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# On Absolute Matrix Summability of Orthogonal Series

**Xhevat Z. Krasniqi[1], Hüseyin Bor[2], Naim L. Braha[3] and Marjan Dema[4]**

[1,3]Department of Mathematics and Computer Sciences
University of Prishtina
Avenue "Mother Theresa " 5
Prishtinë 10000, Kosovë
xheki00@hotmail.com
nbraha@yahoo.com

[2]P.O. Box 121 , 06502 Bahçelievler
Ankara, Turkey
hbor33@gmail.com

[4]Faculty of Electrical and Computer Engineering
University of Prishtina
Bregu i Diellit, p.n., Prishtinë 10000, Kosovë
marjan.dema@uni-pr.edu

**Abstract.** In this paper, we prove two theorems on $|A|_k, 1 \leq k \leq 2$, summability of orthogonal series. The first one gives a sufficient condition under which an orthogonal series is absolutely summable almost everywhere, and the second one, is a general theorem, which also gives a sufficient condition so that an orthogonal series is absolutely summable almost everywhere, but it involves a positive numerical sequence that satisfies certain additional conditions. Besides, several known and new results are deduced as corollaries of the main results.

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## 1. Introduction

Let $\sum_{n=0}^{\infty} a_n$ be a given infinite series with its partial sums $\{s_n\}$ and let $A := (a_{nv})$ be a normal matrix, i.e. a lower triangular matrix of non-zero diagonal entries. Then $A$ defines the sequence-to-sequence transformation, mapping the sequence $s := \{s_n\}$ to $As := \{A_n(s)\}$, where

$$A_n(s) := \sum_{v=0}^{n} a_{nv} s_v, \ n = 0, 1, 2, \dots$$

The series $\sum_{n=0}^{\infty} a_n$ is said to be summable $|A|_k$, $k \geq 1$, if (see [5])

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)|^k$$

converges, where

$$\bar{\Delta} A_n(s) = A_n(s) - A_{n-1}(s),$$

and we write in brief

$$\sum_{n=0}^{\infty} a_n \in |A|_k.$$

Then, let $p$ denotes the sequence $\{p_n\}$. For two given sequences $p$ and $q$, the convolution $(p * q)_n$ is defined by

$$(p * q)_n = \sum_{m=0}^{n} p_m q_{n-m} = \sum_{m=0}^{n} p_{n-m} q_m.$$

When $(p*q)_n \neq 0$ for all $n$, the generalized Nörlund transform of the sequence $\{s_n\}$ is the sequence $\{t_n^{p,q}\}$ obtained by putting

$$t_n^{p,q} = \frac{1}{(p * q)_n} \sum_{m=0}^{n} p_{n-m} q_m s_m.$$

The series $\sum_{n=0}^{\infty} a_n$ is absolutely summable $(N, p, q)$ if the series

$$\sum_{n=1}^{\infty} |t_n^{p,q} - t_{n-1}^{p,q}|$$

converges, and is written in brief

$$\sum_{n=0}^{\infty} a_n \in |N, p, q|.$$

We note that $|N, p, q|$ summability is introduced by Tanaka [3].

Let $\{\varphi_n(x)\}$ be an orthonormal system defined in the interval $(a, b)$. We assume that $f(x)$ belongs to $L^2(a, b)$ and

$$f(x) \sim \sum_{n=0}^{\infty} c_n \varphi_n(x), \tag{1.1}$$

where $c_n = \int_a^b f(x)\varphi_n(x)dx$, $(n = 0, 1, 2, \dots)$.

Also is written (see [4])

$$R_n := (p * q)_n, \ R_n^j := \sum_{m=j}^{n} p_{n-m} q_m$$

where

$$R_n^{n+1} = 0, \ R_n^0 = R_n,$$

and the following two theorems are proved.

**Theorem 1.1** ([4]). *If the series*

$$\sum_{n=1}^{\infty} \left\{ \sum_{j=1}^{n} \left( \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

*is summable $|N, p, q|$ almost everywhere.*

**Theorem 1.2** ([4]). *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. Let $\{p_n\}$ and $\{q_n\}$ be non-negative. If the series $\sum_{n=1}^{\infty} |c_n|^2 \Omega(n) w^{(1)}(n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |N, p, q|$ almost everywhere, where $w^{(1)}(n)$ is defined by $w^{(1)}(j) := j^{-1} \sum_{n=j}^{\infty} n^2 \left( \frac{R_n^j}{R_n} - \frac{R_{n-1}^j}{R_{n-1}} \right)^2$.*

The main purpose of the present paper is to generalize Theorem 1.1 and Theorem 1.2 for $|A|_k$ summability of the orthogonal series (1.1), where $1 \leq k \leq 2$. Before starting the main results first introduce some further notations.

Given a normal matrix $A := (a_{nv})$, we associate two lower semi matrices $\bar{A} := (\bar{a}_{nv})$ and $\hat{A} := (\hat{a}_{nv})$ as follows:

$$\bar{a}_{nv} := \sum_{i=v}^{n} a_{ni}, \ n, i = 0, 1, 2, \ldots$$

and

$$\hat{a}_{00} = \bar{a}_{00} = a_{00}, \ \hat{a}_{nv} = \bar{a}_{nv} - \bar{a}_{n-1,v}, \ n = 1, 2, \ldots$$

It may be noted that $\bar{A}$ and $\hat{A}$ are the well-known matrices of series-to-sequence and series-to-series transformations, respectively.

The following lemma due to Beppo Levi (see [6]) is often used in the theory of functions.We will need it to prove the main results.

**Lemma 1.3.** *If $f_n(t) \in L(E)$ are non-negative functions and*

$$\sum_{n=1}^{\infty} \int_E f_n(t) dt < \infty, \tag{1.2}$$

*then the series*

$$\sum_{n=1}^{\infty} f_n(t)$$

*converges almost everywhere on $E$ to a function $f(t) \in L(E)$. Moreover, the series (1.2) is also convergent to $f$ in the norm of $L(E)$.*

Throughout this paper $K$ denotes a positive constant that it may depends only on $k$, and be different in different relations.

## 2. Main Results

We prove the following theorems.

**Theorem 2.1.** *If the series*

$$\sum_{n=1}^{\infty}\left\{n^{2-\frac{2}{k}}\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right\}^{\frac{k}{2}}$$

*converges for $1 \le k \le 2$, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n\varphi_n(x)$$

*is summable $|A|_k$ almost everywhere.*

*Proof.* For the matrix transform $A_n(s)(x)$ of the partial sums of the orthogonal series $\sum_{n=0}^{\infty} c_n\varphi_n(x)$ we have

$$\begin{aligned} A_n(s)(x) &= \sum_{v=0}^{n} a_{nv}s_v(x) = \sum_{v=0}^{n} a_{nv}\sum_{j=0}^{v} c_j\varphi_j(x) \\ &= \sum_{j=0}^{n} c_j\varphi_j(x)\sum_{v=j}^{n} a_{nv} = \sum_{j=0}^{n} \bar{a}_{nj}c_j\varphi_j(x) \end{aligned}$$

where $\sum_{j=0}^{v} c_j\varphi_j(x)$ is the partial sum of order $v$ of the series (1.1).

Hence

$$\begin{aligned} \bar{\Delta}A_n(s)(x) &= \sum_{j=0}^{n} \bar{a}_{nj}c_j\varphi_j(x) - \sum_{j=0}^{n-1} \bar{a}_{n-1,j}c_j\varphi_j(x) \\ &= \bar{a}_{nn}c_n\varphi_n(x) + \sum_{j=0}^{n-1}\left(\bar{a}_{n,j} - \bar{a}_{n-1,j}\right)c_j\varphi_j(x) \\ &= \hat{a}_{nn}c_n\varphi_n(x) + \sum_{j=0}^{n-1} \hat{a}_{n,j}c_j\varphi_j(x) = \sum_{j=0}^{n} \hat{a}_{n,j}c_j\varphi_j(x). \end{aligned}$$

Now let us clarify the reason of restriction $1 \le k \le 2$. From the definition of the $|A|_k$ summability is $k \ge 1$. Since we shall use the Hölder's inequality with $p = \frac{2}{k}$ which must be greater than 1, then $k < 2$ (for $k = 2$ we apply only the orthogonality, until for $k = 1$ we apply Schwarz's inequality). This is why we are focused only for $1 \le k \le 2$. The case when $k > 2$ remains open.

Using the Hölder's inequality and orthogonality to the latter equality, we have that

$$\begin{aligned}\int_a^b |\bar{\Delta} A_n(s)(x)|^k dx &\le (b-a)^{1-\frac{k}{2}} \left(\int_a^b |A_n(s)(x) - A_{n-1}(s)(x)|^2 dx\right)^{\frac{k}{2}} \\ &= (b-a)^{1-\frac{k}{2}} \left(\int_a^b \Big|\sum_{j=0}^n \hat{a}_{n,j} c_j \varphi_j(x)\Big|^2 dx\right)^{\frac{k}{2}} \\ &= (b-a)^{1-\frac{k}{2}} \left[\sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2\right]^{\frac{k}{2}}.\end{aligned}$$

Thus, the series

$$\sum_{n=1}^{\infty} n^{k-1} \int_a^b |\bar{\Delta} A_n(s)(x)|^k dx \le (b-a)^{1-\frac{k}{2}} \sum_{n=1}^{\infty} n^{k-1} \left[\sum_{j=0}^n |\hat{a}_{n,j}|^2 |c_j|^2\right]^{\frac{k}{2}} \tag{2.1}$$

converges since the last does by the assumption. From this fact and since the functions $|\bar{\Delta} A_n(s)(x)|^k$ are non-negative, then by the Lemma 1.3 the series

$$\sum_{n=1}^{\infty} n^{k-1} |\bar{\Delta} A_n(s)(x)|^k$$

converges almost everywhere. With this the proof of Theorem 2.1 is completed. □

If we put

$$w^{(k)}(A; j) := \frac{1}{j^{\frac{2}{k}-1}} \sum_{n=j}^{\infty} n^{\frac{2}{k}} |\hat{a}_{n,j}|^2 \tag{2.2}$$

then the following theorem holds true.

**Theorem 2.2.** *Let $1 \le k \le 2$ and $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty} \frac{1}{n\Omega(n)}$ converges. If the following series $\sum_{n=1}^{\infty} |c_n|^2 \Omega^{\frac{2}{k}-1}(n) w^{(k)}(A; n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty} c_n \varphi_n(x) \in |A|_k$ almost everywhere, where $w^{(k)}(A; n)$ is defined by (2.2).*

*Proof.* Applying Hölder's inequality to the inequality (2.1) we get that

$$\begin{aligned}\sum_{n=1}^{\infty} n^{k-1}\int_a^b |\bar{\Delta}A_n(s)(x)|^k dx \leq \\ &\leq K\sum_{n=1}^{\infty} n^{k-1}\left[\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right]^{\frac{k}{2}} \\ &= K\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))^{\frac{2-k}{2}}}\left[n\Omega^{\frac{2}{k}-1}(n)\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right]^{\frac{k}{2}} \\ &\leq K\left(\sum_{n=1}^{\infty}\frac{1}{(n\Omega(n))}\right)^{\frac{2-k}{2}}\left[\sum_{n=1}^{\infty}n\Omega^{\frac{2}{k}-1}(n)\sum_{j=0}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right]^{\frac{k}{2}} \\ &\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\sum_{n=j}^{\infty}n\Omega^{\frac{2}{k}-1}(n)|\hat{a}_{n,j}|^2\right\}^{\frac{k}{2}} \\ &\leq K\left\{\sum_{j=1}^{\infty}|c_j|^2\left(\frac{\Omega(j)}{j}\right)^{\frac{2}{k}-1}\sum_{n=j}^{\infty}n^{\frac{2}{k}}|\hat{a}_{n,j}|^2\right\}^{\frac{k}{2}} \\ &= K\left\{\sum_{j=1}^{\infty}|c_j|^2\Omega^{\frac{2}{k}-1}(j)w^{(k)}(A;j)\right\}^{\frac{k}{2}},\end{aligned}$$

which is finite by virtue of the hypothesis of the theorem. Now the the proof is an immediate result of the Lemma 1.3. □

The following corollaries follow from the main results ($k = 1$).

**Corollary 2.3.** *If the series*

$$\sum_{n=0}^{\infty}\left\{\sum_{j=1}^{n}|\hat{a}_{n,j}|^2|c_j|^2\right\}^{\frac{1}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n\varphi_n(x)$$

*is summable $|A|$ almost everywhere.*

**Corollary 2.4.** *Let $\{\Omega(n)\}$ be a positive sequence such that $\{\Omega(n)/n\}$ is a non-increasing sequence and the series $\sum_{n=1}^{\infty}\frac{1}{n\Omega(n)}$ converges. If the series $\sum_{n=1}^{\infty}|c_n|^2\Omega(n)w^{(1)}(A;n)$ converges, then the orthogonal series $\sum_{n=0}^{\infty}c_n\varphi_n(x) \in |A|$ almost everywhere, where $w^{(1)}(A;j) = j^{-1}\sum_{n=j}^{\infty}n^2|\hat{a}_{n,j}|^2$.*

## 3. Applications of the main results

We can specialize the matrix $A = (a_{nv})$ obtaining these means as follows

1. $(C,1)$ mean, when $a_{n,v} = \frac{1}{n+1}$;
2. Harmonic means, when $a_{n,v} = \frac{1}{(n-v+1)\log n}$;
3. $(C,\alpha)$ means, when $a_{n,v} = \frac{\binom{n-v+\alpha+1}{\alpha-1}}{\binom{n+\alpha}{\alpha}}$;
4. $(H,p)$ means, when $a_{n,v} = \frac{1}{\log^{p-1}(n+1)} \prod_{m=0}^{p-1} \log^m(v+1)$;
5. Nörlund means $(N,p_n)$, when $a_{n,v} = \frac{p_{n-v}}{P_n}$ where $P_n = \sum_{v=0}^{n} p_v$;
6. Riesz means $(R,p_n)$, when $a_{n,v} = \frac{p_v}{P_n}$;
7. Generalized Nörlund means $(N,p,q)$, when $a_{n,v} = \frac{p_{n-v}q_v}{R_n}$ where $R_n = \sum_{v=0}^{n} p_v q_{n-v}$.

Let us prove now that some of known results are included in Theorem 2.1. Namely, for $a_{n,v} = \frac{p_{n-v}}{P_n}$ we get

$$\begin{aligned}
\hat{a}_{n,j} &= \bar{a}_{n,j} - \bar{a}_{n-1,j} \\
&= \frac{1}{P_n}\sum_{i=j}^{n} p_{n-i} - \frac{1}{P_{n-1}}\sum_{i=j}^{n-1} p_{n-1-i} \\
&= \frac{1}{P_nP_{n-1}}\left(P_{n-1}P_{n-j} - P_nP_{n-1-j}\right) \\
&= \frac{1}{P_nP_{n-1}}\left((P_n - p_n)P_{n-j} - P_n(P_{n-j} - p_{n-j})\right) \\
&= \frac{p_n}{P_nP_{n-1}}\left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right)p_{n-j}.
\end{aligned}$$

Hence, using Theorem 2.1 for $k=1$ the following result holds.

**Corollary 3.1** ([1]). *If the series*

$$\sum_{n=0}^{\infty} \frac{p_n}{P_nP_{n-1}}\left\{\sum_{j=1}^{n} p_{n-j}^2\left(\frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}}\right)^2 |c_j|^2\right\}^{\frac{1}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n\varphi_n(x)$$

*is summable $|N,p|$ almost everywhere.*

Also, for $a_{n,v} = \frac{q_v}{Q_n}$ one can find that

$$\hat{a}_{n,j} = \bar{a}_{n,j} - \bar{a}_{n-1,j} = -\frac{q_n Q_{j-1}}{Q_n Q_{n-1}}.$$

Therefore, using again Theorem 2.1 for $k = 1$ we obtain

**Corollary 3.2** ([2]). *If the series*

$$\sum_{n=0}^{\infty} \frac{q_n}{Q_n Q_{n-1}} \left\{ \sum_{j=1}^{n} Q_{j-1}^2 |c_j|^2 \right\}^{\frac{1}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

*is summable $|R, q|$ almost everywhere.*

**Remark 3.3.** *Theorem 1.1 and Theorem 1.2 are included in Theorem 2.1 and Theorem 2.2 , respectively. It is enough to consider in Theorem 2.1 and Theorem 2.2, the case $k = 1$, and the matrix $A = (a_{nv})$ with its entries $a_{n,v} = \frac{p_{n-v} q_v}{R_n}$.*

Some other consequences of the main results are corollaries formulated below.

**Corollary 3.4.** *If for $1 \leq k \leq 2$ the series*

$$\sum_{n=0}^{\infty} \left( \frac{n^{1-\frac{1}{k}} p_n}{P_n P_{n-1}} \right)^k \left\{ \sum_{j=1}^{n} p_{n-j}^2 \left( \frac{P_n}{p_n} - \frac{P_{n-j}}{p_{n-j}} \right)^2 |c_j|^2 \right\}^{\frac{k}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

*is summable $|N, p|_k$ almost everywhere.*

**Remark 3.5.** *We note that:*

1. *If $p_n = 1$ for all values of $n$ , then $|N, p|_k$ summability reduces to $|C, 1|_k$ summability*
2. *If $k = 1$ and $p_n = 1/(n + 1)$ ,then $|N, p|_k$ is equivalent to $|R, \log n, 1|$ summability.*

**Corollary 3.6.** *If for $1 \leq k \leq 2$ the series*

$$\sum_{n=0}^{\infty} \left( \frac{n^{1-\frac{1}{k}} q_n}{Q_n Q_{n-1}} \right)^k \left\{ \sum_{j=1}^{n} Q_{j-1}^2 |c_j|^2 \right\}^{\frac{k}{2}}$$

*converges, then the orthogonal series*

$$\sum_{n=0}^{\infty} c_n \varphi_n(x)$$

*is summable* $|R, q|_k$ *almost everywhere.*

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