Closure Properties of Matrix Operators
with Application to Continuous-Time Markov Chains

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Abstract

Sufficient conditions are given for an operator defined by an infinite matrix on a subspace of a sequence space to be a closable operator. It is shown by example that these conditions are not necessary. The results are applied to the operators associated with the Q-matrix of a continuous-time Markov Chain, yielding sufficient conditions for the Feller Minimal process to be the unique solution to the Kolgomorov Backward Equations.

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1 Introduction

In this paper we consider operators defined by infinite matrices with domains being subspaces of real or complex sequence spaces. For brevity, we refer to such an operator as a matrix operator. This terminology is employed by a number of authors, for example see [6], [7], [9], [15], [18], [19], [20], and [29]. We note however that some authors use the same phrase for something altogether different, as in [3].

In this paper, we treat the question of whether or not a matrix operator is closable. These results are then applied to the operators associated with continuous-time Markov Chains.

There has been considerable attention given to matrix operators on sequence spaces. For some more examples, see [4], [5], [11], [12], [24], [25], and [26]. However, it is often assumed, as in [11] for example, that the operator is defined on the whole of a sequence space which possesses with some convenient properties, such as being a BK space. It is well-known that in order
to be defined on the whole of a BK sequence space, an matrix operator must be bounded. See [28], section 4.2, Theorem 8, p. 57. Another property that is often assumed is that the domain is normal. That is, if a vector belongs to a space, then the space is assumed to contain all vectors each of whose components has magnitude less than that of the given vector. See [16], §7, or [21], p. 20.

There are however important applications to matrix operators defined on domains that are dense subsets of sequence spaces. One application of matrix operators are the operators on subspaces of $\ell_1$ defined by the Q-matrix associated with a continuous-time Markov chain having a discrete state space. See for example [2], [13], [14], and [22]. The natural domain for an operator defined by a matrix is of course the set of vectors in the space whose product with the matrix is defined as a vector in the desired codomain. This natural domain is a sequence space in its own right, but it might not be a Banach space. In such a case, the operator need not be bounded. The natural domain also might not be normal.

In section 2 we establish notation. In section 3, we give some results concerning operators that are not necessarily defined by matrices. Most of these results are used later in the paper.

In section 4 we give a sufficient condition for an matrix operator to be closed.

In section 5, we restrict the domain of a matrix operator to a certain subspace (generally but not always nontrivial) on which the value of the operator can be expressed as an infinite linear combination of its columns. We then discuss the closure properties of this restricted operator.

In the last two sections only, we restrict our attention to real sequence spaces. In section 6, we apply the results of sections 3, 4, and 5 to Q-matrices.

In section 7 we give some examples. One example is of a matrix for which the natural domain of the corresponding operator is not normal. Another example shows that the sufficient conditions given in section 3 and 4 for closability are not necessary. The examples also show that the restricted domain discussed in section 5 might be either trivial, the natural domain, or something strictly in between.

## 2 Notation

In this section we establish most of the notational conventions we follow in this paper. Much of our notation is similar to that of [21].

$\omega$ denotes the vector space of all sequences of real or complex numbers with the usual algebraic operations. A vector in $\omega$ is an infinite sequence, denoted with an arrow, e.g. $\vec{x}$, and is identified with a column matrix.
\[ \phi = \{ \vec{x} = (x_i)_{i=1}^\infty : x_i = 0 \text{ for all but finitely many indices } i \}. \]  

\( \lambda \) and \( \mu \) always denote sequence spaces, i.e. vector subspaces of \( \omega \).

If \( B \) is a vector space, its algebraic dual is denoted by \( B^\ast \). If \( B \) is a Banach space, its topological dual is denoted by \( B' \). If \( B \) is a sequence space, its \( \alpha \) dual (or K"othe dual) is denoted by \( B^\alpha \).

All matrices used are of countably infinite dimension, indexed with natural numbers, e.g. \( M = (m_{i,j})_{i,j \in \mathbb{N}} \). The \( j \) th column is denoted \( M^j = (m_{i,j})_{j=1}^\infty \) and is considered to be a vector. The \( i \) th row of \( M \) is denoted \( M_i = (m_{i,j})_{j=1}^\infty \) and is thought of as the transpose of a vector. The transpose of a matrix \( M \) is denoted \( M'^t \). We are only concerned with matrices having finite real or complex entries. Hence, all Q matrices considered are stable.

All operators considered are linear and the domain of such an operator is assumed to be a subspace. \( I \) denotes the identity operator on whatever space is being considered. If \( T \) is a linear operator acting on a vector space \( B \), the domain of \( T \) is denoted \( D(T) \), and the closure of \( T \) is denoted by \( \overline{T} \). If \( B \) and \( C \) are vector spaces, \( D_i \) are subspaces of \( B \) and \( T_i : D_i \to C \), \( i = 1, 2 \), then \( T_1 \subseteq T_2 \) means that \( T_1 \) is a restriction of \( T_2 \). In this case, we also write \( T_1 = T_2|_{D_1} \).

If \( \tau \) is a topology on a vector space \( B \) and \( \{ \vec{x}(n) \}_{n=1}^\infty \subseteq B \) and converges to \( \vec{x} \in B \) in the topology \( \tau \), we write \( \vec{x}(n) \xrightarrow{\tau} \vec{x} \). If \( A \subseteq B^\ast \) then \( \sigma(B, A) \) denotes the weak topology on \( B \) defined by \( A \), that is, the coarsest topology making all members of \( A \) continuous. The norm topology on a Banach space \( B \) is denoted \( \tau(B) \). If \( \tau_1 \) is a topology on a vector space \( B \), \( \tau_2 \) is a topology on a vector space \( C \), \( D \) is a subspace of \( B \), and \( T : D \to C \), then we’ll say \( T \) is sequentially continuous with respect to \( \tau_1 - \tau_2 \) if \( \{ \vec{x}(n) \}_{n=1}^\infty \subseteq D, \vec{x} \in D \) and \( \vec{x}(n) \xrightarrow{\tau_2} \vec{x} \) implies that \( T(\vec{x}(n)) \xrightarrow{\tau_2} T(\vec{x}) \).

We want to make a careful distinction between evaluation of an operator and multiplication by a matrix. Hence we depart from the usual convention of denoting the evaluation of an operator by juxtaposition. The value of an operator \( T \) at vector \( \vec{x} \) is written \( T(\vec{x}) \) and the product of a matrix \( M \) with a vector \( \vec{x} \) is written \( M \bullet \vec{x} \) or \( M \bullet (\vec{x}) \) if parentheses are needed for grouping.

If an assertion contains a matrix product, then it is understood that part of what is being asserted is that the product exists, i.e. that every series occurring in every term of the product converges. If \( M \) is a matrix and \( \mu \) is a sequence space, then we denote \( \mu_M = \{ \vec{x} \in \omega : M \bullet \vec{x} \in \mu \} \). This is consistent with the notation used in [5], [12], [26], and [28]. Hence if a matrix defines an operator from \( \lambda \) to \( \mu \), then its natural domain is \( \lambda \cap \mu_M \). We also define \( \mu_{[M]} = \{ \vec{x} \in \mu_M : \text{for each index } i, \sum_{j=1}^\infty |m_{i,j}x| < \infty \} \). In other words, for a vector to belong to \( \mu_{[M]} \), we insist that all series occurring in the matrix product converge absolutely. The domain \( \lambda \cap \mu_{[M]} \) is somewhat smaller than the natural domain, but very useful for the applications to Markov Chains that we will consider in 6. We note that what we’ve defined as \( \mu_{[M]} \) is the same as \( \mu_{[M]^p} \) as defined on page 61 of [12] with \( p = 1 \).
It is well known (see [28], p. 64 - 65) that if $\lambda$ is a BK space containing $\phi$, then $\lambda^\alpha$ is a BK space as well, provided that $\lambda^\alpha$ is equipped with the norm it inherits from $\lambda'$. That norm turns out to be the usual one in many spaces of interest. We assume that is normed that way. This implies that for all $\vec{x}, \vec{y} \in \lambda, |\vec{y} \bullet \vec{x}| \leq \|\vec{x}\|_{\lambda} \|\vec{y}\|_{\lambda^\alpha}$. From this it follows that for each $\vec{x} \in \lambda$ and for each index $i$, $|x_i| \leq \|\vec{x}\|_{\lambda}$.

For each natural number $j$, $\vec{e}^{(j)} = (\delta_{i,j})_{i=1}^\infty$, regarded as a vector, or column matrix. If a sequence space $\lambda$ is understood from context to be under consideration, then for each natural number $j$, $\vec{e}^{(j)*}$ is the corresponding member of $\lambda^\alpha$, that is $\vec{e}^{(j)*} : \lambda \rightarrow \mathbb{R}$ by $\vec{e}^{(j)*}(\vec{x}) = (\vec{e}^{(j)})^t \bullet \vec{x} = x_j$. The zero vector is denoted $\vec{0}$.

In this paper, $Q$ always denotes a $Q$-matrix. See [2], pp. 8–13 for the definition and basic properties. Following the notation used in [2] (see section 1.4), we denote by $Q_1$ the operator defined by the $Q$ matrix on the largest domain possible in the real sequence space $\ell_1$ for which we have absolute convergence of all occurring series, by $Q_0$ the restriction of $Q_1$ to sequences having only finitely many non-zero terms, and by $\Omega$ the infinitesimal generator of the probability transition matrix for the Markov Chain. In particular $\Omega_F$ denotes the infinitesimal generator for the Feller minimal process. The usual notation for these operators in the literature on Markov Chains is to take the input vector to be a row vector which is multiplied by the $Q$ matrix on the right side of the vector. However, in order to keep our notation consistent throughout this paper, we use the transpose of the $Q$ matrix and multiply a column vector on the left. Hence $Q_1 : \ell_1 \cap (\ell_1)[Q^1] \rightarrow \ell_1$ by $Q_1(\vec{x}) = Q^t \bullet \vec{x}$ and $Q_0 = Q_1|_{\phi}$. The parenthesis notation for evaluation and the dot notation for product distinguishes between the operator $Q_1$ and the first row of the matrix $Q$. Since there is no 0th row, $Q_0$ is unambiguous.

The only stochastic processes considered are continuous-time processes with denumerable state space identified with $\mathbb{N}$, and are always discussed in terms of their $Q$ matrix.

The end of a proof is indicated by $\Box$.

3 Results Concerning Operators on Banach Spaces

The results of this section concern operators that need not be defined by matrices. We use the same terminology as in [2], pp. 33–35.

**Lemma 3.1** For $i = 1, 2$, let $\Omega_i : D(\Omega_i) \rightarrow B$, where $B$ is a Banach space, and $D(\Omega_1) \subseteq D(\Omega_2)$. Let $r > 0$ and suppose moreover that $rI - \Omega_1$ is surjective and $rI - \Omega_2$ is injective. Then $\Omega_1 = \Omega_2$. 
Proof. By assumption $\Omega_1 \subseteq \Omega_2$. We need to show that $D(\Omega_1) = D(\Omega_2)$. Suppose to the contrary that $D(\Omega_2) \setminus D(\Omega_1) \neq \emptyset$ and let $\vec{x} \in D(\Omega_2) \setminus D(\Omega_1)$. Let $\vec{y} = (rI - \Omega_2)(\vec{x})$.

Since $rI - \Omega_1$ is surjective, there is some $\vec{z} \in D(\Omega_1)$ for which $(rI - \Omega_1)(\vec{z}) = \vec{y}$. But $rI - \Omega_1 \subseteq rI - \Omega_2$. Hence $(rI - \Omega_2)(\vec{x}) = \vec{y} = (rI - \Omega_1)(\vec{z}) = (rI - \Omega_2)(\vec{z})$. Since $rI - \Omega_2$ is injective, $\vec{x} = \vec{z}$. But $\vec{z} \in D(\Omega_1)$ and $\vec{x} \notin D(\Omega_1)$. This contradiction establishes our result. □

3.1

**Theorem 3.2** For $i = 1, 2$, let $\Omega_i$ be infinitesimal generators for continuous contraction semigroups defined on the same Banach space. If $\Omega_1 \subseteq \Omega_2$ then $\Omega_1 = \Omega_2$.

Proof. This is immediate from Lemma 3.1 and the definition of infinitesimal generator from [2], p. 35. □

**Theorem 3.3** Let $X$ and $Y$ be Banach Spaces. Let $D$ be a subspace of $X$, let $A$ be a total subset of $Y'$, and let $T : D \rightarrow Y$ be an operator which is sequentially continuous with respect to $\tau(X) - \sigma(Y, A)$. Then $T$ is closable.

Proof. Let $\{\vec{x}(n)\}_{n=1}^{\infty} \subseteq D$ and $\vec{y} \in Y$ and suppose that $\vec{x}(n) \xrightarrow{\tau(X)} \vec{0} \text{ and } T(\vec{x}(n)) \xrightarrow{\tau(Y)} \vec{y}$. The closability of $T$ will follow from well-known facts (See [10], p. 54, or [17], p. 8) provided we can show that $\vec{y} = \vec{0}$. Since $T$ is assumed to be sequentially continuous with respect to $\tau(X) - \sigma(Y, A)$, we have that $T(\vec{x}(n)) \xrightarrow{\sigma(Y,A)} T(\vec{0}) = \vec{0}$. Hence, for an arbitrary $f \in A, (f \circ T)(\vec{x}(n)) \rightarrow 0$. Therefore, for an arbitrary natural number $n$,

$$|f(\vec{y})| \leq |f(\vec{y}) - (f \circ T)(\vec{x}(n))| + |(f \circ T)(\vec{x}(n))|$$

$$= |f(\vec{y} - T(\vec{x}(n)))| + |(f \circ T)(\vec{x}(n))||$$

$$\leq \|f\|_{Y'} \|\vec{y} - T(\vec{x}(n))\|_{Y} + |(f \circ T)(\vec{x}(n))| \quad (1)$$

The last expression in computation (1) converges to zero as $n$ goes to infinity. So $f(\vec{y}) = 0$. Since $f$ is an arbitrary member of $A$, which is total, $\vec{y} = \vec{0}$. Hence $T$ is closable. □

**Theorem 3.4** Let $\lambda, \mu$ be BK spaces such that $\phi$ is a subspace of $\lambda$, and let $T : \phi \rightarrow \mu$. Then $T$ is not a closed operator.

Proof. $\phi$ has a denumerable Hamel basis. It follows that range $R(T)$ of $T$ has a countable Hamel basis, and therefore so does $\phi \times R(T)$. Since the graph $G(T)$ of $T$ is a subspace of $\phi \times R(T)$, $G(T)$ has a countable Hamel basis. Since $\phi$ is the image of $G(T)$ under a linear (projection) map, $G(T)$
J. J. Coffey has a denumerable Hamel basis. Hence, $G(T)$ cannot be a Banach space by well-known facts (e.g. see [1], p. 2), and so $G(T)$ is not a closed subspace of $\lambda \times \mu$. □

**Corollary 3.5** Let $\lambda, \mu$ be BK spaces such that $\phi$ is a subspace of $\lambda$. Let $D$ be a subspace of $\lambda$ such that $\phi \subseteq D$, and let $\Omega : D \to \mu$ be an infinitesimal generator for a continuous contraction semigroup of operators. Then $D \neq \phi$.

Proof. It is a well-known fact that an infinitesimal generator for a continuous contraction semigroup is closed. See [23], p. 399, or [27], p. 356. □

### 4 Results Concerning Operators Defined by Matrices

Throughout this section, let $\lambda$ and $\mu$ be BK spaces. Let $M$ be a matrix and suppose that $D \subseteq \lambda \cap \mu$. Let $T : D \to \mu$ by $T(\vec{x}) = M \bullet \vec{x}$.

The first result in this section shows that for the sort of matrices we are concerned with in this section, the distinction between $\lambda \cap \mu_M$ and $\lambda \cap \mu_M$ need not concern us.

**Lemma 4.1** Suppose that for each index $i$, $(M_i)^t \in \lambda^\alpha$. Then $\lambda \cap \mu_M = \lambda \cap \mu_M$.

Proof. Obviously $\lambda \cap \mu_M \subseteq \lambda \cap \mu_M$. Suppose that $\vec{x} \in \lambda \cap \mu_M$. Then $M \bullet \vec{x} \in \mu$, i.e. for each index $i$, $\sum m_{ij}x_j$ converges, and if we consider the vector $\vec{y}$ with $i$th component $y_i$ then $\vec{y} \in \mu$. However, since $(M_i)^t \in \lambda^\alpha$, we also have that the series $\sum m_{ij}x_j$ converges absolutely. So $\vec{x} \in \lambda \cap \mu_M$. □

The following result is a generalization of [24], Lemma 1(b), p. 37. It’s also a generalization of Lemma 2 from p. 38 of the same reference, in the sense that it can be applied to transformations defined by matrices on $\ell_\infty$, but we assert nothing here about the conjugate of a transformation.

**Theorem 4.2** Suppose that for each index $i$, $(M_i)^t \in \lambda^\alpha$, and that $D = \lambda \cap \mu_M$. Then $T$ is closed.

Proof. Suppose $\vec{x} \in \lambda, \vec{y} \in \mu$, $\{\vec{x}(n)\}_{n=1}^\infty \subseteq \lambda \cap \mu_M$, and as $n \to \infty$ we have $\vec{x}(n) \to \vec{x}$ and $T(\vec{x}(n)) \to \vec{y}$ in the norm topologies of $\lambda$ and $\mu$ respectively. Then $\|\vec{x}(n) - \vec{x}\|_\lambda \to 0$ and $\|M \bullet \vec{x}(n) - \vec{y}\|_\mu \to 0$.

Fix the index $i$. Since $\lambda$ and $\mu$ are both BK spaces, $x_i(n) \to x_i$, and $\left(T(\vec{x}(n))\right)_i = M_i \bullet \vec{x}(n) \to y_i$. Since $(M_i)^t \in \lambda^\alpha$, $M_i \bullet \vec{x}$ exists. Moreover, for each natural number $n$,
\[ |M_i \cdot \vec{x}(n) - M_i \cdot \vec{x}| = |M_i \cdot (\vec{x}(n) - \vec{x})| \leq \|M_i\|_{\lambda^\alpha} \|\vec{x}(n) - \vec{x}\|_\lambda. \] (2)

If we now let \( n \to \infty \), the last expression in computation 2 goes to zero. So \( M_i \cdot \vec{x}(n) \to M_i \vec{x} \) and therefore \( M_i \cdot \vec{x} = y_i \). In other words, \( \vec{x} \in D \) and \( T(\vec{x}) = \vec{y} \).

In order to discuss transformations whose domain is not all of \( \lambda \cap \mu_M \), we make the following observation.

**Corollary 4.3** Suppose that the domain of \( T \) is a subspace \( D \) of \( \lambda \cap \mu_M \) and for each index \( i \), \( (M_i)^t \in \lambda^\alpha \). Then \( T \) is closable.

Proof. This is an immediate consequence of Theorem 4.2. It could also be deduced from Theorem 3.3. \( \square \)

## 5 The Column Domain of a Matrix

A well-known fact of linear algebra is that the product of a finite matrix by a column vector can be written as a linear combination of the columns of the matrix. In this section we show that this can often be generalized to infinite matrices by taking a suitable restriction of the domain of the operator.

Throughout this section, we make the same assumptions about \( \lambda, \mu, M \), and \( T \) that we did in section 4, except that we are more specific about the domain of each operator.

**Definition 5.1** \( C(\lambda, \mu; M) = \{ \vec{x} \in \lambda : \sum_{j=1}^{\infty} |x_j|\|M^j\|_\mu < \infty \} \). If for some index \( j \), \( M^j \not\in \mu \), then it’s understood that \( \|M^j\|_\mu = \infty \) and having \( \vec{x} \in C(\lambda, \mu; M) \) entails \( x_j = 0 \). We denote \( C(\lambda, \lambda; M) = C(\lambda; M) \).

**Remark.** It’s apparent that \( C(\lambda, \mu; M) \) is a subspace of \( \lambda \), and is normal.

For a given index \( j \), \( C^{(j)} \in C(\lambda, \mu; M) \) is equivalent to \( M^j \in \mu \); and therefore \( \phi \subseteq C(\lambda, \mu; M) \) is equivalent to the condition that for all indices \( j \), \( M^j \in \mu \).

**Theorem 5.2** a.) \( C(\lambda, \mu; M) \subseteq \lambda \cap \mu_{[M]} \)

b.) For every \( \vec{x} \in C(\lambda, \mu; M) \), \( T(\vec{x}) = \sum_{j=1}^{\infty} x_j M^j \), where the series is understood to converge in the norm topology on \( \mu \).

Proof. Let \( \vec{x} \in C(\lambda, \mu; M) \). Define a sequence \( \{\vec{y}(n)\}_{n=1}^{\infty} \subseteq \mu \) by \( \vec{y}(n) = \sum_{j=1}^{n} x_j M^j \). Since \( \sum_{j=1}^{\infty} |x_j|\|M^j\|_\mu < \infty \), \( \vec{z} := \lim_{n \to \infty} \vec{y}(n) = \sum_{j=1}^{\infty} x_j M^j \) exists in \( \mu \).
Consider a fixed index $i$.

$$
\sum_{j=1}^{\infty} |m_{i,j} x_j| \leq \sum_{j=1}^{\infty} |x_j| \|M^j\|_{\mu} < \infty.
$$

So $M_i \cdot \vec{x}$ exists and all series in the matrix product converge absolutely. But also,

$$
z_i = (\vec{e}^{(i)})^t \cdot (\lim_{n \to \infty} \vec{y}(n)) = \lim_{n \to \infty} (\vec{e}^{(i)})^t \cdot (\vec{y}(n)) = \lim_{n \to \infty} \sum_{j=1}^{n} x_j m_{i,j} = \sum_{j=1}^{\infty} x_j m_{i,j} = M_i \cdot \vec{x}.
$$

Hence $\vec{z} = M \cdot \vec{x}$. So $\vec{x} \in \lambda \cap \mu_{[M]}$ and $T(\vec{x}) = \sum_{j=1}^{\infty} x_j M^j \quad \Box$

Remark. If $\vec{x} \in \lambda \cap \mu_M$, then for $n \geq i$,

$$
\vec{e}^{(i)*} \left( \sum_{i=1}^{n} x_j M^j \right) = \sum_{i=1}^{n} x_j m_{i,j}.
$$

As $n \to \infty$,

$$
\sum_{j=1}^{\infty} x_j m_{i,j} \to \sum_{j=1}^{\infty} x_j m_{i,j} = M_i \cdot \vec{x} = \vec{e}^{(i)*}(T(\vec{x})).
$$

Hence $\sum_{j=1}^{n} x_j M^j$ converges to $T(\vec{x})$ in the $\sigma(\mu, \phi)$ topology. For $\vec{x} \in C(\lambda, \mu; M)$, the convergence is in the $\tau(\mu)$ topology.

**Theorem 5.3** Suppose that $\phi \subseteq C(\lambda, \mu; M)$ and that $\lambda$ has AK.

Then $T|_{\phi}$ is closable if and only if $T|_{C(\lambda, \mu; M)}$ is closable, and in this case $\overline{T|_{\phi}} = \overline{T|_{C(\lambda, \mu; M)}}$.

Proof. Suppose that $T|_{C(\lambda, \mu; M)}$ is closable. Since $\phi \subseteq C(\lambda, \mu; M)$ by hypothesis, $T|_{\phi}$ is closable and $\overline{T|_{\phi}} \subseteq \overline{T|_{C(\lambda, \mu; M)}}$.

Suppose that $T|_{\phi}$ is closable. Let $\vec{x} \in C(\lambda, \mu; M)$. Define a sequence by $\vec{x}(n) = \sum_{i=1}^{n} x_i \vec{e}^{(i)}$. Then $\vec{x}(n) \in \phi \subseteq \overline{D(T|_{\phi})}$ and $\lambda$ having AK implies that $\vec{x}(n) \xrightarrow{\tau(\lambda)} \vec{x}$. Also $T|_{\phi}(\vec{x}(n)) = M \cdot \vec{x}(n) = \sum_{i=1}^{n} x_i M^i$. Since $\vec{x} \in C(\lambda, \mu; M)$, $\sum_{i=1}^{n} x_i M^i \xrightarrow{\tau(\mu)} \sum_{i=1}^{\infty} x_i M^i = T|_{C(\lambda, \mu; M)}(\vec{x})$. So $\vec{x} \in \overline{D(T|_{\phi})}$ and $\overline{T|_{\phi}(\vec{x})} = \overline{T|_{C(\lambda, \mu; M)}(\vec{x})}$. Hence $T|_{C(\lambda, \mu; M)} \subseteq \overline{T|_{\phi}}$ and therefore $T|_{C(\lambda, \mu; M)}$ is closable and $\overline{T|_{C(\lambda, \mu; M)}} \subseteq \overline{T|_{\phi}} \quad \Box$

The next result is very basic and rather specific, but it is important for the applications we have in mind. It is also useful for verifying examples.
Lemma 5.4 $C(\ell_1; M) = \left\{ \vec{x} \in \ell_1 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |m_{i,j}x_j| < \infty \right\}$.

Proof. 

$$\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |m_{i,j}x_j| = \sum_{j=1}^{\infty} |x_j| \sum_{i=1}^{\infty} |m_{i,j}| = \sum_{j=1}^{\infty} |x_j| \|M_j\|_1.$$ 

\[\square\]

6 Applications to Q-Matrices

In this section we consider a stable q-matrix $Q$ for a continuous-time Markov chain. We take the state space to be the set of natural numbers. The associated operators are defined as in the last two paragraphs of section 2. We note that the probability transition matrix for a Markov chain with a given stable Q-matrix is continuous contraction semigroup. (See [2], Theorem 1.4.3.) When we speak of solutions to the Kolgomorov Backward or Forward equations, we are only concerned with probability transition functions. We ignore pseudo-solutions. (For the definition, see [2], p. 83.)

Theorem 6.1 a.) $Q_0$ is not closed.

b.) $Q_0$ is closable.

Proof. a.) This is immediate from Corollary 3.5.

b.) It is well known that $Q_0 \subseteq \Omega_F$. Since an infinitesimal generator for a continuous contraction semigroup is closed, $Q_0$ is closable. \[\square\]

Lemma 6.2 As usual with Q-matrices, denote $q_j = -q_{j,j}$. Then

$$C(\ell_1; Q^t) = \left\{ \vec{x} \in \ell_1 : \sum_{j=1}^{\infty} q_j |x_j| < \infty \right\}.$$ 

Proof. By Lemma 5.4,

$$C(\ell_1; Q^t) = \left\{ \vec{x} \in \ell_1 : \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |q_{i,j}^t x_j| < \infty \right\} = \left\{ \vec{x} \in \ell_1 : \sum_{j=1}^{\infty} |x_j| \sum_{i=1}^{\infty} |q_{i,j}^t| < \infty \right\}. \quad (4)$$

By the basic properties of a Q-matrix,

$$\sum_{i=1}^{\infty} |q_{i,j}^t| = \sum_{i=1}^{\infty} |q_{j,i}|$$
\begin{equation}
q_j + \sum_{i \neq j} q_{j,i}.
\end{equation}

Also \( \sum_{i=1}^{\infty} q_{j,i} \leq 0 \). Therefore

\begin{equation}
q_j \leq \sum_{i=1}^{\infty} |q_{i,j}| \leq 2q_j.
\end{equation}

\hfill \Box

Denote \( Q_2 = Q_1|_{C(\ell_1, Q^t)} \)

\Thm{6.3} \( Q_2 \) is closable and \( \overline{Q_0} = \overline{Q_2} \)

Proof. It’s apparent from Lemma 6.2 that \( \phi \subseteq C(\ell_1; M) \). The assertion is now immediate from Theorem 6.1 and Theorem 5.3. \hfill \Box

\Thm{6.4} Let \( \Omega \) be an infinitesimal generator for a solution to the Kolgomorov Backward Equations. Then \( \overline{Q_2} \subseteq \Omega \).

Proof. By[22], Theorem 2, p. 13, \( Q_0 \subseteq \Omega \). Since \( \Omega \) is closed, \( \overline{Q_0} \subseteq \Omega \). By Theorem 6.3, \( Q_2 \subseteq \Omega \). \hfill \Box

\Cor{6.5} \( \overline{Q_2} \) is an infinitesimal generator for a continuous contraction semigroup if and only \( \overline{Q_2} = \Omega_F \). In this case, the Feller minimal process is the unique solution to the Kolgomorov Backwards Equations.

Proof. It is obvious that if \( \overline{Q_2} = \Omega_F \) then \( \overline{Q_2} \) is an infinitesimal generator for a continuous contraction semigroup. Conversely, suppose that \( \overline{Q_2} \) is an infinitesimal generator for a continuous contraction semigroup. It is well known (see [2], Theorem 2.2.2., pp. 70–71) that the Feller minimal process is a solution to the Kolgomorov Backwards Equations, so by Theorem 6.4, \( \overline{Q_2} \subseteq \Omega_F \). By Lemma 3.1, \( \overline{Q_2} = \Omega_F \). In this case, a similar argument shows that if \( \Omega \) be an infinitesimal generator for a solution to the Kolgomorov Backwards Equations, then \( \overline{Q_2} = \Omega \). Hence \( \Omega = \Omega_F \). \hfill \Box

\Cor{6.6} If \( Q \) is conservative and \( \overline{Q_2} \) is an infinitesimal generator for a continuous contraction semigroup, then \( Q \) is regular.

\Thm{6.7} Let \( Q \) be a stable \( q \)-matrix in which each column is bounded (not necessarily uniformly). Then \( Q_1 \) is a closed operator.

Proof. The columns of \( Q \) correspond to the rows of \( Q^t \). Hence this assertion is immediate from Lemma 4.1 and Theorem 4.2. \hfill \Box
Theorem 6.8 If \( C(\ell_1; Q^t) = \ell_1 \cap (\ell_1)[Q^t] \), then the Feller minimal process is the unique solution to the Kolgomorov Backwards Equations and also to the Kolgomorov Forward Equations.

Proof. It is well-known that the Feller minimal process is a solution to both the Kolgomorov Backwards Equations and the Kolgomorov Forward Equations. Since it’s a solution to the Kolgomorov Backwards Equations, \( \overline{Q_2} \subseteq \Omega_F \) by Theorem 6.3. Since it’s a solution to the Kolgomorov Forward Equations, [22], Theorem 4, p. 14, shows that \( \Omega_F \subseteq Q_1 \). Therefore \( \overline{Q_2} \subseteq \Omega_F \subseteq Q_1 \).

Since \( C(\ell_1; Q^t) = \ell_1 \cap (\ell_1)[Q^t] \) by hypothesis, \( \overline{Q_2} = \Omega_F = Q_1 \). The assertion that \( \Omega_F \) is the unique solution to the Kolgomorov Backward Equations now follows from Cor. 6.5. If \( \Omega \) is the infinitesimal generator for any solution to the Kolgomorov Forward Equations, then \( \Omega \subseteq Q_1 \). But if \( \Omega_F = Q_1 \), then \( Q_1 \) is an infinitesimal generator for a continuous contraction semigroup, and the result follows from Theorem 3.2. \( \Box \)

Remark. Clearly, if \( \Omega_F = Q_1 \), then \( Q_1 \) is a closed operator. The converse is not true. The \( Q \) matrix for a birth-death chain is easily seen to satisfy the hypotheses of Theorem 6.7, and so \( Q_1 \) is a closed operator. However there are well-known examples (See [2], section 3.2.) of Birth-Death chains for which the Feller minimal process is not the unique solution to the Kolgomorov Forward Equations. By Theorem 3.2, \( Q_1 \) cannot be an infinitesimal generator in this case.

Theorem 6.9 The following are equivalent.

a.) \( \Omega_F = Q_1 \)

b.) \( Q_1 \) is an infinitesimal generator for a continuous contraction semigroup.

c.) For every \( r > 0 \), \( rI - Q_1 \) is one-to-one.

d.) For some \( r > 0 \), \( rI - Q_1 \) is one-to-one.

Under these conditions the Feller Minimal Solution is the unique solution to the Kolgomorov Forward Equations.

Proof. It’s obvious that (a.) implies (b.), since \( \Omega_F \) is an infinitesimal generator for a continuous contraction semigroup.

The fact that (b.) implies (c.) is obvious from the definition of infinitesimal generator for a continuous contraction semigroup. (See [2], p.35.)

(c.) trivially implies (d.).

The fact that (d.) implies (a.) follows immediately from Lemma 3.1.

The last assertion is a easy consequence of Lemma 3.1 and the fact that every solution to the Kolmogorov Forward Equations is a restriction of \( Q_1 \). \( \Box \)
7 Examples

Example 7.1 $\ell_1 \cap (\ell_1)[M] \neq \{\vec{0}\}$; $\phi$ is not a subspace of $\ell_1 \cap (\ell_1)[M]$; $C(\ell_1; M) = \{\vec{0}\}$; and $\ell_1 \cap (\ell_1)[M]$ is not normal.

Let the matrix $M$ be defined by

$$m_{i,j} = \begin{cases} 1 & \text{if } i \geq j \\ 0 & \text{if } i < j \end{cases}$$

for natural numbers $i$ and $j$. For each index $j$, $M^j \not\in \ell^1$, so $C(\ell^1; M) = \{\vec{0}\}$. For each natural number $i$, and each $\vec{x} \in \ell_1$, $\sum_{j=1}^{\infty} |m_{i,j}x_j| = \sum_{j=1}^{i} |x_j| < \infty$.

Also $\sum_{i=1}^{\infty} \left| \sum_{j=1}^{\infty} m_{i,j}x_j \right| = \sum_{i=1}^{\infty} \left| \sum_{j=1}^{i} x_j \right|$. Therefore

$$\ell_1 \cap (\ell_1)[M] = \left\{ \vec{x} \in \ell_1 : \sum_{i=1}^{\infty} \left| \sum_{j=1}^{i} x_j \right| < \infty \right\}.$$

It’s apparent that for each natural number $j$, $\vec{e}^{(j)} \not\in \ell_1 \cap (\ell_1)[M]$, so $\phi$ is not a subspace of $\ell_1 \cap (\ell_1)[M]$.

Let

$$\vec{x} = (1, -1, \frac{1}{4}, -\frac{1}{9}, \frac{1}{9}, -\frac{1}{9}, \ldots).$$

and let $y_j = |x_j|$. When $i$ is even, $\sum_{j=1}^{i} x_j = 0$. When $i$ is odd, $\sum_{j=1}^{i} x_j = \left(\frac{i+1}{2}\right)^{-2}$. So

$$\sum_{i=1}^{\infty} \left| \sum_{j=1}^{i} x_j \right| = \sum_{k=1}^{\infty} k^{-2} < \infty.$$

Hence $\vec{x} \in \ell_1 \cap (\ell_1)[M]$. However, when $i$ is even, $\sum_{j=1}^{i} y_j = 2 \sum_{j=1}^{\frac{i}{2}} j^{-2}$, which does not approach 0 as $i \to \infty$. Therefore

$$\sum_{i=1}^{\infty} \left| \sum_{j=1}^{i} y_j \right| = \infty$$

and so $\vec{y} \not\in \ell_1 \cap (\ell_1)[M]$. Therefore $\ell_1 \cap (\ell_1)[M]$ is not normal.

Example 7.2 A matrix operator may or may not be closable, and the conditions given in this paper for closability are not necessary.

Let $\vec{a} = (a_i)_{i=1}^{\infty}$ and $\vec{b} = (b_i)_{i=1}^{\infty}$ be sequences of strictly positive real numbers.
Define a matrix M by

\[
m_{i,j} = \begin{cases} 
a_1b_1 & \text{if } i = 1 \text{ and } j = 1 
a_j & \text{if } i = 1 \text{ and } j > 1 
b_i & \text{if } i = j > 1 
0 & \text{otherwise.}
\end{cases}
\]

We observe that for this matrix,

\[
\phi \subset \mathbb{C}(\ell_1; M) = \ell \cap (\ell_1)[M] = \{ \vec{x} \in \ell_1 : \sum_{i=2}^{\infty} (a_i + b_i)|x_i| < \infty \}.
\]

Consider \(\phi\) as a subspace of \(\ell_1\), and a operator \(T : \phi \to \ell_1\) by \(T : \vec{x} \to M \cdot \vec{x}\).

We will show that depending on how the sequences are chosen, \(T\) may or may not be closable. This will show that a matrix operator need not be closable. In one case, \(\vec{a}\) will be unbounded and \(T\) will be closable. This will show that the sufficient condition for closability given in Corollary 4.3 is not necessary.

Suppose that \(\vec{x}(n) \in \phi\), \(\|\vec{x}(n)\|_1 \to 0\), and \(\vec{y} \in \ell_1\). \(T\) is closable if and only if \(\|M \cdot \vec{x}(n) - \vec{y}\|_1 \to 0\) implies that \(\vec{y}\) is necessarily the zero vector.

Since \(\vec{x} \in \phi\), \(\vec{x}(n) = \sum_{j=1}^{k(n)} x_j \vec{e}(j)\) where \(k(n)\) is a whole number. By our hypotheses,

\[
\|\vec{x}(n)\|_1 = \sum_{j=1}^{k(n)} |x_j| \to 0 \quad (6)
\]

We compute,

\[
\|M \cdot \vec{x}(n) - \vec{y}\|_1 = \sum_{i=1}^{\infty} \|M_i \cdot \vec{x}(n) - y_i\| = |a_1b_1x_1(n) + \sum_{j=1}^{\infty} a_jx_j(n) - y_1| + \sum_{i=2}^{\infty} |b_i x_i(n) - y_i| \quad (7)
\]

For each index \(i\), \(|x_i(n)| \leq \|x(n)\|_1 \to 0\). Also, for \(i \geq 2\), \(|b_i x_i(n) - y_i| \leq \|M \cdot \vec{x}(n) - \vec{y}\|_1\). If \(\|M \cdot \vec{x}(n) - \vec{y}\|_1 \to 0\), then for \(i \geq 2\), \(y_i = \lim_{n \to \infty} b_i x_i(n) = 0\) and \(y_1 = \lim_{n \to \infty} \sum_{j=2}^{\infty} a_j x_j(n)\). We conclude that for a matrix of this form, if \(\|M \cdot \vec{x}(n) - \vec{y}\|_1 \to 0\), then necessarily \(\vec{y} = y_1 \vec{e}(1)\). For the remainder of this example, we restrict our attention of vectors \(\vec{y}\) of this form.
Using this vector $\vec{y}$, we have
\[
\|M \cdot \vec{x}(n) - \vec{y}\|_1 = |a_1 b_1 x_1(n) + \sum_{j=2}^{\infty} a_j x_j(n) - y_1| \\
+ \sum_{i=2}^{\infty} b_i |x_i(n)|
\]  
(8)

From equation (8), we see that having $\|M \cdot \vec{x} - \vec{y}\|_1 \to 0$ is equivalent to having both of the following:
\[
\lim_{n \to \infty} \sum_{i=2}^{\infty} b_i x_i(n) = 0
\]  
(9)

and
\[
y_1 = \lim_{n \to \infty} \sum_{i=2}^{\infty} a_i x_i(n)
\]  
(10)

We conclude that the transformation $T$ is closable if and only if the components of $\vec{a}$ and $\vec{b}$ are chosen so that having conditions (6), (9), and (10) imply $y_1 = 0$. In order to exhibit examples of transformations with various properties, we distinguish several cases.

Case 1: $\vec{a} \in \ell_\infty$. In this case, the conditions (6) and (10) imply $y_1 = 0$, so $T$ is closable. Note that this is also implied by Corollary 4.3, since the transpose of each row of $M$ belongs to $\ell_1$.

Case 2: $\vec{a} \not\in \ell_\infty$ and $\vec{b} \in \ell_\infty$. In this case, conditions (6) and (9) are equivalent. However, since $\{a_i\}_{i=1}^{\infty}$ is unbounded, there is a strictly increasing sequence $\{j_n\}_{n=1}^{\infty}$ of natural numbers such that $a_{j_n} \uparrow \infty$. Let $y_1 > 0$ and define $\vec{x}(n) = \frac{y_1}{a_{j_n}} e(j_n)$. Note that in this case, $k(n) = j_n$. Then $\sum_{j=1}^{k(n)} |x_j(n)| = \frac{y_1}{a_{j_n}} \to 0$ so conditions (6) and (9) hold. For $j > 1$, $\sum_{j=2}^{k(n)} a_j x_j(n) = y_1$. Hence condition (10) holds, but $y_1 \not= 0$. Therefore $T$ is not closable.

Case 3: $\vec{a} = \vec{b} \in \ell_\infty$. (We do not intend these cases to be exhaustive of all possibilities.) Clearly conditions (9) and (10) imply $y_1 = 0$. Therefore $T$ is closable. Since $\vec{a}$ is unbounded, $M$ does not satisfy the hypotheses of Corollary 4.3. Hence the hypotheses to these two results are sufficient for closability, but not necessary. A computation similar to that of case 2 shows that $T$ is not continuous with respect to $\tau(\ell_1) - \sigma(\ell_1, \phi)$. Therefore, for a fixed total subset $A$ of $Y'$, the hypotheses of Theorem 3.3 are sufficient but not necessary for closability.
Example 7.3 \( \phi \not\subseteq C(\ell_1; M) \not\subseteq \ell_1 \cap (\ell_1)[M] \).

Let M be defined by

\[
m_{i,j} = \begin{cases} 
0 & \text{if } i \geq j \\
1 & \text{if } i < j
\end{cases}
\]

By the remark following 5.1, \( \phi \subseteq C(\ell_1; M) \). For a given vector \( \vec{x} \),

\[
\sum_{i=1}^{\infty} \sum_{j=1}^{\infty} |m_{i,j}x_j| = \sum_{j=1}^{\infty} |x_j|.
\]

By Lemma 5.4,

\[
C(\ell_1; M) = \{ \vec{x} \in \ell_1 : \sum_{j=1}^{\infty} j|x_j| < \infty \}.
\]

Let \( x_j = j^{-3} \). Then \( \vec{x} \in C(\ell_1; M) \setminus \phi \).

In this example,

\[
\ell_1 \cap (\ell_1)[M] = \{ \vec{x} \in \ell_1 : \sum_{i=1}^{\infty} \left| \sum_{j=i}^{\infty} x_j \right| < \infty \}.
\]

Let \( \vec{y} = \left( 1, -1, \frac{1}{4}, -\frac{1}{9}, \ldots \right) \). Then by computations similar to those of Example 7.1, \( \vec{y} \in \ell_1 \cap (\ell_1)[M] \setminus C(\ell_1; M) \).

References


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