A Note on Subclass of Analytic Functions
with Respect to $k$-Symmetric Points

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Abstract

In the present paper, we introduce a new subclass $K_k^a,s(\lambda, \delta, \alpha, \beta)$ of analytic functions with respect to $k$-symmetric points defined by differential operator. The integral representation and several coefficient inequalities for functions belonging to this class are obtained.

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1 Introduction

Let $A$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

(1)

Also let $S$ denote the subclass of $A$ consisting of all functions which are univalent in $U$. We denote by $S^*, K, C$ and $C^*$ the familiar subclasses of $A$ consisting of functions which are, respectively, starlike, convex, close-to-convex and quasi-convex in $U$ (see, for details,[3] and [14]; see also [13] and [9])

$$S^* = \{ f \in A : \Re \left\{ \frac{zf'(z)}{f(z)} \right\} > 0 \ (z \in U) \},$$

$$K = \{ f \in A : \Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} > 0 \ (z \in U) \},$$

$$C = \{ f \in A : g \in S^*, \Re \left\{ \frac{zf'(z)}{g(z)} \right\} > 0 \ (z \in U) \},$$

$$C^* = \{ f \in A : g \in K, \Re \left\{ \frac{(zf'(z))'}{g'(z)} \right\} > 0 \ (z \in U) \}.$$

A function $f \in A$ is said to be starlike with respect to symmetrical points in $U$, if it satisfies

$$\Re \left\{ \frac{zf''(z)}{f(z) - f(-z)} \right\} > 0, \ z \in U,$$

This class was introduced and studied by Sakaguchi in 1959 [15]. Some related classes are studied by Shanmugam et al.[17]. In 1979, Chand and Singh [5] defined the class of starlike functions with respect to $k$-symmetric points of order $\alpha(0 \leq \alpha < 1)$. Later, the class $S_k(\alpha, \beta)$ was studied by Gao and Zhou [4] such that

$$\frac{zf'(z)}{f_k(z)} < \frac{1 + \beta z}{1 - \alpha \beta z}.$$
where $0 \leq \alpha \leq 1, 0 < \beta \leq 1$, $k$ is a fixed positive integer and $f_k(z)$ is given by the following

$$f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} f(\varepsilon^\nu z)$$

$$= z + \sum_{\nu=0}^{\infty} a_{k(\nu+1)+1} z^{k(\nu+1)+1}, \quad (\varepsilon = \exp(2\pi i/\varepsilon), \ z \in U).$$

Note also that the class $S_2(\alpha, \beta)$ was studied by Sudharsan [8].

We now define differential operator as follows:

$$D^{\sigma,s}_{\lambda,\delta} f(z) = z + \sum_{n=2}^{\infty} n^s (C(\delta, n)[1 + \lambda(n - 1)])^\sigma a_n z^n,$$

where $\lambda \geq 0, \delta \in N_0$ and $\sigma, s \in N_0$.

Here $D^{\sigma,s}_{\lambda,\delta} f(z)$ can be written, in terms of convolution as

$$\psi(z) = \left[ \frac{\lambda z}{(1 - z)^2} - \frac{\lambda z}{1 - z} + \frac{z}{1 - z} \right] * \left[ \frac{z}{(1 - z)^{\delta+1}} \right], \quad z \in U$$

$$D^{\sigma,s}_{\lambda,\delta} f(z) = \underbrace{\psi(z) * \ldots * \psi(z)}_{\sigma\text{-times}} \sum_{n=1}^{\infty} n^s z^n * f(z) = D_\delta * \ldots * D_\delta \ast D^{\sigma,s}_{\lambda} f(z).$$

Note that $D^{0,1}_{\lambda,\delta} f(z) = D^{1,0}_{1,0} f(z) = z f'(z)$ and $D^{0,0}_{\lambda,\delta} f(z) = f(z)$. When $\sigma = 0$, we get the Sălăgean differential operator [16]. When $\lambda = s = 0, \sigma = 1$, we obtain the Ruscheweyh operator [12]. When $s = 0, \sigma = 1$, we obtain the Al-Shaqsi and Darus differential operator [1] and when $\delta = s = 0$, we obtain the Al-Oboudi differential operator [2].

In the present paper, we introduce the following class of analytic functions with respect to $k$-symmetric points, and obtain some interesting results.

Applying the operator $D^{\sigma,s}_{\lambda,\delta} f(z)$

$$D^{\sigma,s}_{\lambda,\delta} f_k(z) = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu} D^{\sigma,s}_{\lambda,\delta} f(\varepsilon^\nu z), \quad \varepsilon^k = 1,$$

where $k$ is a fixed positive integer.
Definition 1.1 Let $K_{k}^{\sigma,s}(\lambda,\delta,\alpha,\beta)$ denote the class of functions in $A$ satisfying the inequality
\[
\frac{|(z(D_{\lambda,\delta}^{\sigma,s}f(z)))'|}{(D_{\lambda,\delta}^{\sigma,s}f(z))'} - 1 < \beta \frac{\alpha(z(D_{\lambda,\delta}^{\sigma,s}f(z)))'|}{(D_{\lambda,\delta}^{\sigma,s}f(z))'} + 1
\]
where $0 \leq \alpha \leq 1, 0 < \beta \leq 1, \lambda \geq 0, \delta, \sigma, s \in N_0$ and $k$ is a fixed positive integer.

We note that $K_{k}^{0,0}(\lambda,\delta,\alpha,\beta) [11]$.

2 Coefficient Estimates

First, we need a lemma of Lakshminarasimhan [7].

Lemma 2.1 Let $H(z)$ be analytic in $U$ and satisfies the condition
\[
\left| \frac{1 - H(z)}{1 + \alpha H(z)} \right| < \beta
\]
where $0 \leq \alpha \leq 1, 0 < \beta \leq 1$, with $H(0) = 1$. Then we have
\[
H(z) = \frac{1 - z\phi(z)}{1 + \alpha z\phi(z)}
\]
where $\phi(z)$ is analytic in $U$ and $|\phi(z)| \leq \beta$ for $z \in U$. Conversely any function $H(z)$ given by (2.2) above is analytic in $U$ and satisfies (2.1).

Next we give the following lemma, which shall be used to obtain the coefficient estimates for functions to be in the class $K_{k}^{\sigma,s}(\lambda,\delta,\alpha,\beta)$.

Lemma 2.2 Let $f$ and $g$ belong to $A$ and satisfy
\[
\left| \frac{z(D_{\lambda,\delta}^{\sigma,s}f(z))'}{D_{\lambda,\delta}^{\sigma,s}g(z)} - 1 \right| < \beta \left| \frac{\alpha z(D_{\lambda,\delta}^{\sigma,s}f(z))'}{D_{\lambda,\delta}^{\sigma,s}g(z)} + 1 \right|
\]
where $0 \leq \alpha \leq 1, 0 < \beta \leq 1, \lambda \geq 0, \delta, \sigma, s \in N_0$ and $k$ is a fixed positive integer with $f$ given by (1.1) and $g(z) = z + \sum_{n=2}^{\infty} b_n z^n$. Then for $n \geq 2$,
\[
[n^s (C(\delta,n)[1 + \lambda(n-1)])^\sigma |na_n - b_n|^2
\leq 2(\alpha \beta^2 + 1) \sum_{j=2}^{n-1} j^{s+1} (C(\delta,j)[1 + \lambda(j-1)])^\sigma |a_j| |b_j|, \ (|a_1| = |b_1| = 1). \quad (3)
\]
**Proof:** Using the same method given by Sudharsan et al. [8], we shall prove this lemma. By Lemma 2.1 we have

\[
\frac{z(D_{\lambda,\delta}^{\sigma,s}f(z))'}{D_{\lambda,\delta}^{\sigma,s}g(z)} = \frac{1 - z\phi(z)}{1 + \alpha z\phi(z)}
\]  

(4)

where \( \lambda \geq 0, \delta, \sigma, s \in N_0 \) and \( \phi(z) \) is analytic in \( U \) and \( |\phi(z)| \leq \beta \) for \( z \in U \). Then

\[
[\alpha z(D_{\lambda,\delta}^{\sigma,s}f(z))' + D_{\lambda,\delta}^{\sigma,s}g(z)]z\phi(z) = D_{\lambda,\delta}^{\sigma,s}g(z) - z(D_{\lambda,\delta}^{\sigma,s}f(z))'.
\]

Now if

\[
\psi(z) = z\phi(z) = \sum_{n=1}^{\infty} t_n z^n,
\]

then

\[
|\psi(z)| \leq \beta |z|.
\]

Therefore

\[
\left[ (1 + \alpha)z + \sum_{n=2}^{\infty} n^s \left(C(\delta, n)[1 + \lambda(n - 1)]\right)^\sigma [\alpha n a_n + b_n]z^n \right] \left[ \sum_{n=1}^{\infty} t_n z^n \right]
\]

\[
= \sum_{n=2}^{\infty} n^s \left(C(\delta, n)[1 + \lambda(n - 1)]\right)^\sigma [b_n - n a_n]z^n.
\]

(5)

Equating the coefficients of \( z^n \) in (2.5), we have

\[
\sum_{n=2}^{\infty} n^s \left(C(\delta, n)[1 + \lambda(n - 1)]\right)^\sigma [b_n - n a_n]z^n
\]

\[
= (1 + \alpha)t_{n-1} + 2^s (G(\delta, 2)(1 + \lambda)) \sigma [\alpha 2a_2 + b_2] t_{n-2} + ... \]

\[
+(n - 1)^s (G(\delta, n - 1)[1 + \lambda(n - 2)]) \sigma [\alpha(n - 1)a_{n-1} + b_{n-1}] t_1.
\]

Thus the coefficient combination on the right side of (2.5) depends only upon the coefficients combination of

\[
2^s (G(\delta, 2)(1 + \lambda)) \sigma [\alpha 2a_2 + b_2] + ...
\]
\[+(n - 1)^\sigma (G(\delta, n - 1)[1 + \lambda(n - 2)])^\sigma [\alpha(n - 1)a_{n-1} + b_{n-1}]\]
on the left side. Hence for \(n \geq 2\) we can write
\[
(1 + \alpha)z + \sum_{j=2}^{n-1} j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma [j\alpha a_j + b_j]z^j \psi(z)
\]
\[
= \sum_{j=2}^{n} j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma [b_j - j a_j]z^j. \quad (6)
\]

Squaring the moduli of both sides of (2.6) and integrating along \(|z| = r < 1\), and on using the fact that \(|\psi(z)| \leq \beta |z|\), we obtain
\[
\sum_{j=2}^{n} [j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma (|b_j - j a_j|)]^2 r^{2j}.
\]
\[
< \beta^2 \left[ (1 + \alpha)^2 r^2 + \sum_{j=2}^{n-1} [j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma (|j\alpha a_j + b_j|)]^2 r^{2j} \right]
\]

Letting \(r \rightarrow 1\) on the last inequality, we obtain
\[
\sum_{j=2}^{n} [j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma (|b_j - j a_j|)]^2.
\]
\[
< \beta^2 \left[ (1 + \alpha)^2 + \sum_{j=2}^{n-1} [j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma (|j\alpha a_j + b_j|)]^2 \right].
\]

This implies that
\[
[n^s (C(\delta, n)[1 + \lambda(n - 1)])^\sigma (|n a_n - b_n|)]^2.
\]
\[
\leq \beta^2 \left[ (1 + \alpha)^2 + \sum_{j=2}^{n-1} [j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma (|j\alpha a_j + b_j|)]^2 \right]
\]
\[
- \sum_{j=2}^{n-1} [j^s (C(\delta, j)[1 + \lambda(j - 1)])^\sigma (|b_j - j a_j|)]^2.
\]
\[
\leq \beta^2(1 + \alpha)^2 + (\beta^2 \alpha^2 - 1) \sum_{j=2}^{n-1} \left[j^{s+2} (C(\delta, j)(1 + \lambda(j - 1)))^\sigma \left(|ja_j + b_j|\right)^2 |a_j|^2 \right. \\
+ (\beta^2 - 1) \sum_{j=2}^{n-1} \left[j^s (C(\delta, j)(1 + \lambda(j - 1)))^\sigma \left(|ja_j - b_j|\right)^2 |b_j|^2 \right. \\
+ 2(\alpha \beta^2 + 1) \sum_{j=2}^{n-1} j^{s+1} (C(\delta, j)(1 + \lambda(j - 1)))^\sigma |b_j||a_j|.
\]

Then

\[
[n^s (C(\delta, n)(1 + \lambda(n - 1)))^\sigma |na_n - b_n|^2 \leq 2(\alpha \beta^2 + 1) \sum_{j=2}^{n-1} j^{s+1} (C(\delta, j)(1 + \lambda(j - 1)))^\sigma |a_j||b_j|, \quad (|a_1| = |b_1| = 1).
\]

Then the proof is complete.

Now we give two meaningful conclusions about the class \(K^s_{k^s}(\lambda, \delta, \alpha, \beta)\).

**Theorem 2.3** The function \(f \in K^s_{k^s}(\lambda, \delta, \alpha, \beta)\) if, and only if,

\[
\frac{(z(D_{\lambda, \delta}^s f(z))')'}{(D_{\lambda, \delta}^s f_k(z))')} \prec \frac{1 + \beta z}{1 - \alpha \beta z},
\]

where “\(\prec\)” stands for the subordination (see [6]).

**Proof:** Suppose that \(f \in K^s_{k^s}(\lambda, \delta, \alpha, \beta)\), then from (1.5) we have

\[
\left|\frac{(z(D_{\lambda, \delta}^s f(z))')'}{(D_{\lambda, \delta}^s f_k(z))')} - 1\right|^2 < \beta^2 \left|\frac{\alpha(z(D_{\lambda, \delta}^s f(z))')'}{(D_{\lambda, \delta}^s f_k(z))'} + 1\right|^2.
\]

Expanding it, we get

\[
(1 - \alpha^2 \beta^2) \left|\frac{(z(D_{\lambda, \delta}^s f(z))')'}{(D_{\lambda, \delta}^s f_k(z))'}\right|^2 - 2(1 + \alpha \beta^2) \Re\left\{\frac{(z(D_{\lambda, \delta}^s f(z))')'}{(D_{\lambda, \delta}^s f_k(z))'}\right\} < \beta^2 - 1.
\]
If \( \alpha \neq 0 \) or \( \beta \neq 0 \), we have

\[
\left| \frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} \right|^2 - \frac{2(1 + \alpha \beta^2)}{(1 - \alpha^2 \beta^2)} \Re \left\{ \frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} \right\}
\]

\[+ \left[ \frac{(1 + \alpha \beta^2)}{(1 - \alpha^2 \beta^2)} \right]^2 < \beta^2 - \frac{1}{(1 - \alpha^2 \beta^2)} + \left[ \frac{1 + \alpha \beta^2}{(1 - \alpha^2 \beta^2)} \right]^2,
\]

that is,

\[
\left| \frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} - \frac{1 + \alpha \beta^2}{(1 - \alpha^2 \beta^2)} \right|^2 < \frac{\beta^2(1 + \alpha)^2}{(1 - \alpha^2 \beta^2)^2}
\]

or equivalently,

\[
\left| \frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} - \frac{1 + \alpha \beta^2}{(1 - \alpha^2 \beta^2)} \right| < \frac{\beta(1 + \alpha)}{(1 - \alpha^2 \beta^2)}.
\]

This tells us that the value region of \( G(z) = \frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} \) is contained in the disk whose center is \( \frac{(1 + \alpha \beta^2)}{(1 - \alpha^2 \beta^2)} \) and radius is \( \frac{\beta(1 + \alpha)}{(1 - \alpha^2 \beta^2)} \). And we know that the function \( \omega = p(z) = \frac{1 + \beta z}{1 - \alpha \beta z} \) maps the unit disk to the disk:

\[
\left| \omega - \frac{(1 + \alpha \beta^2)}{(1 - \alpha^2 \beta^2)} \right| < \frac{\beta(1 + \alpha)}{(1 - \alpha^2 \beta^2)}.
\]

Notice that \( G(0) = p(0), G(U) \subset p(U) \) and \( p(z) \) is univalent in \( U \), we obtain the following conclusion:

\[
\frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} < p(z) = \frac{1 + \beta z}{1 - \alpha \beta z}.
\]

Conversely, let

\[
\frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} < \frac{1 + \beta z}{1 - \alpha \beta z},
\]

then

\[
\frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} = \frac{1 + \beta \omega(z)}{1 - \alpha \beta \omega(z)}.
\]

(8)
where \( \omega(z) \) is analytic in \( U \), and \( \omega(0) = 0, |\omega(z)| < 1 \). A calculation, we can easily obtain from (2.8) that
\[
\left| \frac{(zD_{\lambda, \delta}^{s,s}f(z))'}{(D_{\lambda, \delta}^{s,s}f_k(z))'} - 1 \right| < \beta \left| \frac{\alpha(zD_{\lambda, \delta}^{s,s}f(z))'}{(D_{\lambda, \delta}^{s,s}f_k(z))'} + 1 \right|
\]
that is, \( f \in K_{k}^{\sigma,s}(\lambda, \delta, \alpha, \beta) \).

If \( \alpha = \beta = 1 \), then inequality (1.5) becomes
\[
\left| \frac{(zD_{\lambda, \delta}^{s,s}f(z))'}{(D_{\lambda, \delta}^{s,s}f_k(z))'} - 1 \right| < \left| \frac{(zD_{\lambda, \delta}^{s,s}f(z))'}{(D_{\lambda, \delta}^{s,s}f_k(z))'} + 1 \right|.
\]

It is obvious that
\[
\frac{(zD_{\lambda, \delta}^{s,s}f(z))'}{(D_{\lambda, \delta}^{s,s}f_k(z))'} < \frac{1 + z}{1 - z}.
\]
Therefore, the proof of Theorem 2.3 is complete.

**Remark 1:** From Theorem 2.3 we know that
\[
\Re \left\{ \frac{(zD_{\lambda, \delta}^{s,s}f(z))'}{(D_{\lambda, \delta}^{s,s}f_k(z))'} \right\} > 0,
\]
(9)
since \( \Re \{ \frac{1 + \beta z}{1 - \alpha \beta z} \} > 0 \).

**Theorem 2.4** Let \( f \in K_{k}^{\sigma,s}(\lambda, \delta, \alpha, \beta) \), then \( D_{\lambda, \delta}^{s,s}f_k \in K \)

**Proof:** For \( f \in K_{k}^{\sigma,s}(\lambda, \delta, \alpha, \beta) \), we can obtain inequality (2.9) from Theorem 2.3. Substituting \( z \) by \( \varepsilon^\mu z \), where \( \varepsilon^k = 1 (\mu = 0, 1, ..., k - 1) \) in (2.9), then (2.9) is also true, that is,
\[
\Re \left\{ \frac{(D_{\lambda, \delta}^{s,s}f(\varepsilon^\mu z))' + \varepsilon^\mu z(D_{\lambda, \delta}^{s,s}f(\varepsilon^\mu z))''}{(D_{\lambda, \delta}^{s,s}f_k(\varepsilon^\mu z))'} \right\} > 0.
\]
(10)
According to the definition of \( f_k \) and \( \varepsilon^k = 1 \), we know that \( f_k'(\varepsilon^\mu z) = f_k'(z) \). Then inequality (2.10) becomes
\[
\Re \left\{ \frac{(D_{\lambda, \delta}^{s,s}f(\varepsilon^\mu z))' + \varepsilon^\mu z(D_{\lambda, \delta}^{s,s}f(\varepsilon^\mu z))''}{(D_{\lambda, \delta}^{s,s}f_k(z))'} \right\} > 0.
\]
(11)
Let \( \mu = 0, 1, ..., k - 1 \) in (2.11) respectively, and summing up all of them, we can get

\[
\Re\left\{ \frac{\sum_{\mu=0}^{k-1} (D_{\lambda,\delta}^{s} f(\varepsilon^{\mu} z)')' + z \sum_{\mu=0}^{k-1} \varepsilon^{\mu} (D_{\lambda,\delta}^{s} f(\varepsilon^{\mu} z)''')}{(D_{\lambda,\delta}^{s} f_k(z))'} \right\} > 0, 
\]

or equivalently,

\[
\Re\left\{ \frac{(z(D_{\lambda,\delta}^{s} f_k(z))')'}{(D_{\lambda,\delta}^{s} f_k(z))'} \right\} > 0, 
\]

that is, \( D_{\lambda,\delta}^{s} f_k \in K \).

**Remark 2:** From Theorem 2.4 and inequality (2.9), we know that, if \( f \in K_{k}^{s,s}(\lambda, \delta, \alpha, \beta) \), then \( D_{\lambda,\delta}^{s} f \) is a quasi-convex function.

Now we give the following theorem.

**Theorem 2.5** Let \( f \in K_{k}^{s,s}(\lambda, \delta, \alpha, \beta) \), then we have

1. \( \iota \geq 2 \),

\[
\left\{ k(\iota - 1)((\iota - 1)k + 1)^{s+1} (C(\delta, (\iota - 1)k + 1)[1 + \lambda k(\iota - 1)])^{\alpha} \right\}^{2} |a_{(\iota - 1)k + 1}|^{2}
\]

\[
\leq 2(\alpha^{2} + 1) \sum_{i=1}^{\iota - 1} \left[ ((i - 1)k + 1)^{s+1} (C(\delta, (i - 1)k + 1)[1 + \lambda k(i - 1)])^{\alpha} \right]^{3} |a_{(i - 1)k + 1}|^{2}
\]

\( (a_1 = 1) \);

2. For \( n \geq 2, n \neq (\iota - 1)k + 1 \),

\[
\left\{ n^{s+1} (C(\delta, n)[1 + \lambda (n - 1)])^{\alpha} \right\}^{4} |a_{n}|^{2}
\]

\[
\leq 2(\alpha^{2} + 1) \sum_{i=1}^{\left\lfloor \frac{n-2}{k} + 1 \right\rfloor} \left[ ((i - 1)k + 1)^{s+1} (C(\delta, (i - 1)k + 1)[1 + \lambda k(i - 1)])^{\alpha} \right]^{3} |a_{(i - 1)k + 1}|^{2},
\]

\( (a_1 = 1) \);

where \( \left\lfloor \frac{n-2}{k} + 1 \right\rfloor \) denotes the biggest integer \( \leq \left\lfloor \frac{n-2}{k} + 1 \right\rfloor \).
**Proof:** It is easy to know that the condition (1.5) can be written as

\[
\left| \frac{z(z(D_{\lambda,\delta}^{\sigma,s}f(z))')'}{z(D_{\lambda,\delta}^{\sigma,s}f_k(z))'} - 1 \right| < \beta \left| \frac{\alpha z(z(D_{\lambda,\delta}^{\sigma,s}f(z))')'}{z(D_{\lambda,\delta}^{\sigma,s}f_k(z))'} + 1 \right|
\]

Now suppose that \( f \in K_k^{\sigma,s}(\lambda, \delta, \alpha, \beta) \), it is well-known that

\[
\frac{(z(D_{\lambda,\delta}^{\sigma,s}f(z))')'}{(D_{\lambda,\delta}^{\sigma,s}g(z))')} = \frac{(z(D_{\lambda,\delta}^{\sigma,s} * f)'(z))'}{(D_{\lambda,\delta}^{\sigma,s} * f)'_k(z)} = \frac{z(D_{\lambda,\delta}^{\sigma,s} * z f'(z))'}{(D_{\lambda,\delta}^{\sigma,s} * z g'(z))(z)} = \frac{z(D_{\lambda,\delta}^{\sigma,s} * z f'(z))'}{D_{\lambda,\delta}^{\sigma,s} z g'(z)},
\]

then

\[D_{\lambda,\delta}^{\sigma,s}f \in C^* \iff D_{\lambda,\delta}^{\sigma,s}z f' \in C.\]

Now from Theorem 2.4 and using the same method as before, we have

\[D_{\lambda,\delta}^{\sigma,s}f_k \in K \iff D_{\lambda,\delta}^{\sigma,s}z f'_k \in S^*.\]

So \( D_{\lambda,\delta}^{\sigma,s}z f' \) and \( D_{\lambda,\delta}^{\sigma,s}f'_k \) satisfy the condition of Lemma 2.2. At the same time, by the definition of \( f_k \) we have

\[f_k(z) = z + \sum_{i=2}^{\infty} a_{k(i-1)+1} z^{k(i-1)+1},\]

and

\[D_{\lambda,\delta}^{\sigma,s}f(z) = z + \sum_{i=2}^{\infty} [k(i - 1) + 1]^s (C(\delta, k(i-1) + 1)[1 + \lambda k(i-1)])^\sigma a_{k(i-1)+1} z^{k(i-1)+1}.
\]

Using Lemma 2.2, let \( n = (\iota - 1)k + 1 \) in (2.3), we get (1), if \( n \neq (\iota - 1)k + 1, n \geq 2 \), from (2.3), we get (2).

### 3 The Integral Representation

In this section, we give the integral representation of functions in the class \( K_k^{\sigma,s}(\lambda, \delta, \alpha, \beta) \).
Theorem 3.1 Let $f \in K^\sigma_s(\lambda, \delta, \phi)$. Then

$$D_{\lambda, \delta}^{\sigma, s} f_k(z) = \int_0^z \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon_{\mu}} \frac{(1 + \alpha)\beta \omega(t)}{t(1 - \alpha \beta \omega(t))} \, dt \right\} \, d\eta, \quad (1)$$

where $D_{\lambda, \delta}^{\sigma, s} f_k(z)$ is defined by equality (1.4), $\omega(z)$ is analytic in $U$ and $\omega(0) = 0$, $|\omega(z)| < 1$.

Proof: Suppose that $f \in K^\sigma_s(\lambda, \delta, \alpha, \beta)$, it follows from Theorem 2.3 we have

$$\frac{(z(D_{\lambda, \delta}^{\sigma, s} f(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} = 1 + \beta \omega(z), \quad (2)$$

where $\omega(z)$ is analytic in $U$ such that $\omega(0) = 0$, $|\omega(z)| < 1$. Substituting $z$ by $\varepsilon^\mu z$, in (3.2) respectively ($\mu = 0, 1, ..., k-1; \varepsilon^k = 1$), we have

$$\frac{(D_{\lambda, \delta}^{\sigma, s} f(\varepsilon^\mu z))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(\varepsilon^\mu z))'} = 1 + \beta \omega(\varepsilon^\mu z) = 1 - \alpha \beta \omega(\varepsilon^\mu z). \quad (3)$$

It is easy to know that $(D_{\lambda, \delta}^{\sigma, s} f(\varepsilon^\mu z))' = (D_{\lambda, \delta}^{\sigma, s} f(z))'$. Summing up (3.3) we can obtain

$$\frac{(z(D_{\lambda, \delta}^{\sigma, s} f_k(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{1 + \beta \omega(\varepsilon^\mu z)}{1 - \alpha \beta \omega(\varepsilon^\mu z)}. \quad (4)$$

It then follows from equality (3.4) to get

$$\frac{(z(D_{\lambda, \delta}^{\sigma, s} f_k(z)))'}{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'} - \frac{1}{z} = \frac{1}{k} \sum_{\mu=0}^{k-1} \frac{(1 + \alpha)\omega(\varepsilon^\mu z)}{z(1 - \alpha \beta \omega(\varepsilon^\mu z))}. \quad (5)$$

Integrating equality (3.5), we have

$$\log \{(D_{\lambda, \delta}^{\sigma, s} f_k(z))'\} = \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^z \frac{(1 + \alpha)\omega(\varepsilon^\mu \zeta)}{\zeta(1 - \alpha \beta \omega(\varepsilon^\mu \zeta))} \, d\zeta$$

$$= \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{(1 + \alpha)\omega(t)}{t(1 - \alpha \beta \omega(t))} \, dt,$$

that is,

$$(D_{\lambda, \delta}^{\sigma, s} f_k(z))' = \exp \left\{ \frac{1}{k} \sum_{\mu=0}^{k-1} \int_0^{\varepsilon^\mu z} \frac{(1 + \alpha)\omega(t)}{t(1 - \alpha \beta \omega(t))} \, dt \right\}. \quad (6)$$

Therefore, integrating equality (3.6) we can obtain equality (3.1).
Theorem 3.2 Let \( f \in K^\sigma_k(\lambda, \delta, \alpha, \beta) \). Then
\[
D^\sigma_k f(z) = \int_0^z \frac{1}{\xi} \int_0^\xi \exp\left\{ -k \sum_{\mu=0}^{k-1} \int_0^\varepsilon z (1 + \alpha) \omega(t) \frac{dt}{t(1 - \alpha \beta \omega(t))} \right\} \frac{1 + \beta \omega(\eta)}{1 - \alpha \beta \omega(\eta)} \frac{d\eta}{d\xi},
\]
(7)
where \( \omega(z) \) is analytic in \( U \) and \( \omega(0) = 0, |\omega(z)| < 1 \).

Proof: Let \( f \in K^\sigma_k(\lambda, \delta, \alpha, \beta) \). It follows from equalities (3.2) and (3.6) that
\[
(z(D^\sigma_k f(z))'' = (D^\sigma_k f(z))' \frac{1 + \beta \omega(z)}{1 - \alpha \beta \omega(z)}
\]
\[
= \exp\left\{ -k \sum_{\mu=0}^{k-1} \int_0^\varepsilon z (1 + \alpha) \omega(t) \frac{dt}{t(1 - \alpha \beta \omega(t))} \right\} \frac{1 + \beta \omega(z)}{1 - \alpha \beta \omega(z)}.
\]
(8)
Integrating equality (3.8) we can obtain
\[
(D^\sigma_k f(z))' = \frac{1}{z} \int_0^z \exp\left\{ -k \sum_{\mu=0}^{k-1} \int_0^\varepsilon z (1 + \alpha) \omega(t) \frac{dt}{t(1 - \alpha \beta \omega(t))} \right\} \frac{1 + \beta \omega(\eta)}{1 - \alpha \beta \omega(\eta)} d\eta.
\]
(9)
Therefore, integrating equality (3.9) we can obtain equality (3.7).

4 Sufficient Condition

At last, we give the sufficient condition for functions belonging to the class \( K^\sigma_k(\lambda, \delta, \alpha, \beta) \).

Theorem 4.1 Let the function \( f \) be defined by (1.1). If for \( 0 \leq \alpha \leq 1, 0 < \beta \leq 1, \lambda \geq 0, \delta, \sigma, s \in N_0, \) and
\[
\sum_{n=1}^{\infty} (nk + 1)^{s+1} (G(\delta, nk + 1)[1 + \lambda nk])^\sigma [(1 + \alpha \beta)(nk + 1) + \beta - 1] |a_{nk+1}|
\]
\[+ \sum_{n=2}^{\infty} n^{s+2} (G(\delta, n)[1 + \lambda(n-1)])^\sigma (1 + \alpha \beta) |a_n| \leq (1 + \alpha) \beta.
\]
(1)
Then \( f \in K^\sigma_k(\lambda, \delta, \alpha, \beta) \).
Proof. Suppose that \( f \) be defined by (1.1), then for \(|z| < 1\) we have

\[
M = \left| (z(D_{\lambda,\delta}^{\sigma,s} f(z))')' - (D_{\lambda,\delta}^{\sigma,s} f(z))' \right| - \beta \left| \alpha(z(D_{\lambda,\delta}^{\sigma,s} f(z))')' + (D_{\lambda,\delta}^{\sigma,s} f(z))' \right|
\]

\[
= \left| 1 + \sum_{n=2}^{\infty} n^{s+2} (G(\delta, n)[1 + \lambda(n - 1)])^\sigma a_n z^{n-1} - 1 - \sum_{n=2}^{\infty} n^{s+1} (G(\delta, n)[1 + \lambda(n - 1)])^\sigma \varphi_n a_n z^{n-1} \right|
\]

\[
- \beta \left| \alpha + \sum_{n=2}^{\infty} \alpha n^{s+2} (G(\delta, n)[1 + \lambda(n - 1)])^\sigma a_n z^{n-1} + 1 + \sum_{n=2}^{\infty} n^{s+1} (G(\delta, n)[1 + \lambda(n - 1)])^\sigma \varphi_n a_n z^{n-1} \right|
\]

where

\[
\varphi_n = \frac{1}{k} \sum_{\nu=0}^{k-1} \varepsilon^{-\nu}, \quad \varepsilon^k = 1.
\]

Thus we have

\[
M \leq \sum_{n=2}^{\infty} n^{s+1} (G(\delta, n)[1 + \lambda(n - 1)])^\sigma (n - \varphi_n) |a_n| r^{n-1}
\]

\[
- \beta \left[ (1 + \alpha) - \sum_{n=2}^{\infty} n^{s+1} (G(\delta, n)[1 + \lambda(n - 1)])^\sigma (\alpha n + \varphi_n) |a_n| r^{n-1} \right]
\]

\[
< \sum_{n=2}^{\infty} n^{s+1} (G(\delta, n)[1 + \lambda(n - 1)])^\sigma [(n - \varphi_n) + \beta(\alpha n + \varphi_n)] |a_n| - (1 + \alpha)\beta. \tag{2}
\]

From the definition of \( \varphi_n \) we know

\[
\varphi_n = \begin{cases} 
1, & n = ik + 1, \\
0, & n \neq ik + 1.
\end{cases} \tag{3}
\]

Substituting (4.3) into inequality (4.2), we get

\[
M < \sum_{n=1}^{\infty} (nk + 1)^{s+1} (G(\delta, nk + 1)[1 + \lambda nk])^\sigma [(1 + \alpha \beta)(nk + 1) + \beta - 1] |a_{nk+1}|
\]
\[ + \sum_{n=2}^{\infty} r^{s+2} (G(\delta, n)[1 + \lambda(n - 1)])^s (1 + \alpha \beta) \ |a_n| - (1 + \alpha) \beta. \]

From (4.1) we know \( M < 0 \). Thus we have

\[
\left| \frac{(z(D_{\alpha, \delta}^s f(z))')'}{(D_{\alpha, \delta}^s f_k(z))'} - 1 \right| < \beta \left| \frac{\alpha(z(D_{\alpha, \delta}^s f(z))')'}{(D_{\alpha, \delta}^s f_k(z))'} + 1 \right|,
\]

that is, \( f \in K_{\alpha, \delta}^s(\lambda, \delta, \alpha, \beta) \). Therefore, the proof of Theorem 4.1 is complete.

Other works on differential operators can be seen in ([18]-[25]) for different problems.

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**References**


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