Numerical Solution of $N^{th}$-Order Fuzzy Differential Equation by Runge-Kutta Method of Order Five

T. Jayakumar, K. Kanagarajan and S. Indrakumar

Department of Mathematics
Sri Ramakrishna Mission Vidyalaya College of Arts and Science
Coimbatore-641020, Tamilnadu, India
jayakumar.thippan68@gmail.com, indrakumar1729@gmail.com

Abstract

In this paper we study a numerical solution for $N^{th}$-order fuzzy differential equations based on Seikkala derivative with initial value problem. The Runge-Kutta method of Order Five is used for the numerical solution of this problem and the convergence and stability of the method is proved. By this method, we can obtain strong fuzzy solution. This method is illustrated by solving examples.

Keywords: Fuzzy Numbers, $N^{th}$-order FDEs, RK method of Order Five

1 Introduction

The topic of fuzzy differential equations (FDEs) have been rapidly growing in recent years. The concept of fuzzy derivative was first introduced by Chang and Zadeh [10], it was followed up by Dubois and Prade [12] by using the extension principle in their approach. Other methods have been discussed by Puri and Ralescu [28] and Goetschel and Voxman [15]. Kandel and Byatt [26] applied the concept of fuzzy differential equation (FDE) to the analysis of fuzzy dynamical problems. The FDE and the initial value problem (Cauchy problem) were rigorously treated by Kaleva [21, 22], Seikkala [30], He and Yi [16], Kloeden [23] and by other researchers (see [6, 20]). The numerical methods for solving fuzzy differential equations are introduced in [1, 2, 3]. Buckley and Feuring [9] introduced two analytical methods for solving $N^{th}$-order linear differential equations with fuzzy initial value conditions. Their first method of solution was to fuzzy if the crisp solution and then check to see if it satisfies the
differential equation with fuzzy initial value conditions; and the second method was the reverse of the first method, they first solved the fuzzy initial value condition and the checked to see if it defined a fuzzy function.

In this paper, a numerical method to solve $N^{\text{th}}$-order linear differential equations with fuzzy initial conditions is presented. The structure of the paper is organized as follows: In Section 2, we give some basic results on fuzzy numbers and define a fuzzy derivative and a fuzzy integral then the fuzzy initial values is treated in Section 3 using the extension principle of Zadeh and the concept of fuzzy derivative. It is shown that the fuzzy initial value problem has a unique fuzzy solution when $f$ satisfies Lipschitz condition which guarantees a unique solution to the deterministic initial value problem. In Section 4, the Runge-Kutta Method of Order Five for solving $N^{\text{th}}$-order fuzzy differential equations is introduced. In Section 5 Convergence and Stability are illustrated. In Section 6 the proposed method is illustrated by solving several examples, and the conclusion is drawn in section 7.

2 Preliminary Notes

An arbitrary fuzzy number is represented by an ordered pair of functions $(\underline{u}(r), \overline{u}(r))$ for all $r \in [0,1]$, which satisfy the following requirements [12]:

(i) $\underline{u}(r)$ is a bounded left continuous non-decreasing function over $[0,1]$,

(ii) $\overline{u}(r)$ is a bounded right continuous non-increasing function over $[0,1]$,

(iii) $\underline{u}(r) \leq \overline{u}(r) \forall r \in [0,1]$,

let $E$ be the set of all upper semi-continuous normal convex fuzzy numbers with bounded $\alpha$-level intervals.

Lemma 2.1 Let $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in (0,1]$ be a given family of non-empty intervals. If

(i) $[\underline{v}(\alpha), \overline{v}(\alpha)] \supset [\underline{v}(\beta), \overline{v}(\beta)]$ for $0 < \alpha \leq \beta$,

and

(ii) $[\lim_{k \to \infty} \underline{v}(\alpha_k), \lim_{k \to \infty} \overline{v}(\alpha_k)] = [\underline{v}(\alpha), \overline{v}(\alpha)]$,

whenever $(\alpha_k)$ is a non-decreasing sequence converging to $\alpha \in (0,1]$, then the family $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in (0,1]$, represent the $\alpha$-level sets of a fuzzy number $v \in E$. Conversely if $[\underline{v}(\alpha), \overline{v}(\alpha)]$, $\alpha \in (0,1]$, are $\alpha$-level sets of a fuzzy number $v \in E$, then the conditions (i) and (ii) hold true.
Definition 2.2 Let $I$ be a real interval. A mapping $v : I \rightarrow E$ is called a fuzzy process and we denoted the $\alpha$-level set by $[v(t)]_\alpha = [\underline{v}(t, \alpha), \overline{v}(t, \alpha)]$. The Seikkala derivative $v'(t)$ of $v$ is defined by

$$[v(t)]_\alpha = [\underline{v}'(t, \alpha), \overline{v}'(t, \alpha)],$$

provided that is a equation defines a fuzzy number $v'(t) \in E$.

Definition 2.3 Suppose $u$ and $v$ are fuzzy sets in $E$. Then their Hausdroff

$$D : E \times E \rightarrow R_+ \cup \{0\},$$

$$D(u, v) = \sup_{\alpha \in [0, 1]} \max \{|\underline{u}(\alpha) - \underline{v}(\alpha)|, |\overline{u}(\alpha) - \overline{v}(\alpha)|\},$$

i.e., $D(u, v)$ is maximal distance between $\alpha$-level sets of $u$ and $v$.

3 Fuzzy Initial Value Problem

Now we consider the initial value problem

$$\begin{cases} x^{(n)}(t) = \psi(t, x, x', \ldots, x^{(n-1)}), \\
x(0) = a_1, \ldots, x^{(n-1)}(0) = a_n, \end{cases}$$

(1)

where $\psi$ is a continuous mapping from $R_+ \times R^n$ into $R$ and $a_i$ ($0 \leq i \leq n$) are fuzzy numbers in $E$. The mentioned $n$th-order fuzzy differential equation by changining variables

$$y_1(t) = x(t), y_2(t) = x'(t), \ldots, y_n(t) = x^{(n-1)}(t),$$

converts to the following fuzzy system

$$\begin{cases} y_1'(t) = f_1(t, y_1, \ldots, y_n), \\
\quad \vdots \\
{y_n}'(t) = f_n(t, y_1, \ldots, y_n), \\
y_1(0) = y_1^{[0]} = a_1, \ldots, y_n(0) = y_n^{[0]} = a_n, \end{cases}$$

(2)

where $f_i$ ($1 \leq i \leq n$) are continuous mapping from $R_+ \times R^n$ into $R$ and $y_{i}^{[0]}$ are fuzzy numbers in $E$ with $\alpha$-level intervals

$$[y_{i}^{[0]}]_\alpha = [y_{i}^{[0]}(\alpha), \overline{y}_{i}^{[0]}(\alpha)] \quad \text{for} \quad i = 1, \ldots, n, \quad \text{and} \quad 0 < \alpha \leq 1.$$

We call $y = (y_1, \ldots, y_n)^t$ is a fuzzy solution of (2) on an interval $I$, if

$$\underline{y}'(t, \alpha) = \min\{f_i(t, u_1, \ldots, u_n); u_j \in [y_j^{[0]}(t, \alpha), \overline{y}_j(t, \alpha)]\} = \underline{f}_i(t, y(t, \alpha)), \quad (3)$$

$$\overline{y}'(t, \alpha) = \max\{f_i(t, u_1, \ldots, u_n); u_j \in [y_j^{[0]}(t, \alpha), \overline{y}_j(t, \alpha)]\} = \overline{f}_i(t, y(t, \alpha)), \quad (4)$$
and

\[ y(0, \alpha) = y^{[0]}(\alpha), \quad \underline{y}_i(0, \alpha) = \underline{y}^{[0]}_i(\alpha). \]

Thus for fixed \( \alpha \) we have a system of initial value problem in \( \mathbb{R}^{2n} \). If we can solve it (uniquely), we have only to verify that the intervals, \([y_j(t, \alpha), \underline{y}_j(t, \alpha)]\) define a fuzzy number \( y_i(t) \in E \). Now let \( y^{[0]}(\alpha) = (y^{[0]}_1(\alpha), \ldots, y^{[0]}_n(\alpha))^t \) and \( \underline{y}^{[0]}(\alpha) = (\underline{y}^{[0]}_1(\alpha), \ldots, \underline{y}^{[0]}_n(\alpha))^t \), with respect to the above mentioned indicators, system (2) can be written as with assumption

\[
\begin{aligned}
  \begin{cases} 
    y'(t) = F(t, y(t)), \\
    y(0) = y^{[0]} \in \mathbb{R}^n. 
  \end{cases} 
\end{aligned}
\]  

(5)

With assumption \( y(t, \alpha) = [y(t, \alpha), \underline{y}(t, \alpha)] \) and \( y'(t, \alpha) = [y'(t, \alpha), \underline{y}'(t, \alpha)] \)

where

\[
\begin{aligned}
  y(t, \alpha) &= (y_1(t, \alpha), \ldots, y_n(t, \alpha))^t, \quad (6) \\
  \underline{y}(t, \alpha) &= (\underline{y}_1(t, \alpha), \ldots, \underline{y}_n(t, \alpha))^t, \quad (7) \\
  y'(t, \alpha) &= (y'_1(t, \alpha), \ldots, y'_n(t, \alpha))^t, \quad (8) \\
  \underline{y}'(t, \alpha) &= (\underline{y}'_1(t, \alpha), \ldots, \underline{y}'_n(t, \alpha))^t. \quad (9)
\end{aligned}
\]

and with assumption \( F(t, y(t, \alpha)) = [F(t, y(t, \alpha)), \underline{F}(t, y(t, \alpha))] \)

where

\[
\begin{aligned}
  F(t, y(t, \alpha)) &= (f_1(t, y(t, \alpha)), \ldots, f_n(t, y(t, \alpha)))^t, \quad (10) \\
  \underline{F}(t, y(t, \alpha)) &= (\underline{f}_1(t, y(t, \alpha)), \ldots, \underline{f}_n(t, y(t, \alpha)))^t. \quad (11)
\end{aligned}
\]

\( y(t) \) is a fuzzy solution of (5) on an interval \( I \) for all \( \alpha \in (0, 1] \), if

\[
\begin{aligned}
  \begin{cases} 
    y'(t, \alpha) = F(t, y(t, \alpha)); \\
    \underline{y}(t, \alpha) = \underline{F}(t, y(t, \alpha)); \\
    y(0, \alpha) = y^{[0]}(\alpha), \quad \underline{y}(0, \alpha) = \underline{y}^{[0]}(\alpha). 
  \end{cases} 
\end{aligned}
\]  

(12)

or

\[
\begin{aligned}
  \begin{cases} 
    y'(t, \alpha) = F(t, y(t, \alpha)), \\
    y(0, \alpha) = y^{[0]}(\alpha). 
  \end{cases} 
\end{aligned}
\]  

(13)

Now we show that under the assumption for functions \( f_i \), for \( i = 1, \ldots, n \) how we can obtain a unique fuzzy solution for system (2).

**Theorem 3.1** If \( f_i(t, u_1, \ldots, u_n) \) for \( i = 1, \ldots, n \) are continuous function of \( t \) and satisfies the Lipschitz condition in \( u = (u_1, \ldots, u_n)^t \) in the region \( D = \{(t, u) | t \in [0, 1], \ -\infty < u_i < \infty \ for \ i = 1, \ldots, n \} \) with constant \( L_i \) then the initial value problem (2) has a unique fuzzy solution in each case.

**Proof.** See [5, 18].
4 The Runge-Kutta Method of Order Five

With before assumptions, the initial values problem (2) has a unique solution, such as \( y = (y_1, \ldots, y_n)^t \in E^n \) for found an approximate solution for (2) with the Runge-Kutta method of Order Five, we first define

\[
\underline{y}(t_{n+1}; \alpha) - \underline{y}(t_n; \alpha) = \sum_{i=1}^{5} w_i k_i(t_n, y(t_n; \alpha)),
\]

(14)

\[
\underline{y}(t_{n+1}; \alpha) - \underline{y}(t_n; \alpha) = \sum_{i=1}^{5} w_i k_i(t_n, y(t_n; \alpha)),
\]

\[
[k_i(t, y(t; \alpha))]_\alpha = [k_{i,1}(t, y(t; \alpha)), k_{i,2}(t, y(t; \alpha))], \quad i = 1, 2, 3, 4, 5.
\]

\[
k_{i,1}(t_n, y(t_n; \alpha)) = h.f(t_n + a_i h, y_1(t_n) + \sum_{j=1}^{i-1} b_{ij} k_{j,1}(t_n, y(t_n; \alpha))),
\]

(15)

\[
k_{i,2}(t_n, y(t_n; \alpha)) = h.f(t_n + a_i h, y_2(t_n) + \sum_{j=1}^{i-1} b_{ij} k_{j,2}(t_n, y(t_n; \alpha))),
\]

\[
k_{1,1}(t, y(t; \alpha)) = \min \{ h.f(t, s_1, \ldots, s_n) | s_i \in [y_1(t; \alpha), y_2(t; \alpha)] \},
\]

\[
k_{1,2}(t, y(t; \alpha)) = \max \{ h.f(t, s_1, \ldots, s_n) | s_i \in [y_1(t; \alpha), y_2(t; \alpha)] \},
\]

\[
k_{2,1}(t, y(t; \alpha)) = \min \{ h.f(t + \frac{h}{3}, s_1, \ldots, s_n) | s_i \in [z_{1,1}(t, y(t; \alpha), h), z_{1,2}(t, y(t; \alpha), h)] \},
\]

\[
k_{2,2}(t, y(t; \alpha)) = \max \{ h.f(t + \frac{h}{3}, s_1, \ldots, s_n) | s_i \in [z_{1,1}(t, y(t; \alpha), h), z_{1,2}(t, y(t; \alpha), h)] \},
\]

\[
k_{3,1}(t, y(t; \alpha)) = \min \{ h.f(t + \frac{h}{2}, s_1, \ldots, s_n) | s_i \in [z_{2,1}(t, y(t; \alpha), h), z_{2,2}(t, y(t; \alpha), h)] \},
\]

\[
k_{3,2}(t, y(t; \alpha)) = \max \{ h.f(t + \frac{h}{2}, s_1, \ldots, s_n) | s_i \in [z_{2,1}(t, y(t; \alpha), h), z_{2,2}(t, y(t; \alpha), h)] \},
\]

\[
k_{4,1}(t, y(t; \alpha)) = \min \{ h.f(t + h, s_1, \ldots, s_n) | s_i \in [z_{3,1}(t, y(t; \alpha), h), z_{3,2}(t, y(t; \alpha), h)] \},
\]

\[
k_{4,2}(t, y(t; \alpha)) = \max \{ h.f(t + h, s_1, \ldots, s_n) | s_i \in [z_{3,1}(t, y(t; \alpha), h), z_{3,2}(t, y(t; \alpha), h)] \},
\]

\[
k_{5,1}(t, y(t; \alpha)) = \min \{ h.f(t + h, s_1, \ldots, s_n) | s_i \in [z_{4,1}(t, y(t; \alpha), h), z_{4,2}(t, y(t; \alpha), h)] \},
\]

\[
k_{5,2}(t, y(t; \alpha)) = \max \{ h.f(t + h, s_1, \ldots, s_n) | s_i \in [z_{4,1}(t, y(t; \alpha), h), z_{4,2}(t, y(t; \alpha), h)] \}.
\]
where in the Runge-Kutta method of Order Five,

\[
\begin{align*}
    z_{1,1}(t,y(t;\alpha),h) &= y_1(t;\alpha) + \frac{1}{3} k_{1,1}(t,y(t;\alpha)), \\
    z_{1,2}(t,y(t;\alpha),h) &= y_2(t;\alpha) + \frac{1}{3} k_{1,2}(t,y(t;\alpha)), \\
    z_{2,1}(t,y(t;\alpha),h) &= y_1(t;\alpha) + \frac{1}{6} k_{1,1}(t,y(t;\alpha),h) + \frac{1}{6} k_{2,1}(t,y(t;\alpha),h), \\
    z_{2,2}(t,y(t;\alpha),h) &= y_2(t;\alpha) + \frac{1}{6} k_{1,2}(t,y(t;\alpha),h) + \frac{1}{6} k_{2,2}(t,y(t;\alpha),h), \\
    z_{3,1}(t,y(t;\alpha),h) &= y_1(t;\alpha) + \frac{1}{8} k_{1,1}(t,y(t;\alpha),h) + \frac{3}{8} k_{3,1}(t,y(t;\alpha),h), \\
    z_{3,2}(t,y(t;\alpha),h) &= y_2(t;\alpha) + \frac{1}{8} k_{1,2}(t,y(t;\alpha),h) + \frac{3}{8} k_{3,2}(t,y(t;\alpha),h). \\
\end{align*}
\]  

(17)

Define,

\[
\begin{align*}
    F(t,y(t;\alpha),h) &= k_{1,1}(t,y(t;\alpha),h) + 4k_{4,1}(t,y(t;\alpha),h) + k_{5,1}(t,y(t;\alpha),h), \\
    G(t,y(t;\alpha),h) &= k_{1,2}(t,y(t;\alpha),h) + 4k_{4,2}(t,y(t;\alpha),h) + k_{5,2}(t,y(t;\alpha),h),
\end{align*}
\]  

(18)

and suppose that the discrete equally spaced grid points \( \{t_0 = 0, t_1, \ldots, t_N = T\} \) is a partition for interval \([0,T]\). If the exact and approximate solution in the \( i \)-th \( \alpha \) cut at \( t_m, 0 \leq m \leq N \) are denoted by \([y_i^{[m]}(\alpha), y_i^{[m]}(\alpha)]\) and \([Y_i^{[m]}(\alpha), \bar{Y}_i^{[m]}(\alpha)]\) respectively, then the numerical method for solution approximation in the \( i \)-th coordinate \( \alpha \) cut, with the Runge-Kutta method of Order Five is

\[
\begin{align*}
    Y_i^{[m+1]}(\alpha) &= Y_i^{[m]}(\alpha) + \frac{4}{3} F_i(t_m,Y_m(\alpha),h), \\
    \bar{Y}_i^{[m+1]}(\alpha) &= \bar{Y}_i^{[m]}(\alpha) + \frac{4}{3} F_i(t_m,Y_m(\alpha),h), \\
    Y_i^{[0]}(\alpha) &= y_i^{[0]}(\alpha), \\
    \bar{Y}_i^{[0]}(\alpha) &= \bar{y}_i^{[0]}(\alpha),
\end{align*}
\]  

(19)

where \([Y_i(t)]_\alpha = [Y_i(t,\alpha), \bar{Y}_i(t,\alpha)]\), \( Y^{[m]}(\alpha) = [Y_i^{[m]}(\alpha), \bar{Y}_i^{[m]}(\alpha)]\]

\[
    Y^{[m]}(\alpha) = (Y^{[m]}(\alpha), \ldots, Y^{[m]}(\alpha))^t, \text{ and}
\]

\[
    \bar{Y}^{[m]}(\alpha) = (\bar{Y}_1^{[m]}(\alpha), \ldots, \bar{Y}_n^{[m]}(\alpha))^t.
\]  

(20)
Now we input
\[ F^*(t, Y'^{[m]}(\alpha), h) = \frac{1}{6}(F_1(t, Y'^{[m]}(\alpha), h), \ldots, F_n(t, Y'^{[m]}(\alpha), h))^t, \]
\[ G^*(t, Y'^{[m]}(\alpha), h) = \frac{1}{6}(G_1(t, Y'^{[m]}(\alpha), h), \ldots, G_n(t, Y'^{[m]}(\alpha), h))^t. \]

The Runge-Kutta method of Order Five for solutions approximation \( \alpha \)-cut of differential equation (13) is as follows
\[ Y'^{[m+1]}(\alpha) = Y'^{[m]}(\alpha) + hH(t_m, Y'^{[m]}(\alpha), h), \quad Y'^{[0]}(\alpha) = y'^{[0]}(\alpha) \]
where
\[ H(t_m, Y'^{[m]}(\alpha), h) = [F^*(t_m, Y'^{[m]}(\alpha), h), G^*(t_m, Y'^{[m]}(\alpha), h)], \]
and
\[ F^*(t_m, Y'^{[m]}(\alpha), h) = \frac{1}{6}[k_1(t_m, Y'^{[m]}(\alpha), h) + 4k_4(t_m, Y'^{[m]}(\alpha), h)] \]
\[ + k_5(t_m, Y'^{[m]}(\alpha), h)], \]
\[ G^*(t_m, Y'^{[m]}(\alpha), h) = \frac{1}{6}[\overline{k}_1(t_m, Y'^{[m]}(\alpha), h) + 4\overline{k}_4(t_m, Y'^{[m]}(\alpha), h)] \]
\[ + \overline{k}_5(t_m, Y'^{[m]}(\alpha), h)] \]
and also
\[ k_j(t, Y'^{[m]}(\alpha), h) = (k_{1j}(t, Y'^{[m]}(\alpha), h), \ldots, k_{nj}(t, Y'^{[m]}(\alpha), h))^t, \]
\[ \overline{k}_j(t, Y'^{[m]}(\alpha), h) = (\overline{k}_{1j}(t, Y'^{[m]}(\alpha), h), \ldots, \overline{k}_{nj}(t, Y'^{[m]}(\alpha), h))^t. \]

5 Convergence and Stability

**Definition 5.1** A one-step method for approximating the solution of a differential equation
\[
\begin{align*}
&y'(t) = F(t, y(t)), \\
y(0) = y^0 \in \mathbb{R}^n,
\end{align*}
\]
which \( F \) is a \( n^{th} \)-ordered as follows \( f = (f_1, \ldots, f_n)^t \) and 
\( f_i : R_+ \times \mathbb{R}^n \rightarrow R \ (1 \leq i \leq n) \), is a method which can be written in the form
\[ Y'^{[n+1]} = Y'^{[n]} + h\psi(t_n, Y'^{[n]}, h), \]
where the increment function \( \psi \) is determined by \( F \) and is a function \( t_n, Y'^{[n]} \) and \( h \) only.
Theorem 5.2 If $\psi(t, y, h)$ satisfies a Lipschitz condition in $y$, then the method given by (27) is stable.

Theorem 5.3 In relation (5), if $F(t, y)$ satisfies a Lipschitz condition in $y$, then the method given by (22) is stable.

Theorem 5.4 If

$$Y^{[m+1]}(\alpha) = Y^{[m]}(\alpha) + h\psi(t_m, Y^{[m]}(\alpha), h), \quad Y^{[0]} = y^{[m]}(\alpha)$$

where $\psi(t_m, Y^{[m]}(\alpha), h) = [\psi_1(t_m, Y^{[m]}(\alpha), h), \psi_2(t_m, Y^{[m]}(\alpha), h)]$ is a numerical method for approximation of differential equation (13), and $\psi_1$ and $\psi_2$ are continuous in $t, y, h$ for $0 \leq t \leq T, 0 \leq h \leq h_0$ and all $y$, and if they satisfy a Lipschitz condition in the region $D = \{(t, u, v, h)\mid 0 \leq t \leq T, -\infty < u_i \leq v_i, -\infty < v_i \leq +\infty, 0 \leq h \leq h_0 \ i = 1, \ldots, n\}$, necessary and sufficient conditions for convergence above mentioned method is

$$\psi(t, y(t, \alpha), h) = F(t, y(t, \alpha)).$$

Proof. See [5, 18].

6 Corollary

The Runge-Kutta proposed method by (22) and is convergent to the solution of the system (13) respectively.

7 Numerical Example

Example 7.1 Consider the following fuzzy differential equation with fuzzy initial value

$$\begin{align*}
  y'''(t) &= 2y''(t) + 3y'(t) \quad (0 \leq t \leq 1), \\
  y(0) &= (3 + \alpha, 5 - \alpha), \\
  y'(0) &= (-1 - \alpha, -3 + \alpha), \\
  y''(0) &= (8 + \alpha, 10 - \alpha).
\end{align*}$$

the eigen value -eigen vector solution is as follows:

$$y(t; r) = \left(-\frac{1}{3} + \frac{7}{12}e^{3t} + \frac{11}{4} + \alpha\right)e^{-t}, -\frac{1}{3} + \frac{7}{12}e^{3t} + \left(\frac{19}{4} - \alpha\right)e^{-t}\right)$$
The solution of Runge-Kutta method of Order Five is as follows and Figure 1 and Table 1 show the obtain the results:

\[
\begin{align*}
Y^{[m+1]}_1 &= Y^{[m]}_1 + \left( h + \frac{h^3}{2} + \frac{h^4}{4} + \frac{7h^5}{48} \right)Y^{[m]}_2 + \left( \frac{h^2}{2} + \frac{h^3}{3} + \frac{7h^4}{24} + \frac{5h^5}{36} \right)Y^{[m]}_3, \\
Y^{[m+1]}_2 &= Y^{[m]}_2 + \left( \frac{3h^2}{2} + \frac{h^3}{2} + \frac{7h^4}{8} + \frac{5h^5}{12} \right) Y^{[m]}_2 \\
&\quad + \left( h + h^2 + \frac{7h^3}{6} + \frac{5h^4}{6} + \frac{h^5}{144} \right) Y^{[m]}_3, \\
Y^{[m+1]}_3 &= Y^{[m]}_3 + \left( 3h + 3h^2 + \frac{7h^3}{2} + \frac{5h^4}{2} + \frac{h^5}{48} \right) Y^{[m]}_2 \\
&\quad + \left( 2h + \frac{7h^2}{2} + \frac{10h^3}{3} + \frac{h^4}{24} + \frac{1h^5}{72} \right) Y^{[m]}_3, \\
Y^{[m+1]}_4 &= Y^{[m]}_4 + \left( 3h + 3h^2 + \frac{7h^3}{2} + \frac{5h^4}{2} + \frac{h^5}{48} \right) Y^{[m]}_2 \\
&\quad + \left( 2h + \frac{7h^2}{2} + \frac{10h^3}{3} + \frac{h^4}{24} + \frac{1h^5}{72} \right) Y^{[m]}_3.
\end{align*}
\]

**Table 1.**

<table>
<thead>
<tr>
<th>(r)</th>
<th>(RK)-Nystrom</th>
<th>(RK)-OrderFive</th>
<th>Exact</th>
</tr>
</thead>
<tbody>
<tr>
<td>(y_1(t_i; r))</td>
<td>(y_2(t_i; r))</td>
<td>(Y_1(t_i; r))</td>
<td>(Y_2(t_i; r))</td>
</tr>
<tr>
<td>0.1</td>
<td>12.4005</td>
<td>13.0627</td>
<td>12.4313</td>
</tr>
<tr>
<td>0.2</td>
<td>12.4373</td>
<td>13.0259</td>
<td>12.4681</td>
</tr>
<tr>
<td>0.3</td>
<td>12.4741</td>
<td>12.9891</td>
<td>12.5049</td>
</tr>
<tr>
<td>0.4</td>
<td>12.5109</td>
<td>12.9523</td>
<td>12.5417</td>
</tr>
<tr>
<td>0.5</td>
<td>12.5477</td>
<td>12.9155</td>
<td>12.5785</td>
</tr>
<tr>
<td>0.6</td>
<td>12.5845</td>
<td>12.8788</td>
<td>12.6152</td>
</tr>
<tr>
<td>0.7</td>
<td>12.6212</td>
<td>12.8420</td>
<td>12.6520</td>
</tr>
<tr>
<td>0.8</td>
<td>12.6580</td>
<td>12.8052</td>
<td>12.6888</td>
</tr>
<tr>
<td>0.9</td>
<td>12.6948</td>
<td>12.7684</td>
<td>12.7256</td>
</tr>
</tbody>
</table>
8 Conclusion

In this paper we have applied iterative solution of Runge-Kutta method of Order Five for Numerical solution of fuzzy differential equation. It is clear that the method introduced in section 4 performs better than Runge-Kutta Nystrom method.

References


Numerical solution of $N^{th}$-order fuzzy differential equation


Received: July, 2012