modification.

Let \((X, U_1)\), \((Y, U_2)\) and \((Z, U_3)\) are uniform spaces. Families \(\{d_i^1: i \in I\}\) being indexing set, \(\{d_i^2: i \in I\}\) and \(\{d_i^3: i \in I\}\) of pseudometrics on \(X\), \(Y\) and \(Z\) respectively, are called associated families for uniformities \(U_1\), \(U_2\) and \(U_3\) respectively, if families

\[
\beta_1 = \{V_1(i, r): i \in I, r > 0\} \\
\beta_2 = \{V_2(i, r): i \in I, r > 0\} \\
\beta_3 = \{V_3(i, r): i \in I, r > 0\}
\]

where

\[
V_1(i, r) = \{(x, x'): x, x' \in X, d_i^1(x, x') < r \} \\
V_2(i, r) = \{(y, y'): y, y' \in Y, d_i^2(y, y') < r \} \\
V_3(i, r) = \{(z, z'): z, z' \in Z, d_i^3(z, z') < r \}
\]

(1.2)

are sub bases for uniformities \(U_1\), \(U_2\) and \(U_3\) respectively, we may assume that \(\beta_1\), \(\beta_2\), \(\beta_3\) themselves are a base by adjoining finite intersections of members of \(\beta_1\), \(\beta_2\) and \(\beta_3\) if necessary. The corresponding families of pseudometrics are called an augmented associated families for \(U_1\), \(U_2\) and \(U_3\). An associated family for \(U_1\), \(U_2\) and \(U_3\) will be denoted by \(\mathcal{D}_1\), \(\mathcal{D}_2\) and \(\mathcal{D}_3\) respectively.

Let \(A\), \(B\) and \(C\) is nonempty subsets of a uniform space \(X\), \(Y\), and \(Z\) respectively. Define

\[
P_i^*(A) = \sup \{d_i^1(x, x'): x, x' \in A, i \in I\} \\
P_i^*(B) = \sup \{d_i^2(y, y'): y, y' \in B, i \in I\} \\
P_i^*(C) = \sup \{d_i^3(z, z'): z, z' \in C, i \in I\}
\]

(1.3)

where

\[
\{d_i^1(x, x'): x, x' \in A, i \in I\} = P_1^* \\
\{d_i^2(y, y'): y, y' \in B, i \in I\} = P_2^* \\
\{d_i^3(z, z'): z, z' \in C, i \in I\} = P_3^*.
\]
Then $P_1^*(A)$, and $P_2^*(B)$ and $P_3^*(C)$ are called an augmented diameter of $A$, $B$ and $C$. Further $A$, $B$ and $C$ are said to be $P_1^*(A) < \infty$, $P_2^*(B) < \infty$, $P_3^*(C) < \infty$. Let

$$2^X = \{A: A is a nonempty $P_1^*$ bounded subset of $X\}$$

$$2^Y = \{B: B is a nonempty $P_2^*$ bounded subset of $Y\}$$

$$2^Z = \{C: C is a nonempty $P_3^*$ bounded subset of $Z\}$$ \tag{1.4}$$

For each $i \in I$ and $A_1, A_2 \in 2^X$, $B_1, B_2 \in 2^Y$ and $C_1, C_2 \in 2^Z$, define

$$\delta_1^i(A_1, A_2) = \sup \{d_1^i(x, x'): x \in A_1, x' \in A_2\}$$

$$\delta_2^i(B_1, B_2) = \sup \{d_2^i(y, y'): y \in B_1, y' \in B_2\}$$

$$\delta_3^i(C_1, C_2) = \sup \{d_3^i(z, z'): z \in C_1, z' \in C_2\}$$ \tag{1.5}$$

Let $(X, U_1)$, $(X, U_2)$ and $(X, U_3)$ be uniform spaces and let $U_1 \subseteq U_1$, $U_2 \subseteq U_2$ and $U_3 \subseteq U_3$ be arbitrary entourages. For each $A \in 2^X$, $B \in 2^Y$ and $C \in 2^Z$, define

$$U_1[A] = \{x' \in X: (x, x') \in U_1 \text{ for some } x \in A\}$$

$$U_2[B] = \{y' \in Y: (y, y') \in U_2 \text{ for some } y \in B\}$$

$$U_3[C] = \{z' \in Z: (z, z') \in U_3 \text{ for some } z \in C\}$$ \tag{1.6}$$

The uniformities $2^{U_1}$ on $2^X$, $2^{U_2}$ on $2^Y$ and $2^{U_3}$ on $2^Z$ are defined by bases

$$2^{U_1} = \{\mathcal{U}_1: U_1 \subseteq U_1\},$$

$$2^{U_2} = \{\mathcal{U}_2: U_2 \subseteq U_2\},$$

$$2^{U_3} = \{\mathcal{U}_3: U_3 \subseteq U_3\},$$ \tag{1.7}$$

where

$$\mathcal{U}_1 = \{(A_1, A_2) \in 2^X \times 2^X: A_1 \times A_2 \subseteq U_1\} \cup \Delta,$$

$$\mathcal{U}_2 = \{(B_1, B_2) \in 2^Y \times 2^Y: B_1 \times B_2 \subseteq U_2\} \cup \Delta,$$

$$\mathcal{U}_3 = \{(C_1, C_2) \in 2^Z \times 2^Z: C_1 \times C_2 \subseteq U_3\} \cup \Delta$$ \tag{1.8}$$

where $\Delta$ denotes the diagonal of $X \times X$, $Y \times Y$ and $Z \times Z$. The augmented associated families $P_1^*$, $P_2^*$ and $P_3^*$ also induce uniformities $U_1$ on $2^X$, $U_2$ on $2^Y$ and $U_3$ on $2^Z$ defined by bases

$$\beta_1^* = \{V_1^*(i, r): i \in I, r > 0\}$$

$$\beta_2^* = \{V_2^*(i, r): i \in I, r > 0\}$$

$$\beta_3^* = \{V_3^*(i, r): i \in I, r > 0\}$$ \tag{1.9}$$

Where

$$V_1^*(i, r) = \{(A_1, A_2): A_1, A_2 \in 2^X: \delta_1^i(A_1, A_2) < r\} \cup \Delta$$

$$V_2^*(i, r) = \{(B_1, B_2): B_1, B_2 \in 2^Y: \delta_2^i(B_1, B_2) < r\} \cup \Delta$$

$$V_3^*(i, r) = \{(C_1, C_2): C_1, C_2 \in 2^Z: \delta_3^i(C_1, C_2) < r\} \cup \Delta$$ \tag{1.10}$$

Uniformities $2^{U_1}$ and $U_1^*$ on $2^X$ are uniformly isomorphic, uniformities $2^{U_2}$ and $U_2^*$ on $2^Y$ are uniformly isomorphic and uniformities $2^{U_3}$ and $U_3^*$ on $2^Z$ are uniformly isomorphic. The space $(2^X, U_1^*)$ is thus a uniform space called the
hyperspaces of \((X, U_1)\). The \((2^Y, U_2^*)\) is also a uniform space called the hyperspace of \((Y, U_2)\). The \((2^Z, U_3^*)\) is also a uniform space called the hyperspace of \((Z, U_3)\).

Now, let \(\{A_n : n = 1, 2, \ldots\}\) be a sequence of nonempty subsets of uniform space \((X, U)\). We say that sequence \(\{A_n\}\) converges to subset \(A\) of \(X\) if
1. each point \(a\) in \(A\) is the limit of a convergent sequence \(\{a_n\}\), where \(a_n\) is in \(A_n\) for \(n = 1, 2, \ldots\).
2. for arbitrary \(\varepsilon > 0\), there exists an integer \(N\) such that \(A_n \subseteq A_\varepsilon\) for \(n > N\), where
   \[
   A_\varepsilon = \bigcup_{x \in A} U(x) = \{y \in X : d(x, y) < \varepsilon \text{ for some } x \in A, i \in I\} \tag{1.11}
   \]

A is then said to be a limit of the sequence \(\{A_n\}\).

It follows easily from the definition that if \(A\) is the limit of a sequence \(\{A_n\}\), then \(A\) is closed.

**Lemma 1.4.** If \(\{A_n\}\) and \(\{B_n\}\) are sequences of bounded, nonempty subsets of a complete uniform space \((X, U)\) which converge to the bounded subsets \(A\) and \(B\) respectively, then sequence \(\{\delta(A_n, B_n)\}\) converges to \(\delta(A, B)\).

### 2. Main Result

We prove the following theorem.

**Theorem 2.1.** Let \((X, U_1)\), \((Y, U_2)\) and \((Z, U_3)\) be complete Hausdorff uniform spaces defined by \(\{d_{1i}, i \in I\} = P_1^*\), \(\{d_{2i}, i \in I\} = P_2^*\) and \(\{d_{3i}, i \in I\} = P_3^*\), \((2^X, U_1^*)\), \((2^Y, U_2^*)\) and \((2^Z, U_3^*)\) hyperspaces, let \(F: X \to 2^Y\), \(G: Y \to 2^Z\) and \(H: Z \to 2^X\) satisfying the inequalities

\[
\delta_1(HGFx, HGFx') \leq c_1 \max \{d_{1i}(x, x'), \delta_1(x, HGFx), \delta_1(x', HGFx'), \\
\delta_2(Fx, Fx'), \delta_3(GFx, GFx')\} \tag{2.1}
\]

\[
\delta_2(FHGy, FHGy') \leq c_2 \max \{d_{2i}(y, y'), \delta_2(y, FHGy), \delta_2(y', FHGy'), \\
\delta_3(Gy, Gy'), \delta_4(HGy, HGy')\} \tag{2.2}
\]

\[
\delta_3(GFHz, GFHz') \leq c_3 \max \{d_{3i}(z, z'), \delta_3(z, GFHz), \delta_3(z', GFHz'), \\
\delta_4(Hz, Hz'), \delta_5(FHz, FHHz')\} \tag{2.3}
\]
for all \( i \in I \) and \( x, x' \in X, y, y' \in Y \) and \( z, z' \in Z \) where \( 0 \leq c_i < 1 \), If \( F \) and \( G \) are continuous, then \( HGF \) has a unique fixed point \( u \) in \( X \), \( FHG \) has a unique fixed point \( v \) in \( Y \) and \( GFH \) has a unique fixed point \( w \) in \( Z \). Further \( Fu = \{ v \} \), \( Gv = \{ w \} \) and \( Hw = \{ u \} \).

**Proof:** Let \( x_1 \) be an arbitrary point in \( X \). Define sequences \( \{ x_n \} \) in \( X \), \( \{ y_n \} \) in \( Y \) and \( \{ z_n \} \) in \( Z \), respectively, as follows. Choose a point \( y_1 \) in \( Fx_1 \), then a point \( z_1 \) in \( Gy_1 \) and then a point \( x_2 \) in \( Hz_1 \). In general, having chosen \( x_n \) in \( X \), \( y_n \) in \( Y \) and \( z_n \) in \( Z \), choose a point \( x_{n+1} \) in \( Hz_n \), then a point \( y_{n+1} \) in \( Fx_{n+1} \) and then a point \( z_{n+1} \) in \( Gyz_{n+1} \) for \( n = 1, 2, \ldots \).

Let \( U_1 \in U_1 \) be an arbitrary entourage. Since \( \beta_1 \) is a base for \( U_1 \), there exists \( V_1(i, r) \in \beta_1 \) such that \( V_1(i, r) \subseteq U_1 \), we have by using inequality (2.1)

\[
\begin{align*}
d_1^1(x_{n+1}, x_{n+2}) & \leq \delta_2(HGFx_n, HGFx_{n+1}) \\
& \leq c_1 \max\{d_1^1(x_n, x_{n+1}), \delta_1(x_n, HGFx_n), \delta_1(x_{n+1}, HGFx_{n+1}), \\
& \quad \delta_2(Fx_n, Fx_{n+1}), \delta_3(GFx_n, GFx_{n+1})\} \\
& \leq c_1 \max\{\delta_1(HGFx_{n-1}, HGFx_n), \delta_1(HGFx_n, HGFx_{n+1}), \\
& \quad \delta_2(FHGy_{n-1}, FHGy_n), \delta_3(GFHz_{n-1}, GFHz_n)\} \\
& \leq c_1 \max\{\delta_1(HGFx_{n-1}, HGFx_n), \delta_1(HGFx_n, HGFx_{n+1}), \\
& \quad \delta_3(GFHz_{n-1}, GFHz_n)\} \tag{2.4}
\end{align*}
\]

Similarly let \( U_2 \in U_2 \) be an arbitrary entourage, since \( \beta_2 \) is a base for \( U_2 \) there exists \( V_2(i, r) \in \beta_2 \) such that \( V_2(i, r) \subseteq U_2 \), by using inequality (2.2), we have

\[
\begin{align*}
d_1^2(y_{n+1}, y_{n+2}) & \leq \delta_2(FHGy_n, FHGy_{n+1}) \\
& \leq c_1 \max\{d_1^2(y_n, y_{n+1}), \delta_2(y_n, FHGy_n), \delta_2(y_{n+1}, FHGy_{n+1}), \\
& \quad \delta_3(Gy_n, Gy_{n+1}), \delta_1(HGy_n, HGy_{n+1})\} \\
& \leq c_1 \max\{\delta_2(FHGy_{n-1}, FHGy_n), \delta_2(FHGy_n, FHGy_{n+1}), \\
& \quad \delta_3(GFHz_{n-1}, GFHz_n), \delta_1(HGFx_n, HGFx_{n+1})\} \\
& \leq c_1 \max\{\delta_2(FHGy_{n-1}, FHGy_n), \delta_3(GFHz_{n-1}, GFHz_n), \\
& \quad \delta_1(HGFx_n, HGFx_{n+1})\} \tag{2.5}
\end{align*}
\]
Again, let $U_3 \in U_3$ be an arbitrary entourage. Since $\beta_3$ is a base for $U_3$. There exists $V_3(i, r) \in \beta_3$ such that $V_3(i, r) \subseteq U_3$, on using inequality (2.4). Further, on using inequality (2.3), we have

\[
d_3^1(z_{n+1}, z_{n+2}) \leq \delta_3(GFH_z, GFHz_n, GFHz_{n+1})
\]
\[
\leq c_1 \max \{ \delta_3(z_n, GFHz_n), \delta_3(z_{n+1}, GFHz_{n+1}), \delta_1(Hz_n, Hz_{n+1}), \delta_2(FHz_n, FHz_{n+1}) \}
\]
\[
\leq c_1 \max \{ \delta_3(GFHz_{n-1}, GFHz_n), \delta_3(GFHz_{n}, GFHz_{n+1}), \delta_4(HGFx_n, HGFx_{n+1}), \delta_2(FHGy_n, FHGy_{n+1}) \}
\]
\[
\leq c_1 \max \{ \delta_3(GFHz_{n-1}, GFHz_n), \delta_4(HGFx_n, HGFx_{n+1}), \delta_2(FHGy_n, FHGy_{n+1}), \delta_3(GFHz, GFHz_2) \}
\]

On using inequalities (2.4) and (2.5).

It now follows easily by induction on using inequalities (2.4), (2.5) and (2.6) that

\[
d_1^1(x_{n+1}, x_{n+2}) \leq c_1^{n-1} \max \{ \delta_4(HGFz_1, HGFz_2), \delta_2(FHGy_1, FHGy_2), \delta_3(GFHz_1, GFHz_2) \}
\]
\[
d_2^1(y_{n+1}, y_{n+2}) \leq c_1^{n-1} \max \{ \delta_4(HGFx_1, HGFx_2), \delta_2(FHGy_1, FHGy_2), \delta_3(GFHz_1, GFHz_2) \}
\]
\[
d_3^1(z_{n+1}, z_{n+2}) \leq c_1^{n-1} \max \{ \delta_4(HGFx_1, HGFx_2), \delta_2(FHGy_1, FHGy_2), \delta_3(GFHz_1, GFHz_2) \}
\]

Then for $m = 1, 2, \ldots$ and arbitrary $\varepsilon > 0$, we have

\[
d_1^1(x_{n+1}, x_{n+m+1}) \leq \delta_4(HGFx_n, HGFx_{n+m})
\]
\[
\leq \delta_4(HGFx_n, HGFx_{n+1}) + \ldots + \delta_4(HGFx_{n+m-1}, HGFx_{n+m})
\]
\[
\leq (c_1^{n-1} + c_1^n + \ldots + c_1^{n+m-2}) \times \max \{ \delta_4(HGFx_1, HGFx_2), \delta_2(FHGy_1, FHGy_2), \delta_3(GFHz_1, GFHz_2) \}
\]
\[
< \varepsilon
\]

(2.7)

for $n$ greater than some $N$. Since $c_i < 1$, it follows that there exists $p$ such that $d_1^1(x_n, x_m) < r$ and hence $(x_n, x_m) \in U_1$ for all $n, m \geq p$. Therefore, sequence $\{x_n\}$ is Cauchy sequence in the $d_1^1$ uniformity on $X$. 
Let \( S_p = \{x_n; n \geq p\} \) for all positive integers \( p \) and let \( B_i \) be the filter basis \( \{S_p = 1, 2, \ldots\} \). Then, since \( \{x_n\} \) is a \( d^1_i \)-Cauchy sequence for each \( i \in I \), it is easy to see that the filter basis \( B_i \) is a Cauchy filter in the uniform space \((X, U_i)\). To see this, we first note that family \( \{V_i(i, r) \mid i \in I, r > 0\} \) is a base for \( U_1 \) as \( P^1_i = \{d^1_i : i \in I\} \).

Now, since \( \{x_n\} \) is a \( d^1_i \)-Cauchy sequence in \( X \), there exists a positive integer \( p \) such that \( d^1_i(x_m, x_n) < r \) for \( m \geq p, n \geq p \). This implies that \( S_p \times S_p \subset V(i, r) \). Thus, given any \( U_1 \in U_1 \), we can find an \( S_p \in B_i \) such that \( S_p \times S_p \subset U_1 \). Hence \( B_i \) is a Cauchy filter in \((X, U_i)\). Since \((X, U_i)\) is a complete hausdorff space, the Cauchy filter \( B_i = \{S_p\} \) converges to a unique point \( u \in X \), similarly, the Cauchy filter \( B_2 = \{S_k\} \) converges to a unique point \( v \in Y \) and the Cauchy filter \( B_3 = \{S_q\} \) converges to unique point \( w \in Z \).

Further, inequality (2.7) gives

\[
\delta_1(u, HGS_p) + \delta_1(S_{m+1}, HGS_p) \\
= d^1(u, S_{m+1}) + \delta_1(HGFS_m, HGS_p) \\
\leq d^1(u, S_{m+1}) + \varepsilon
\]

for \( m, n \geq p \), letting \( m \) tend to infinity, it follows that

\[
\delta_1(u, HGS_p) \leq \varepsilon
\]

for \( n \geq p \) and so

\[
\lim_{n \to \infty} HGS_p = \{u\} = \lim_{n \to \infty} HGS_k
\]

Since \( \varepsilon \) is arbitrary, similarly,

\[
\lim_{n \to \infty} FHGS_k = \{v\} = \lim_{n \to \infty} FHS_q
\]
\[
\lim_{n \to \infty} GFHS_q = \{w\} = \lim_{n \to \infty} GFS_p
\]

Using the continuity of \( F \) and \( G \), we obtain

\[
\lim_{n \to \infty} S_k = \lim_{n \to \infty} FS_p = Fu = \{v\}
\]
\[
\lim_{n \to \infty} S_q = \lim_{n \to \infty} GS_k = GV = \{w\}
\]
and then we see that $GFS_p = \{w\}$ \hfill (2.13)

Now, let $W \in U_1$ be an arbitrary entourage. Since $\beta_1$ is a base for $U_1$. There exists $V_1(j, t) \in \beta_1$ such that $V_1(j, t) \subseteq W$. We now have

$$
\delta_1(HGFS_p, HGFu) \leq c_i \max \{d_1(S_p, u), \delta_1(S_p, HGFS_p), \delta_1(u, HGFu), \\
\delta_2(Fu, FS_p), \delta_2(GFu, GFS_p)\}
$$

Since $F$ and $G$ are continuous, it follows on letting $p$ tend to infinity, we have using (2.8)

$$
\delta_1(HGFu, u) \leq c_i \delta_1(u, HGFu)
$$

Since $c_i < 1$ we have $\delta_1(u, HGFu) = 0 < t$. Hence, $(u, HGFu) \in V_1(j, t) \subseteq W$. Again, since $W$ is arbitrary and $X$ is Hausdorff, we must have $HG Fu = Hw = \{u\}$, proving that $u$ is a fixed point of $HG$. Further, we now have $HG Fu = Hw = \{u\}$.

Then using equation (2.13) and then

$$
FHGv = FHGFu = Fu = \{v\} \\
GFHw = GFHGv = Gv = \{w\}
$$

Hence, $v$ and $w$ are fixed points of $FHG$ and $GFH$ respectively, further, we see that $HGv = \{u\}$, $FHw = \{v\}$.

To prove the uniqueness of $u$, suppose that $HG$ has a second fixed point $u'$. Then, using inequality (2.1), we have

$$
\delta_1(u', HGFu') \leq \delta_1(HG Fu', HG Fu') \\
\leq c_i \max \{d_1(u', u'), \delta_1(u', HG Fu'), \delta_2(Fu', Fu'), \delta_3(G Fu', G Fu')\} \\
\leq c_i \max \{\delta_2(Fu', Fu'), \delta_3(G Fu', G Fu')\}
$$

Next, using inequality (2.2), we have

$$
\delta_2(Fu', Fu') \leq \delta_2(FHGFu', FHGFu') \\
\leq c_i \max \{d_2(Fu', Fu'), \delta_2(Fu', FHGFu'), \delta_3(G Fu', G Fu'), \delta_1(HG Fu', HG Fu')\}
$$
\[ \leq c_i \max \{ \delta_3'(GFu', GFu'), \delta_1'(HGFu', HGFu') \} \quad (2.15) \]

And using inequality (2.3), we have

\[
\delta_3(GFu', GFu') \leq \delta_3(GFHFu', GFHFu') \\
\leq c_i \max \{ d_3'(GFu', GFu'), \delta_3(GFu', GFHFu'), \\
\delta_1'(HGFu', HGFu'), \delta_2'(FHGFu', FHGFu') \} \\
\leq c_i \max \{ \delta_1'(HGFu', HGFu'), \delta_2'(FHGFu', FHGFu') \} \quad (2.16)
\]

It now follows easily from inequalities (2.15) and (2.16) that

\[
\delta_2'(FHGFu', FHGFu') \leq c_i \delta_1'(HGFu', HGFu') \quad (2.17)
\]

and then

\[
\delta_3(GFu', GFu') \leq c_i \delta_3(HGFu', HGFu') \quad (2.18)
\]

Using inequalities (2.14), (2.17) and (2.18), we now have

\[
\delta_1'(u', HGFu') \leq \delta_1'(HGFu', HGFu') \\
\leq c_i^2 \delta_1'(HGFu', HGFu') \\
\leq c_i \delta_1'(HGFu', HGFu') \quad (2.19)
\]

and so \( HGFu' \) is a singleton and \( HGFu' = \{ u' \} \).
Since \( c_i < 1 \), it then follows from inequality (2.18) that \( GFu' \) is a singleton and from inequality (2.17) that \( Fu' \) is a singleton. Using inequality (2.1) again, we have

\[
d_1'(u, u') = \delta_1(HGFu, HGFu') \\
\leq c_i \max \{ d_1'(u, u'), d_1'(u, u'), \delta_1'(Fu, Fu'), \\
\delta_3'(GFu, GFu') \} \\
= c_i \max \{ d_1'(u, u'), \delta_3'(FU, FU'), \delta_3'(GFu, GFu') \} \\
= c_i d_1'(u, u') \quad (2.19)
\]

Since \( c_i < 1 \), it follows that \( u = u' \) and the uniqueness of \( u \) follows. Similarly, it can be proved that \( v \) is the unique fixed point of \( FHG \) and \( w \) is the unique fixed point of \( GFH \). We finally prove that we also have \( Hw = \{ w \} \). To do this note that
$Hw = H(GFHw) = HGF(Hw)$ and so $Hw$ is a fixed point of $HGF$. Since $u$ is the unique fixed point of $HGF$, it follows that $Hw = \{u\}$.

This completes the proof of the theorem.

**COROLLARY 2.2.** Let $(X, U_1)$, $(Y, U_2)$ and $(Z, U_3)$ be complete Hausdorff uniform spaces. If $T$ is a continuous mapping of $X$ into $Y$, $S$ is a continuous mapping of $Y$ into $Z$ and $R$ is a mapping of $Z$ into $X$ satisfying the inequalities

$$d_1^i(RST, RST') \leq c_i \max \{d_1^i(x, x'), d_1^i(x, RSTx), d_1^i(x', RST'x'), d_2^i(Tx, Tx'), d_3^i(STx, ST'x')\}$$

$$d_2^i(TRSy, TRSy') \leq c_i \max \{d_1^i(y, y'), d_1^i(y, TRSy), d_1^i(y', TRSy'), d_2^i(Sy, Sy'), d_3^i(RSy, RSy')\}$$

$$d_3^i(TRSz, TRSz') \leq c_i \max \{d_2^i(z, z'), d_2^i(z, TRSz), d_2^i(z', TRSz'), d_1^i(Rz, Rz'), d_2^i(TRz, TRz')\}$$

for all $x, x' \in X$ and $y, y' \in Y$ and $z, z' \in Z$, $i \in I$ where $0 \leq c_i < 1$, then $RST$ has a unique fixed point $u$ in $X$, $TRS$ has a unique fixed point $v$ in $Y$ and $STR$ has a unique fixed point $w$ in $Z$. Further, $Tu = v$, $Sv = w$ and $Rw = u$.

**Theorem 2.3.** Let $(X, U_1)$, $(Y, U_2)$ and $(Z, U_3)$ be compact uniform spaces defined by $\{d_1^i : i \in I\} = P_1^*$, $\{d_2^i : i \in I\} = P_2^*$, $\{d_3^i : i \in I\} = P_3^*$ and $(2^X, U_1^*)$, $(2^Y, U_2^*)$ and $(2^Z, U_3^*)$ hyperspaces. If $F$ is a continuous mapping of $X$ into $2^Y$, $G$ is continuous mapping of $Y$ into $2^Z$ and $H$ is a continuous mapping of $Z$ into $2^X$ satisfying

$$\delta_1^i(HGFx, HGF'x) < \max \{d_1^i(x, x'), \delta_1^i(x, HGFx), \delta_1^i(x', HGF'x), \delta_2^i(Fx, F'x), \delta_3^i(GFx, GF'x)\}$$

$$\delta_2^i(FHGy, FHG'y) < \max \{d_1^i(y, y'), \delta_1^i(y, FHGy), \delta_1^i(y', FHG'y), \delta_2^i(Gy, G'y), \delta_3^i(HGy, HG'y)\}$$

$$\delta_3^i(GFHz, GFH'z) < \max \{d_2^i(z, z'), \delta_2^i(z, GFHz), \delta_2^i(z', GFH'z), \delta_1^i(Hz, H'z), \delta_2^i(FHz, F'H'z)\}$$

for all distinct $x, x'$ in $X$, all distinct $y, y'$ in $Y$, and all distinct $z, z'$ in $Z$. Then $HGF$ has a unique fixed point $u$ in $X$, $FHG$ has a unique fixed point $v$ in $Y$, and $GFH$ has a unique fixed point $w$ in $Z$. Further $Fu = \{v\}$, $Gv = \{w\}$ and $Hw = \{u\}$.

**Proof:** In a similar process as in theorem [2.1] one can easily prove this theorem.
**COROLLARY 2.4.** Let \((X, U_1), (Y, U_2)\) and \((Z, U_3)\) be compact Hausdorff uniform spaces. If \(T\) is a continuous mapping of \(X\) into \(Y\), \(S\) is a continuous mapping of \(Y\) into \(Z\) and \(R\) is a continuous mapping of \(Z\) into \(X\) satisfying the inequalities.

\[
d_1^i(RSTx, RSTx') < \max \{d_1^i(x, x'), d_1^i(x, RSTx), d_1^i(x', RSTx'), d_2^i(Tx, Tx'), d_3^i(STx, STx')\}
\]

\[
d_2^j(TRSy, TRSy') < \max \{d_2^j(y, y'), d_2^j(y, TRSy), d_2^j(y', TRSy'), d_3^j(Sy, Sy'), d_1^j(RSy, RSy')\},
\]

\[
d_3^j(STRz, STRz') < \max \{d_3^j(z, z'), d_3^j(z, STRz), d_3^j(z', STRz'), d_1^j(Rz, Rz'), d_2^j(TRz, TRz')\}
\]

for all \(x, x' \in X\), \(y, y' \in Y\) and \(z, z' \in Z\), \(i \in I\) for which the right hand sides of the inequalities are positive, then \(RST\) has a unique fixed point \(u\) in \(X\), \(TRS\) has a unique fixed point \(v\) in \(Y\), and \(STR\) has a unique fixed point \(w\) in \(Z\). Further

\[Tu = v, Sv = w\] and \(Rw = u\).

**REMARKS 2.5.** If we replace the uniform spaces \((X, U_1), (Y, U_2)\) and \((Z, U_3)\) in theorem (2.1), (2.3) and corollaries (2.2), (2.4) by a metric space (i.e., a metrizable uniform space) then the results of the authors [3] and [2] will follow as special cases of our results.

**REFERENCES**


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