Distortion Theorem for Certain Class of Bazilevic Functions

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Abstract

In this work, we give a distortion theorem for certain subclasses of analytic functions that are univalent in the unit disk, and we defined a certain class of Bazilevic function using linear operator. The results generalize and unify similar well known results for several subclasses of univalent functions defined on the unit disk having the form $f(z) = z + \sum_{k=2}^{\infty} a_k z^k$, and normalized by $f(0) = 0$ and $f'(0) = 1$.

1 Introduction

Let $A$ denote the class of functions $f$ in the open unit disk

$$\mathbb{U} = \{z \in C : |z| < 1\},$$

given by the normalized power series

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}),$$

where $a_k$ is a complex number.
For two functions $f \in A$ and $g(z) = z + \sum_{k=2}^{\infty} b_k z^k$, $(z \in \mathbb{U})$, their Hadamard product (or convolution) is defined by

$$(f \ast g)(z) = z + \sum_{k=2}^{\infty} a_k b_k z^k.$$ 

Also, let $\mathcal{P}$ denote the class consisting of analytic functions $p(z)$, such that $p(0) = 1$, with positive real part, i.e, $\Re p(z) > 0$, in $\mathbb{U}$. This class of function is known as the class of Carathéodory functions and have the form

$$p(z) = 1 + \sum_{k=2}^{\infty} a_k z^k, \quad (z \in \mathbb{U}).$$

The authors (1) have recently introduced a new linear operator $D^m_{\lambda}(a, b) f(z)$ as the following:

**Definition 1.1** We define a linear operator $D^m_{\lambda}(a, b) : A \rightarrow A$ by the following Hadamard product:

$$D^m_{\lambda}(a, b) f(z) := z + \sum_{k=2}^{\infty} \left( \frac{1 + \lambda(k-1) + l}{1 + l} \right)^m (a)_{k-1} (b)_{k-1} a_k z^k,$$

(2)

where $(z \in \mathbb{U}, b \neq 0, -1, -2, -3, ...), \lambda \geq 0, m \in \mathbb{Z}, l \geq 0,$

and $(x)_k$ denotes the Pochhammer symbol (or the shifted factorial) defined by:

$$(x)_k = \begin{cases} 1 & \text{for } k = 0, \\ x(x+1)(x+2)...(x+k-1) & \text{for } k \in \mathbb{N} = \{1, 2, 3, ...\}. \end{cases}$$

Using the operator above, we give the definition of a wider class of family of Bazilevic functions as follows.

**Definition 1.2** Let $T^\alpha_m(\lambda, \beta, l, a, b)$ denote the subclass of $A$ consisting of functions $f$ which satisfy the inequality

$$\Re \left\{ \frac{D^m_{\lambda}(a, b) f(z)^\alpha}{(1 + \lambda(k-1) + l)^m z^\alpha} \right\} > \beta,$$

for $\lambda, l \geq 0, \alpha > 0$ ($\alpha$ is real), $m \in \mathbb{Z}, 0 \leq \beta < 1$ and all the index meant principal determination only.
Based on Definition 1.2 above, we have the following remark:

**Remark 1.3**

(i) For $l = 0, a = b = 1$, we have

$$\Re \frac{D_{0}^{m, \lambda}(1, 1)f(z)^{\alpha}}{(1 + \lambda(\alpha - 1))^{m}z^{\alpha}} > \beta$$

$$\equiv \Re \frac{D_{0}^{m}f(z)^{\alpha}}{(1 + \lambda(\alpha - 1))^{m}z^{\alpha}} > \beta,$$

where $D_{0}^{m}$ is the Al-Oboudi derivative operator and $f \in T_{m}^{\alpha}(\lambda, \beta, 0, 1, 1) \equiv T_{n}^{\alpha}(\lambda, \beta)$, where $T_{n}^{\alpha}(\lambda, \beta)$ is presumed new subclass of Bazilevic function.

(ii) For $l = 0, a = b = 1, \lambda = 1, n = m$, we have

$$\Re \frac{D_{0}^{m, 1}(1, 1)f(z)^{\alpha}}{\alpha^{m}z^{\alpha}} > \beta$$

$$\equiv \Re \frac{D^{n}f(z)^{\alpha}}{\alpha^{n}z^{\alpha}} > \beta$$

which is the class of functions studied in (2; 3), and $D^{n}$ is the Sâlâegan derivative operator and $f \in T_{m}^{\alpha}(1, \beta, 0, 1, 1) \equiv T_{n}^{\alpha}(\beta)$.

(iii) For $l = 0, a = b = 1, \lambda = 1, \beta = 0, n = m$, we have

$$\Re \frac{D_{0}^{m, 1}(1, 1)f(z)^{\alpha}}{\alpha^{m}z^{\alpha}} > 0$$

$$\equiv \Re \frac{D^{n}f(z)^{\alpha}}{z^{\alpha}} > 0$$

which is the class of functions studied in (6), and $D^{n}$ is the Sâlâegan derivative operator and $f \in B_{n}(\alpha)$.

(iv) For $l = 0, a = b = 1, \lambda = 1, \alpha = 1, m = \beta = 0$, we have

$$\Re \frac{D_{0}^{0, 1}(1, 1)f(z)}{z} > 0$$

$$\equiv \Re \frac{f(z)}{z} > 0$$
which is the class of function studied by Yamaguchi in (5), and \( f \in T_0^1(1, 0, 0, 1, 1) \equiv T_0^1(0) \).

(v) For \( l = 0, a = b = 1, \lambda = 1, \beta = 0, m = 1 \), we have

\[
\mathbb{R} \frac{D_0^{1,1}(1, 1)f(z)^\alpha}{z^\alpha} > 0
\]

\[
\equiv \mathbb{R} \frac{\alpha zf'(z)f(z)^{\alpha-1}}{z^\alpha} > 0
\]

which is the class of functions studied by Singh and Bazilevic (7; 8), respectively.

Our main aim in this paper is to prove a distortion theorem for functions in the class \( T_\alpha^m(\lambda, \beta, l, a, b) \) which generalize and unify certain well known results.

## 2 Distortion Theorem for functions in \( T_\alpha^m(\lambda, \beta, l, a, b) \)

Attention to the so-called coefficient estimate problem for different subclasses of univalent functions has been the main interest among the authors. Closely related to this problem is to determine how large the modulus of a univalent function together with its derivatives can be in a particular subclass. Such results, referred to as distortion theorems provide important information about the geometry of functions in these subclasses. In this section, we give a distortion theorem for functions in \( T_\alpha^m(\lambda, \beta, l, a, b) \).

In order to prove our main theorem, we will need the following result which gives a representation of functions in the class \( T_\alpha^m(\lambda, \beta, l, a, b) \) in terms of an analytic function in the unit disk.

**Theorem 2.1** The function \( f(z), z \in \mathbb{U} \) belongs to the class \( T_\alpha^m(\lambda, \beta, l, a, b) \) if, and only if, there exist a function \( \phi(z) \), analytic in \( \mathbb{U} \) with the property that \( |\phi(z)| \leq 1 \) such that

\[
\frac{D_l^{m,\lambda}(a, b)f(z)^\alpha}{z^\alpha} = (2\beta - 1) + \frac{2(1 - \beta)}{1 + z\phi(z)}.
\]

**Proof:**

Let

\[
A(z) = \frac{D_l^{m,\lambda}(a, b)f(z)^\alpha}{z^\alpha} - \beta,
\]
and we define

\[ B(z) = \frac{(1 - \beta) - A(z)}{(1 - \beta) + A(z)} = \frac{1 - \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha}}{(1 - 2\beta) + \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha}}. \]

Clearly, the function \( B(z) \) ia analytic for \( |z| < 1 \) and since by the normalization on \( f \), that is, \( f(0) = 0 \) and \( f'(0) = 1 \), it follows that \( B(0) = 0 \). Consequently, we have that \( |B(z)| < 1 \) for \( |z| < 1 \).

So by Schwarz’s Lemma, we get that \( |B(z)| < |z| \) for \( |z| < 1 \). That is,

\[ \left| \frac{1 - \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha}}{(1 - 2\beta) + \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha}} \right| < |z| \text{ for } |z| < 1. \]

Equivalently, we obtain

\[ \frac{1 - \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha}}{(1 - 2\beta) + \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha}} = z\phi(z), \quad (1) \]

where \( \phi(z) \) is an analytic function with the property that \( |\phi(z)| < 1 \) for \( |z| < 1 \). Solving equation (1) for \( \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha} \), we obtain

\[ \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha} = (2\beta - 1) + \frac{2(1 - \beta)}{1 + z\phi(z)}. \quad (2) \]

Conversely, suppose that equation (2) holds, then by letting \( z \) be real and approach 1 from below, i.e, \( z \to 1^- \), we get that

\[ \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha} > \beta. \]

Then \( f \in T_{m}^\alpha(\lambda, \beta, l, a, b) \), completes the proof.

**Theorem 2.2** Let \( f \in T_{m}^\alpha(\lambda, \beta, l, a, b) \) and \( z = re^{i\theta} \), where \( 0 \leq \theta \leq 2\pi \), then

\[ \frac{(r - 1)^2 - \frac{1 + \lambda(a - 1) + l}{1 + l}m(1 + r)^2}{2r(1 + r)} \leq \Re \left\{ \frac{D_l^{m,\lambda}(a,b)f(z)^\alpha}{z^\alpha} \right\} \leq \left( \frac{1 + \lambda(\alpha - 1) + l}{1 + l} \right)^m \frac{1 + r}{1 - r}. \]
**Proof:** From Theorem 2.1, we have that \( f \in T_m^\alpha(\lambda, \beta, l, a, b) \) if, and only if,

\[
\frac{D_l^{m,\lambda}(a, b)f(z)\alpha}{z^\alpha} = (2\beta - 1) + \frac{2(1 - \beta)}{1 + z\phi(z)} = \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m P(z)
\]

where \( P(z) \in \mathcal{P} \) is a function with positive real part. It is known that for such functions,

\[
\Re\{P(z)\} \leq \frac{1 + r}{1 - r}, \quad |z| < 1.
\]

On the other hand, taking real parts in equation (3) gives

\[
\Re\{P(z)\} = \frac{2\beta - 1}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m} + \frac{2(1 - \beta)}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m} \Re\left\{\frac{1}{1 + z\phi(z)}\right\} 
\]

\[
\leq \frac{2\beta - 1}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m} + \frac{2(1 - \beta)}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m(1 + r)}
\]

since \(|\phi(z)| \leq 1\).

It is therefore follows that

\[
\frac{1 + r}{1 - r} \geq \frac{2\beta - 1}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m} + \frac{2(1 - \beta)}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m(1 + r)}
\]

and solving for \( \beta \), we obtain

\[
\beta \geq \frac{(r - 1)^2 - \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m(1 + r)}{2r(1 + r)}
\]

from which we conclude that

\[
\Re\left\{\frac{D_l^{m,\lambda}(a, b)f(z)\alpha}{z^\alpha}\right\} \geq \frac{2\beta - 1}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m} + \frac{2(1 - \beta)}{\left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m(1 + r)}
\]

This is in fact the left hand side of the inequality of Theorem 2.2.

The right hand side is straight forward since

\[
\Re\left\{\frac{D_l^{m,\lambda}(a, b)f(z)\alpha}{z^\alpha}\right\} = \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m \Re\{P(z)\} \leq \left(\frac{1 + \lambda(\alpha - 1) + l}{1 + l}\right)^m \frac{1 + r}{1 - r}
\]

This completes the proof of Theorem 2.2.

Special case of Theorem 2.2 by setting \( \lambda = 1, l = 0 \) (9), we have the following corollary:
Corollary 2.3 Let \( f(z) \in T^\alpha_m(1, \beta, 0, 1, 1) \equiv T^\alpha_n(\beta) \) and \( z = re^{i\theta} \), where \( 0 \leq \theta \leq 2\pi \), then
\[
\frac{(r - 1)^2 - \alpha^n(1 + r)^2}{2r(1 + r)} \leq \Re \left\{ \frac{D^n f(z)^\alpha}{z^\alpha} \right\} \leq \alpha^n \frac{1 + r}{1 - r},
\]
where \( D^n \) is the Sălăgean differential operator defined in (4).

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References


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