A New Types of Upper and Lower Continuous Multifunctions in Topological Spaces via e-Open and e*-Open Sets

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Abstract

The purpose of present paper is to introduce and investigate two new classes of Continuous Multifunctions called upper (lower) e-continuous Multifunctions and upper (lower) e*-continuous Multifunctions by using the concepts of e-open and e*-open sets. The class of upper (lower) e-Continuous Multifunctions is stronger than upper (lower) e*-Continuous Multifunctions and generalization of upper (lower) δ-pre Continuous Multifunctions. And the class upper (lower) e*-Continuous Multifunctions is a generalization of upper (lower) β-Continuous Multifunctions and upper (lower) e-Continuous Multifunctions Several characterizations and fundamental properties concerning upper (lower) e-(e*)-Continuous Multifunctions are obtained. Furthermore, the relationships between upper (lower) e-(e*)-Continuous Multifunctions and other well-known types of Continuous Multifunctions are also discussed.

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1. Introduction

It is well known that the concept of continuity play a significant role in
general topology as well as all branches of mathematics and quantum physics, of course its strong forms are important, too. This concept has been extended to the setting of multifunctions and has been generalized by weaker forms of open sets such as $\alpha$-open sets, semi-open sets, preopen sets, $\beta$-open sets, $\delta$-preopen sets. A great number of papers dealing with such functions have appeared, and a good number of them have been extended to the setting of multifunction. This implies that continuous functions and of course continuous multifunction are important tools for studying properties of spaces and for constructing new spaces from previously existing ones, also Generalized open sets play a very important role in General Topology and they are now the research topics of many topologists worldwide. Indeed a significant theme in General Topology and Real Analysis concerns the variously modified forms of Continuity, in 1996, El-Atik [9] introduced the concept of $\gamma$-continuous functions as a generalization of semi-continuous functions due to Levine [15] and pre-continuous functions due to Mashhour et al. [16]. Most of these weaker forms of continuity in ordinary topology such as $\alpha$-continuity, $\delta$-precontinuous, pre-continuity, quasai-continuity, $b$-continuity and $\beta$-continuity have been extended to multifunction (cf. [1,18,21,22-25]). In this paper we introduce some new classes of Continuous Multifunctions, namely upper (lower) e-Continuous Multifunctions and upper (lower) $e^*$-Continuous Multifunctions, and to obtain several characterizations of these continuous multifunctions and present several of their properties. Moreover, the relationships between upper (lower) e-($e^*$) Continuous Multifunctions and some other known types of Continuous Multifunctions are also given.

2. Preliminaries

Throughout this paper, $(X, T)$ and $(Y, T^*)$ (or simply $X$ and $Y$) mean topological spaces on which no separation axioms are assumed unless explicitly stated. For any subset $A$ of $X$, The closure and interior of $A$ are denoted by $\text{Cl}(A)$ and $\text{Int}(A)$, respectively.

A subset $A$ is said to be regular open (resp. regular closed) [29] if $A = \text{Int} (\text{Cl}(A))$ (resp. $A = \text{Cl}(\text{Int}(A))$). A subset $A$ of a space $(X, T)$ is called $\delta$-open [31] if for each $x \in A$ there exists a regular open set $V$ such that $x \in V \subseteq A$. The $\delta$-interior of $A$ is the union of all regular open sets contained in $A$ and is denoted by $\text{Int}_\delta(A)$. The subset $A$ is called $\delta$-open [31] if $A = \text{Int}_\delta(A)$. A point $x \in X$ is called a $\delta$-cluster points of $A$ [31] if $A \cap \text{Int}(\text{Cl}(V)) \neq \emptyset$ for each open set $V$ containing $x$. The set of all $\delta$-cluster points of $A$ is called the $\delta$-closure of $A$ and is denoted by $\text{Cl}_\delta(A)$. If $A = \text{Cl}_\delta(A)$, then $A$ is said to be $\delta$-closed [31]. The complement of $\delta$-closed set is said to be $\delta$-open set.

A subset $A$ of $X$ is said to be $\alpha$-open [19] (resp. semi-open [20], preopen [16], $\beta$-open [2] or semi-preopen [3], $b$-open [4] or $\gamma$-open [9], $\delta$-preopen [28]) if $A \subseteq \text{Int}(\text{Cl}(\text{Int}(A)))$ (resp. $A \subseteq \text{Cl}(\text{Int}(A))$), $A \subseteq \text{Int} (\text{Cl}(\text{Int}(A))) \cup \text{Cl}(\text{Int}(A)), A \subseteq \text{Int} (\text{Cl}_\delta(A))$.

The complement of a semi-open (resp. $\alpha$-open, preopen, $\beta$-open, $b$-open, $\delta$-preopen) set is said to be semi-closed [8], (resp. $\alpha$-closed [17], preclosed [10], $\beta$-
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closed [16], b-closed [9], δ-preclosed [28]. The intersection of all b-closed (resp. semi-closed, α-closed, preclosed, β-closed, δ-preclosed) sets of X containing A is called the b-closure [9] (resp. s-closure[8], α-closure[19], pre-closure[10], β-closure [2], δ-preclosure[28]) of A and are denoted by bCl(A), (resp. SCl(A), αCl(A), PCl(A), βCl(A), δCl(A)).

The family of all b-open (resp. β-open, α-open, semi-open, preopen, δ-preopen) subsets of X containing a point x ∈ X is denoted by BΣ(X, x) (resp. βΣ(X, x), αΣ(X, x), SΣ(X, x), δΣ(X, x)). The family of all b-open (resp. β-open, α-open, semi-open, preopen, δ-preopen) sets in X are denoted by BΣ(X, T) (resp. βΣ(X, T), αΣ(X, T), SΣ(X, T), δΣ(X, T)).

A subset A of a space X is called e-open [11] if A ⊂ Cl(δ-Int(A))∪ Int(δ-Cl(A)). The complement of an e-open set is called e-closed. The intersection of all e-closed sets containing A is called the e-closure of A [11] and is denoted by e-Cl(A). The union of all e-open sets of X contained in A is called the e-interior [11] of A and is denoted by e-Int(A).

A subset A of a space X is called e*-open [12], if A ⊂ Cl(Int(δ-Cl(A))), the complement of an e*-open set is called e*-closed. The intersection of all e*-closed sets containing A is called the e*-closure of A [12] and is denoted by e*Cl(A). The union of all e*-open sets of X contained in A is called the e*-interior [12] of A and is denoted by e*-Int(A).

The family of all e-open (resp. e-closed, e*-open, e*-closed) subsets of X containing a point x ∈ X is denoted by ES(X, x) (resp. EC(X, x), EΣ(X, x), E*Σ(X, x)). The family of all e-open (resp. e-closed, e*-open, e*-closed) sets in X are denoted by ES(X, T) (resp. EC(X, T), EΣ(X, T), E*Σ(X, T)).

By a multifunction F: X → Y, we mean a point to set correspondence from X into Y, also we always assume that F(x) ≠ 0 for all x ∈ X. For a multifunction F: X → Y, the upper and lower inverse of any subset A of Y are denoted by F+(A) and F−(A), respectively, Where F+(A) = {x ∈ X: F(x) ⊂ A} and F−(A) = {x ∈ X: F(x) ∩ A ≠ 0}. In particular, F−(y) = {x ∈ X: y ∈ F(x)} for each point y ∈ Y. A multifunction F: X → Y is said to be a surjection if F(X) = Y. A multifunction F: X → Y is called upper semi-continuous (rename upper continuous) (resp. lower semi-continuous (rename lower continuous) if F+(V) (resp. F−(V)) is open in X for every open set V of Y. [1].

Remark 2.1 Erdal Ekici [11] shows that the notions of e-open set and b-open set and the notions of e-open set and β-open set and the notions of e-open set and semiopen set are independent see example (2.6) [11].

Remark 2.3. From above definitions we have the following diagram in which the converses of implications need not be true, see the examples in [12], [11].
Definition 2.1. A multifunction \( F : (X, T) \to (Y, T^*) \) is said to be:

a) upper semi continuous\([27]\) (resp. upper b-continuous \([1]\), upper almost continuous \([26,23,30]\) or upper precontinuous \([23]\), upper quasi-continuous \([22]\), upper \(\alpha\)-continuous \([18]\), upper \(\beta\)-continuous \([24,25]\), upper \(\delta\)-precontinuous \([21]\)) if for each \( x \in X \), and each open set \( V \) of \( Y \) containing \( F(x) \), there exists an open set \( U \) of \( X \) containing \( x \) (resp. \( U \in B_\Sigma(X, x) \), \( U \in P_\Sigma(X, x) \), \( U \in S_\Sigma(X, x) \), \( U \in \alpha_\Sigma(X, x) \), \( U \in \beta_\Sigma(X, x) \), \( U \in \delta_P\Sigma(X, x) \)) such that \( F(U) \subset V \).

b) Lower semi continuous\([27]\) (resp. lower b-continuous \([1]\), lower almost continuous \([26,23,30]\) or lower precontinuous \([23]\), lower quasi-continuous \([22]\), lower \(\alpha\)-continuous \([18]\), lower \(\beta\)-continuous \([24,25]\), lower \(\delta\)-precontinuous \([21]\)) at a point \( x \in X \), if for each open set \( V \) of \( Y \) such that \( F(x) \cap V \neq \emptyset \) there exists an open set \( U \) of \( X \) containing \( x \) (resp. \( U \in B_\Sigma(X, x) \), \( U \in P_\Sigma(X, x) \), \( U \in S_\Sigma(X, x) \), \( U \in \alpha_\Sigma(X, x) \), \( U \in \beta_\Sigma(X, x) \), \( U \in \delta_P\Sigma(X, x) \)) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \).

c) Upper (lower) semi continuous (resp. upper (lower) b-continuous , upper (lower) precontinuous, upper (lower) quasi-continuous, upper (lower) \(\alpha\)-continuous, upper (lower) \(\beta\)-continuous, upper (lower) \(\delta\)-precontinuous) if \( F \) has this property at each point of \( X \).

3. Characterizations of upper and lower \(e\) \((e^*)\)-Continuous Multifunctions

Definitions 3.1. A multifunction \( F : (X, T) \to (Y, T^*) \) is said to be:

a) Upper \(e\)-continuous (resp. upper \(e^*\)-continuous) if for each \( x \in X \), and each open set \( V \) of \( Y \) such that \( F(x) \subset V \), there exists \( U \in E_\Sigma(X, x) \) (resp. \( U \in E^*_\Sigma(X, x) \)) such that \( F(U) \subset V \).

b) Lower \(e\)-continuous (resp. lower \(e^*\)-continuous) at a point \( x \in X \), if for each open set \( V \) of \( Y \) such that \( F(x) \cap V \neq \emptyset \) there exists \( U \in E_\Sigma(X, x) \) (resp. \( U \in E^*_\Sigma(X, x) \)) such that \( F(u) \cap V \neq \emptyset \) for every \( u \in U \).

c) Upper (lower) \(e\)-continuous (resp. upper (lower) \(e^*\)-continuous) if \( F \) has this property at each point of \( X \).

Definition 3.2. A subset \( U \) of a topological space \( (X, T) \) is called \(e\)-neighborhood (resp. \(e^*\)-neighborhood) of a point \( x \in X \) if there exists an \(e\)-open (resp. \(e^*\)-open) set \( A \) of \( X \) such that \( x \in A \subset U \).

Theorem 3.1. For a multifunction \( F : X \to Y \), the following statements are equivalent:

a) \( F \) is Upper \(e^*\)-continuous;

b) \( F^+(V) \in E^*_\Sigma(X, T) \) for every open set \( V \) of \( Y \);

c) \( F^-(V) \in E^*C(X, T) \) for every closed set \( V \) of \( Y \);

d) \( e^*\-Cl(F^-(B)) \subset F^-(Cl(B)) \) for every \( B \subset Y \);

e) For each point \( x \in X \) and for each neighborhood \( V \) of \( F(x) \), \( F^+(V) \) is an \(e^*\)-Neighborhood of \( x \);
f) For each point $x \in X$ and for each neighborhood $V$ of $F(x)$, there exists an $e^*$-neighborhood $U$ of $x$ such that $F(U) \subset V$;

$g)$ $\text{Cl} (\text{Int}(\delta \text{-Cl}(F^-(B)))) \subset F^-(\text{Cl}(B))$ for every subset $B$ of $Y$;

$h)$ $F^+(\text{Int}(B)) \subset \text{Cl} (\text{Int}(\delta \text{-Cl}(F^+(B))))$ for every subset $B$ of $Y$;

**Proof.** (a) $\Rightarrow$ (b): Let $V$ be any open set of $Y$ and $x \in F^+(V)$, there exists $U \in E^* \Sigma (X, x)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset \text{Cl}(\text{Int}(\delta \text{-Cl}(F^+(V))))$ and hence $F^+(V) \in E^* \Sigma (X, T)$.

(b) $\Leftrightarrow$ (c): from the fact that $F^+(Y \setminus B) = X \setminus F^-(B)$ for every subset $B$ of $Y$. This Proof follows immediately.

(c) $\Rightarrow$ (d): since for any subset $B$ of $Y$, $\text{Cl}(B)$ is closed in $Y$ and $F^-(\text{Cl}(B))$ is $e^*$-closed in $X$. Therefore, we obtain $e^* - \text{Cl}(F^-(B)) \subset F^-(\text{Cl}(B))$.

(d) $\Rightarrow$ (c): Let $V$ be any closed set of $Y$. Then, we have:

$e^* - \text{Cl}(F^-(V)) \subset F^-(\text{Cl}(V)) = F^-(V)$. This shows that $F^-(V)$ is $e^*$-closed in $X$.

(b) $\Rightarrow$ (e): Let $x \in X$ and $V$ be a neighborhood of $F(x)$. Then there exists an open set $A$ of $Y$ such that $F(x) \subset A \subset V$. Therefore, we obtain $x \in F^+(A) \subset F^+(V)$.

Since $F^+(A) \in E^* \Sigma (X, T)$, Then $F^+(V)$ is an $e^*$-neighborhood of $x$.

(e) $\Rightarrow$ (f): Let $x \in X$ and $V$ be a neighborhood of $F(x)$. Put $U = F^+(V)$, Then $U$ is an $e^*$-neighborhood of $x$ and $F(U) \subset V$.

(f) $\Rightarrow$ (a): Let $x \in X$ and $V$ be any open set of $Y$ such that $F(x) \subset V$. Then $V$ is a neighborhood of $F(x)$. There exists an $e^*$-neighborhood $U$ of $x$ such that $F(U) \subset V$. Therefore there exists, $A \in E^* \Sigma (X, T)$ such that $x \in A \subset U$, hence $F(A) \subset V$.

(c) $\Rightarrow$ (g): For any subset $B$ of $Y$, $\text{Cl}(B)$ is closed in $Y$ and by (c), we have $F^-(\text{Cl}(B))$ is $e^*$-closed in $X$. This means,

$\text{Cl}(\text{Int}(\delta \text{-Cl}(F^-(B)))) \subset \text{Cl}(\text{Int}(\delta \text{-Cl}(F^-(\text{Cl}(B)))) \subset F^-(\text{Cl}(B))$.

(g) $\Rightarrow$ (h): By replacing $Y \setminus B$ instead of $B$ in (g), we have

$\text{Cl}(\text{Int}(\delta \text{-Cl}(F^+(Y \setminus B)))) \subset F^+(\text{Cl}(Y \setminus B))$, therefore $F^+(\text{Int}(B)) \subset \text{Cl}(\text{Int}(\delta \text{-Cl}(F^+(B))))$.

(h) $\Rightarrow$ (b): Let $V$ be any open set of $Y$. Then, by using (h) we have $F^+(V) \in E^* \Sigma (X, T)$ and this completes the proof.

**Theorem 3.2.** For a multifunction $F: X \to Y$, the following statements are equivalent:

a) $F$ is lower $e^*$-continuous;

b) $F^-(V) \in E^* \Sigma (X, T)$ for every open set $V$ of $Y$;

c) $F^+(V) \in E^* \Sigma (X, T)$ for every closed set $V$ of $Y$;

d) $e^* - \text{Cl}(F^+(B)) \subset F^+(\text{Cl}(B))$ for every $B \subset Y$;

e) $F(e^* - \text{Cl}(A)) \subset \text{Cl}(F(A))$ for every $A \subset X$;

f) $\text{Cl}(\text{Int}(\delta \text{-Cl}(F^+(B)))) \subset F^+(\text{Cl}(B))$ for every subset $B$ of $Y$;

g) $F^-(\text{Int}(B)) \subset \text{Cl}(\text{Int}(\delta \text{-Cl}(F^-(B))))$ for every subset $B$ of $Y$;

**Proof.** This proof is similar to the proof of Theorem (3.1) thus omitted.

**Theorem 3.3.** For a multifunction $F: X \to Y$, the following statements are equivalent:
a) $F$ is upper e-continuous;
b) $F^{-}(V) \in E\Sigma(X, T)$ for every open set $V$ of $Y$;
c) $F^{-}(V) \in EC(X, T)$ for every closed set $V$ of $Y$;
d) $e-Cl(F^{-}(B)) \subset F^{-}(\text{Cl}(B))$ for every $B \subset Y$;
e) For each point $x \in X$ and for each neighborhood $V$ of $F(x)$, $F^{-}(V)$ is an e-neighborhood of $x$;
f) For each point $x \in X$ and for each neighborhood $V$ of $F(x)$, there exists an e-neighborhood $U$ of $x$ such that $F(U) \subset V$;
g) $\text{Cl}(\delta-\text{Int}(F^{-}(B))) \cap \text{Int}(\delta-\text{Cl}(F^{-}(B))) \subset F^{-}(\text{Cl}(B))$ for every subset $B$ of $Y$;
h) $F^{-}(\text{Int}(B)) \subset \text{Int}(\delta-\text{Cl}(F^{-}(B))) \cup \text{Cl}(\delta-\text{Int}(F^{-}(B)))$ for every subset $B$ of $Y$.

**Proof.** (a) $\Rightarrow$ (b): Let $V$ be any open set of $Y$ and $x \in F^{-}(V)$, there exists $U \in E\Sigma(X, T)$ such that $F(U) \subset V$. Therefore, we obtain $x \in U \subset \text{Cl}(\delta-\text{Int}(U)) \cup \text{Int}(\delta-\text{Cl}(U)) \subset \text{Cl}(\delta-\text{Int}(F^{-}(V))) \cup \text{Int}(\delta-\text{Cl}(F^{-}(V)))$. We have $F^{-}(V) \subset \text{Cl}(\delta-\text{Int}(F^{-}(V))) \cup \text{Int}(\delta-\text{Cl}(F^{-}(V)))$ and hence $F^{-}(V) \in E\Sigma(X, T)$.

(b) $\iff$ (c): from the fact that $F^{-}(Y \setminus B) = X \setminus F^{-}(B)$ for every subset $B$ of $Y$

This proof follows immediately.

(c) $\Rightarrow$ (d): since $\text{Cl}(B)$ is closed in $Y$ for any subset $B$ of $Y$, and $F^{-}(\text{Cl}(B))$ is e-closed in $X$. Therefore, we obtain $e-Cl(F^{-}(B)) \subset F^{-}(\text{Cl}(B))$.

(d) $\Rightarrow$ (c): Let $V$ be any closed set of $Y$. Then we have:

e-Cl(F^{-}(V)) \subset F^{-}(\text{Cl}(V)). This shows that $F^{-}(V)$ is e-closed in $X$.

(e) $\Rightarrow$ (f): Let $x \in X$ and $V$ be a neighborhood of $F(x)$. Put $U = F^{-}(V)$, Then $U$ is an e-neighborhood of $x$.

(f) $\Rightarrow$ (a): Let $x \in X$ and $V$ be any open set of $Y$ such that $F(x) \subset V$. Then $V$ is a neighborhood of $F(x)$. There exists an e-neighborhood $U$ of $x$ such that $F(U) \subset V$. Therefore there exists, $A \in E\Sigma(X, T)$ such that $x \in A \subset U$, hence $F(A) \subset V$.

(g) $\Rightarrow$ (h): By replacing $Y \setminus B$ instead of $B$ in (g), we have:

$h) \Rightarrow$ (b): Let $V$ be any open set of $Y$. Then, by using (h) we have: $F^{-}(V) \in E\Sigma(X, T)$ and this completes the proof.

**Theorem 3.4.** For a multifunction $F: X \rightarrow Y$, the following statements are equivalent:

a) $F$ is lower e-continuous;
b) $F^{-}(V) \in E\Sigma(X, T)$ for every open set $V$ of $Y$;
c) $F^{-}(V) \in EC(X, T)$ for every closed set $V$ of $Y$;
d) $e-Cl(F^{-}(B)) \subset F^{-}(\text{Cl}(B))$ for every $B \subset Y$;
e) $F(e-Cl(A)) \subset \text{Cl}(F(A))$ for every $A \subset X$;
f) $\text{Cl}(\delta-\text{Int}(F^{-}(B))) \cap \text{Int}(\delta-\text{Cl}(F^{-}(B))) \subset F^{-}(\text{Cl}(B))$ for every subset $B$ of $Y$. 

g) $F^{-1}(\text{Int}(B)) \subseteq \text{Int}(\delta^-\text{Cl}(F^{-1}(B))) \cup \text{Cl}(\delta^-\text{Int}(F^{-1}(B)))$ for every subset $B$ of $Y$.

**Proof.** This Proof is Similar to that of theorem (3.3). Thus is omitted.

**Theorem 3.5.** Let $F: (X, T) \rightarrow (Y, T^*)$ and $F^*: (Y, T^*) \rightarrow (Z, T^{**})$ be a multifunctions.
If $F: X \rightarrow Y$ is upper (lower) e-continuous (resp. upper (lower) $e^*$-continuous) multifunction and $F^*: Y \rightarrow Z$ is upper (lower) simecontinuous multifunction, then $F^*oF: X \rightarrow Z$ is an upper (lower) e-continuous (resp. upper (lower) $e^*$-continuous) multifunction.

**Proof.** Let $V$ be any open subset of $Z$, using the definition of $F^*oF$, we obtain $(F^*oF)^+(V) = F^+(F^*+(V))$ (resp. $(F^*oF)^-(V) = F^- (F^*-(V)))$. Since $F^*$ is upper (lower) simecontinuous multifunction, it follows that $F^+(V)$ (resp. $F^-(V)$) is an open set. Since $F$ is upper (lower) e-continuous (resp. upper (lower) $e^*$-continuous) multifunction, it follows that $F^+(F^*+(V))$ (resp. $F^- (F^*-(V))$) is an e-open (resp. $e^*$-open) set. It show that $F^*oF$ is an upper (lower) e-continuous (resp. upper (lower) $e^*$-continuous) multifunction.

**Remark 3.1.** For a multifunction $F: X \rightarrow Y$, the following implications hold:

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Upper Continuity  Upper $\alpha$-Continuity  Upper Pre-Continuity
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Upper $\beta$-Continuity  Upper $e^*$-Continuity
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However the converses are not true in general by Examples (3.1, 3.2, 3.3, 3.4, 3.5) of [1] and the following examples.

**Example 3.1.** Let $X = Y = \{a, b, c\}$, Define a topology $T = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$ on $X$ and a topology $T^* = \{\emptyset, Y, \{a\}, \{b, c\}\}$ on $Y$ and let $F: (X, T) \rightarrow (Y, T^*)$ be a multifunction defined as follows: $F(x) = \{x\}$ for each $x \in X$. Then $F$ is upper $e$-Continuous (resp. upper $e^*$-continuous) but not upper $\delta$-precontinuous because $\{b, c\}$ is open in $(Y, T^*)$ while $F^+(\{b, c\}) = \{b, c\}$ is not $\delta$-preopen in $(X, T)$.

**Example 3.2.** Let $X = Y = \{a, b, c, d\}$, Define a topology $T = \{\emptyset, X, \{a\}, \{c\}, \{a,b\}, \{a,c\}, \{a,b,c\}, \{a,c,d\}\}$ on $X$ and a topology $T^* = \{\emptyset, Y, \{d\}, \{b, d\}, \{a, b, d\}\}$ on $Y$ and let $F: (X, T) \rightarrow (Y, T^*)$ be a multifunction defined as follows: $F(x) = \{x\}$ for each $x \in X$. Then $F$ is upper $e^*$-Continuous but not upper $e$-Continuous because $\{b, d\}$ is open in $(Y, T^*)$ and $F^+(\{b, d\}) = \{b, d\}$ is not $e$-open in $(X, T)$.

**Example 3.3.** Let $(X, T)$ and $(Y, T^*)$ be define as in Example (3.2). And let $F: (X, T) \rightarrow (Y, T^*)$ be a multifunction defined as follows: $F(x) = \{x\}$ for each $x \in X$. Then $F$ is upper $e^*$-Continuous but neither upper $\beta$-continuous nor upper $b$-
continuous since \( \{d\} \) is open in \((Y, T^*)\) and \(F^+(\{d\}) = \{d\}\) is neither \(\beta\)-open nor \(b^*\)-open in \((X, T)\).

Recall that, for a multifunction \(F: X \rightarrow Y\), the graph multifunction \(G_F: X \rightarrow X \times Y\) is defined as follows: \(G_F(x) = \{x\} \times F(x)\) for every \(x \in X\).

**Lemma 3.1.** [20] For a multifunction \(F: (X, T) \rightarrow (Y, T^*)\), the following hold:

a) \(G_{F^+}(A \times B) = A \cap F^+(B)\),

b) \(G_{F^*}(A \times B) = A \cap F^*(B)\), for any subsets \(A \subset X\) and \(B \subset Y\).

**Theorem 3.6.** Let \(F: (X, T) \rightarrow (Y, T^*)\) be a multifunction such that \(F(x)\) is compact for each \(x \in X\). Then \(F\) is upper e-continuous (resp. upper \(e^*\)-continuous) if and only if \(G_F: X \rightarrow X \times Y\) is upper e-continuous (resp. upper \(e^*\)-continuous).

**Proof.** (Necessity) Suppose that \(F: X \rightarrow Y\) is upper e-continuous (resp. upper \(e^*\)-continuous). Let \(x \in X\) and \(H\) be any open set of \(X \times Y\) containing \(G_F(x)\). For each \(y \in F(x)\), there exist open sets \(U(y) \subset X\) and \(V(y) \subset Y\) such that \((x, y) \in U(y) \times V(y) \subset H\). The family of \(\{V(y): y \in F(x)\}\) is an open cover of \(F(x)\) and \(F(x)\) is compact. Therefore, there exist a finite number of points, says, \(y_1, y_2, \ldots, y_n\) in \(F(x)\) such that \(F(x) \subset \bigcup \{ V(y_i): 1 \leq i \leq n \}\). Set \(U = \bigcap \{U(y_i): 1 \leq i \leq n\}\) and \(V = \bigcup \{V(y_i): 1 \leq i \leq n\}\). Then \(U\) and \(V\) are open in \(X\) and \(Y\), respectively, and \(\{x\} \times F(x) \subset U \times V \subset H\). Since \(F\) is upper e-continuous (resp. upper \(e^*\)-continuous), there exists \(U_o \in E^*(\Sigma(X, x))\) (resp. \(U_o \in E^*(\Sigma(X, x))\)) such that \(F(\bigcap U_o) \subset V\). By Lemma (3.1), we have \(U \cap U_o \subset G_F^- \left( U \times V \right)\). Therefore, \(G_F(U \cap U_o) \subset H\). This shows that \(G_F\) is upper e-continuous (resp. upper \(e^*\)-continuous).

(Sufficiency) Suppose that \(G_F: X \rightarrow X \times Y\) is upper e-continuous (resp. upper \(e^*\)-continuous). Let \(x \in X\) and \(V\) be any open set of \(Y\) containing \(F(x)\). Since \(X \times V\) is open in \(X \times Y\) and \(G_F(x) \subset X \times V\), there exists \(U \in E \Sigma(X, x)\) (resp. \(U \in E \Sigma(X, x)\)) such that \(G_F(U) \subset X \times V\). Therefore, by Lemma (3.1) we have \(U \subset G_F^* (X \times V) = F^*(V)\) and hence \(F(U) \subset V\). This shows that \(F\) is upper e-continuous (resp. upper \(e^*\)-continuous).

**Theorem 3.7.** A multifunction \(F: (X, T) \rightarrow (Y, T^*)\) is lower e-continuous (resp. lower \(e^*\)-continuous) if and only if \(G_F: X \rightarrow X \times Y\) is lower e-continuous (resp. lower \(e^*\)-continuous).

**Proof.** (Necessity) Suppose that \(F: X \rightarrow Y\) is lower e-continuous (resp. lower \(e^*\)-continuous). Let \(x \in X\) and \(H\) be any open set of \(X \times Y\) containing \(G_F(x)\). Since \(X \times Y\) is open in \(X \times Y\) and \(G_F(x) \subset X \times V\), there exists \(U \in E \Sigma(X, x)\) (resp. \(U \in E \Sigma(X, x)\)) such that \(G_F(U) \subset X \times V\). Therefore, by Lemma (3.1) we have \(U \subset G_F^- (X \times V) = F^- (V)\) and hence \(F(U) \subset V\). This shows that \(F\) is lower e-continuous (resp. lower \(e^*\)-continuous).

(Sufficiency) Suppose that \(G_F: X \rightarrow X \times Y\) is lower e-continuous (resp. lower \(e^*\)-continuous). Let \(x \in X\) and \(H\) be any open set of \(X \times Y\) such that \(x \in G_F^- (H)\). Since \(H \cap (\{x\} \times F(x)) \neq \emptyset\), there exists \(y \in F(x)\) such that \((x, y) \in H\) and hence \((x, y) \in U \times V \subset H\) for some open sets \(U \subset X\) and \(V \subset Y\). Since \(F(x) \cap V \neq \emptyset\), there exists \(U_o \in E \Sigma(X, x)\) (resp. \(U_o \in E \Sigma(X, x)\)) such that \(U_o \subset F^- (V)\). By Lemma (3.1) we have \(U \cap U_o \subset U \cap F^- (V) = G_F^- (U \times V) \subset G_F^- (H)\). Moreover, \(x \in U \cap U_o \in E \Sigma(X, T)\) (resp. \(x \in U \cap U_o \in E \Sigma(X, T)\)) and hence \(G_F\) is lower e-continuous (resp. lower \(e^*\)-continuous).
(Sufficiency). Suppose that $G_F$ is lower $e$-continuous (resp. lower $e^*$-continuous). Let $x \in X$ and $V$ be an open set in $Y$ such that $x \not\in F^{-1}(V)$. Then $X \times V$ is open in $X \times Y$ and $G_F(x) \cap (X \times V) = \{x\} \times F(x) \cap (X \times V) = \{x\} \times (F(x) \cap V) \neq \emptyset$. Since $G_F$ is lower $e$-continuous (resp. lower $e^*$-continuous), there exists $U \in E \Sigma(X, x)$ (resp. $U \in E^* \Sigma(X, x)$) such that $U \subset F^{-1}(X \times V)$. By Lemma (3.1) we obtain $U \subset F^{-1}(V)$. This shows that $F$ is lower $e$-continuous (resp. lower $e^*$-continuous).

Definitions 3.3. A subset $A$ of a topological space $(X, T)$ is said to be:

a) $\alpha$-regular [14] if for each $a \in A$ and each open set $U$ of $X$ containing $a$, there exists an open set $G$ of $X$ such that $a \in G \subset \text{Cl}(G) \subset U$.

b) $\alpha$-paracompact [32] if every cover of $A$ by open sets of $X$ is refined by a cover of $A$ which consists of open sets of $X$ and is locally finite in $X$;

Lemma 3.2. [14] If $A$ is an $\alpha$-paracompact and $\alpha$-regular set of a topological space $X$ and $U$ is an open neighborhood of $A$, then there exists an open set $G$ of $X$ such that $A \subset G \subset \text{Cl}(G) \subset U$.

Recall that, for a multifunction $F: (X, T) \to (Y, T^*)$, by $\text{Cl}(F): (X, T) \to (Y, T^*)$ [6] we denote a multifunction defined as follows: $(\text{Cl}(F))(x) = \text{Cl}(F(x))$ for each point $x \in X$. Similarly, we can define $e\text{-Cl}(F): X \to Y$, $e^*\text{-Cl}(F): X \to Y$, $\alpha\text{Cl}(F): X \to Y$, $s\text{Cl}(F): X \to Y$, $p\text{Cl}(F): X \to Y$, $b\text{Cl}(F): X \to Y$, $\beta\text{Cl}(F): X \to Y$, and $\delta\text{Cl}(F): X \to Y$.

Lemma 3.3. If $F: (X, T) \to (Y, T^*)$ is a multifunction such that $F(x)$ is $\alpha$-regular and $\alpha$-paracompact for each $x \in X$, then there exists an open set $G$ of $X$ such that $A \subset G \subset \text{Cl}(G) \subset U$.

Proof. The proof is similar to that of Lemma (3.3) of [1].

Theorem 3.8. Let $F: (X, T) \to (Y, T^*)$ be a multifunction such that $F(x)$ is $\alpha$-regular and $\alpha$-paracompact for every $x \in X$. Then the following properties are equivalent:

a) $F$ is upper $e$-continuous (resp. $F$ is upper $e^*$-continuous);
b) $e\text{-Cl}(F)$ is upper $e$-continuous (resp. $e\text{-Cl}(F)$ is upper $e^*$-continuous);
c) $e^*\text{-Cl}(F)$ is upper $e$-continuous (resp. $e^*\text{-Cl}(F)$ is upper $e^*$-continuous);
d) $\alpha\text{Cl}(F)$ is upper $e$-continuous (resp. $\alpha\text{Cl}(F)$ is upper $e^*$-continuous);
e) $s\text{Cl}(F)$ is upper $e$-continuous (resp. $s\text{Cl}(F)$ is upper $e^*$-continuous);
f) $p\text{Cl}(F)$ is upper $e$-continuous (resp. $p\text{Cl}(F)$ is upper $e^*$-continuous);
g) $\delta\text{Cl}(F)$ is upper $e$-continuous (resp. $\delta\text{Cl}(F)$ is upper $e^*$-continuous);
h) $b\text{Cl}(F)$ is upper $e$-continuous (resp. $b\text{Cl}(F)$ is upper $e^*$-continuous);
i) $\beta\text{Cl}(F)$ is upper $e$-continuous (resp. $\beta\text{Cl}(F)$ is upper $e^*$-continuous);
j) $\text{Cl}(F)$ is upper $e$-continuous (resp. $\text{Cl}(F)$ upper $e^*$-continuous).

Proof. We set $G = e\text{-Cl}(F)$, $e^*\text{-Cl}(F)$, $\alpha\text{Cl}(F)$, $s\text{Cl}(F)$, $p\text{Cl}(F)$, $\delta\text{Cl}(F)$, $b\text{Cl}(F)$, $\beta\text{Cl}(F)$ or $\text{Cl}(F)$. Suppose that $F$ is upper $e$-continuous (resp. upper $e^*$-continuous).
continuous). Let \( x \in X \) and \( V \) be any open set of \( Y \) containing \( G(x) \). By lemma (3.3), we have \( x \in G'(V) = F'(V) \) and hence there exists \( U \in E \Sigma(X, x) \) (resp. \( U \in E \Sigma(X, x) \)) such that \( F(U) \subset V \). Since \( F(u) \) is \( \alpha \)-regular and \( \alpha \)-paracompact for each \( u \in U \), By lemma (3.2), There exists an open set \( H \) such that \( F(u) \subset H \subset Cl(H) \subset V \); hence \( G(u) \subset Cl(H) \subset V \) for every \( u \in U \). Therefore, we obtain \( G(U) \subset V \). This shows that \( G \) upper e-continuous (resp. upper e*-continuous).

(Conversely), suppose that \( G \) is upper e-continuous (resp. upper e*-continuous). Let \( x \in X \) and \( V \) be any open set of \( Y \) containing \( F(x) \). By Lemma (3.3) we have \( x \in F'(V) = G'(V) \) and hence \( G(x) \subset V \). There exists \( U \in E \Sigma(X, x) \) (resp. \( U \in E \Sigma(X, x) \)) such that \( F(U) \subset V \). Therefore, we obtain \( U \subset G'(V) = F'(V) \) and hence \( F(U) \subset V \). This shows that \( F \) is upper e-continuous (resp. upper e*-continuous).

**Lemma 3.4.** If \( F: X \to Y \) is a multifunction, then, for each open set \( V \) of \( Y \), \( G^{-}(V) = F^{-}(V) \), where \( G \) denotes \( \mathcal{e}-Cl(F) \), \( \mathcal{e}^{*}-Cl(F) \), \( \alpha Cl(F) \), \( s Cl(F) \), \( p Cl(F) \), \( \delta p Cl(F) \), \( b Cl(F) \), \( \beta Cl(F) \) or \( Cl(F) \).

**Proof.** The proof is similar to that of Lemma (3.4) of [1].

**Theorem 3.9.** For a multifunction \( F: X \to Y \), the following properties are equivalent:

a) \( F \) is lower e-continuous (resp. \( F \) is lower e*-continuous);

b) \( \mathcal{e}-Cl(F) \) is lower e-continuous (resp. \( \mathcal{e}-Cl(F) \) is lower e*-continuous);

c) \( \mathcal{e}^{*}-Cl(F) \) is lower e-continuous (resp. \( \mathcal{e}^{*}-Cl(F) \) is lower e*-continuous);

d) \( \alpha Cl(F) \) is lower e-continuous (resp. \( \alpha Cl(F) \) is lower e*-continuous);

e) \( s Cl(F) \) is lower e-continuous (resp. \( s Cl(F) \) is lower e*-continuous);

f) \( \mathcal{p} Cl(F) \) is lower e-continuous (resp. \( \mathcal{p} Cl(F) \) is lower e*-continuous);

g) \( \delta Cl(F) \) is lower e-continuous (resp. \( \delta Cl(F) \) is lower e*-continuous);

h) \( \beta Cl(F) \) is lower e-continuous (resp. \( \beta Cl(F) \) is lower e*-continuous);

i) \( Cl(F) \) is lower e-continuous (resp. \( Cl(F) \) lower e*-continuous).

**Proof.** By using Lemma (3.4) this is shown similarly as in Theorem (3.8).

Recall that, for two multifunctions \( F: X_1 \to Y_1 \) and \( F^*: X_2 \to Y_2 \), the product multifunction \( F \times F^*: X_1 \times X_2 \to Y_1 \times Y_2 \) is defined as follows: \( (F \times F^*)(x_1, x_2) = F(x_1) \times F^*(x_2) \) for each \( x_1 \in X_1 \) and \( x_2 \in X_2 \).

**Lemma 3.5.** [21] For two multifunctions \( F: X_1 \to Y_1 \) and \( F^*: X_2 \to Y_2 \), the following hold:

\begin{enumerate}
\item \( (F \times F^*)^{-}(A \times B) = F^-(A) \times F^{*-}(B) \).
\item \( (F \times F^*)^{-}(A \times B) = F^-(A) \times F^{*-}(B) \) for any \( A \subset X_1 \) and \( B \subset X_2 \).
\end{enumerate}

**Theorem 3.10.** If \( F: X_1 \to Y_1 \) and \( F^*: X_2 \to Y_2 \) are upper e-continuous (resp. upper e*-continuous) multifunctions, then \( F \times F^*: X_1 \times X_2 \to Y_1 \times Y_2 \) is upper e-continuous (resp. upper e*-continuous).
Proof. Let \((x_1, x_2) \in X_1 \times X_2\) and \(H\) be any open set of \(Y_1 \times Y_2\) containing \(F(x_1) \times F^*(x_2)\). There exist open sets \(V_1\) and \(V_2\) of \(Y_1\) and \(Y_2\), respectively, such that \(F(x_1) \times F^*(x_2) \subseteq V_1 \times V_2 \subseteq H\). Since \(F\) and \(F^*\) are upper \(e\)-continuous (resp. upper \(e^*\)-continuous), there exist \(U_1 \in E(X_1, x_1)\) (resp. \(U_1 \in E^*(X_1, x_1)\)) and \(U_2 \in E(X_2, x_2)\) (resp. \(U_2 \in E^*(X_2, x_2)\)) such that \(F(U_1) \subseteq V_1\) and \(F^*(U_2) \subseteq V_2\). By lemma (3.5) we obtain
\[
U_1 \times U_2 \subseteq F^*(V_1) \times F^*(V_2) = (F \times F^*)(V_1 \times V_2) \subseteq (F \times F^*)(H).\]
Therefore, we have \(U_1 \times U_2 \in E(X_1 \times X_2, (x_1, x_2))\) (resp. \(U_1 \times U_2 \in E^*(X_1 \times X_2, (x_1, x_2))\) and \(F \times F^*(U_1 \times U_2) \subseteq H\). This shows that \(F \times F^*\) is upper \(e\)-continuous (resp. upper \(e^*\)-continuous).

**Theorem 3.11.** If \(F\): \(X_1 \to Y_1\) and \(F^*\): \(X_2 \to Y_2\) are lower \(e\)-continuous (resp. lower \(e^*\)-continuous) multifunctions, then \(F \times F^*\): \(X_1 \times X_2 \to Y_1 \times Y_2\) is lower \(e\)-continuous (resp. lower \(e^*\)-continuous).

**Proof.** This Proof is Similar to that of theorem (3.10).

### 4. Some properties of upper (lower) \(e\) \((e^*)\)-Continuous Multifunctions

**Definition 4.1.** A topological space \((X, T)\) is said to be:

a) \(e\)-closed [5] if every cover of \(X\) by \(e\)-open sets has a finite subcover whose preclosures cover \(X\). Some properties

b) \(e^*\)-closed if every cover of \(X\) by \(e^*\)-open sets has a finite subcover whose preclosures cover \(X\).

**Definitions 4.2.** A space \(X\) is said to be, \(e\)-compact [13] (resp. \(e^*\)-compact if every cover of \(X\) by \(e\)-open (resp. \(e^*\)-open) sets has a finite sub cover. It can be easily seen that every \(e\)-compact (resp. \(e^*\)-compact) space is \(e\)-closed (resp. \(e^*\)-closed) space.

**Theorem 4.1.** Let \(F\): \((X, T) \to (Y, T^*)\) be an upper \(e\)-continuous (resp. upper \(e^*\)-continuous) surjective multifunction such that \(F(x)\) is compact for each \(x \in X\). If \(X\) is \(e\)-closed (resp. \(e^*\)-closed) space, then \(Y\) is compact.

**Proof.** Let \(\{V_\lambda\}_{\lambda \in \Delta}\) be an open cover of \(Y\). For each \(x \in X\), \(F(x)\) is compact and there exists a finite subset \(\Delta(x)\) of \(\Delta\) such that \(F(x) \subseteq \bigcup \{V_\lambda\}_{\lambda \in \Delta(x)}\). Set \(V(x) = \bigcup \{V_\lambda\}_{\lambda \in \Delta(x)}\). Since \(F\) is upper \(e\)-continuous (resp. upper \(e^*\)-continuous), there exists \(U(x) \subseteq E(X, T)\) (resp. \(U(x) \subseteq E^*(X, T)\)) containing \(x\) such that \(F(U(x)) \subseteq V(x)\). The family \(\{U(x)\}_{x \in X}\) is an \(e\)-open (resp. \(e^*\)-open) cover of \(X\) and there exist a finite number of points, say, \(x_1, x_2, \ldots, x_n\) in \(X\) such that \(X = \bigcup \{U(x_i)\}_{1 \leq i \leq n}\). Therefore, we have \(Y = F(X) = \bigcup_{i=1}^{n} F(U(x_i))\).
Corollary 4.1. Let $F: (X, T) \to (Y, T^*)$ be an upper e-continuous (resp. upper $e^*$-continuous) surjective multifunctions such that $F(x)$ is compact for each $x \in X$. If $X$ is e-compact (resp. $e^*$-compact) space, then $Y$ is compact.

Definitions 4.3. Let $A$ be a subset of a topological space $(X, T)$. The e-frontier (resp. $e^*$-frontier) of $A$, denoted by $e$-$Fr(A)$ (resp. $e^*$-$Fr(A)$) is defined by:

$$e$-$Fr(A) = e$-$Cl(A) \cap e$-$Cl(X \setminus A)$ (resp. $e^*$-$Fr(A) = e^*$-$Cl(A) \cap e^*$-$Cl(X \setminus A)$).

Theorem 4.2. Let $F: X \to Y$ be a multifunction. The set of all points $x$ of $X$ is not upper e-continuous (resp. upper $e^*$-continuous) is identical with the union of the e-frontier (resp. $e^*$-frontier) of the upper inverse images of open sets containing $F(x)$.

Proof. Let $x \in X$ at which $F$ is not upper e-continuous (resp. upper $e^*$-continuous). Then, there exists an open set $V$ of $Y$ containing $F(x)$ such that $U \subseteq (X \setminus F^+(V)) \neq \emptyset$ for every $U \in E\Sigma(X, x)$. Therefore, $x \in e$-$Cl(X \setminus F^+(V)) = X \setminus e$-$Int(F^+(V))$ (resp. $x \in e^*$-$Cl(X \setminus F^+(V)) = X \setminus e^*$-$Int(F^+(V))$).

Conversely, if $F$ is upper e-continuous (resp. upper $e^*$-continuous) at $x$, suppose that $V$ is an open set of $Y$ containing $F(x)$ such that $x \in e$-$Fr(F^+(V))$ (resp. $x \in e^*$-$Fr(F^+(V))$). Then, there exists $U \in E\Sigma(X, x)$ (resp. $U \in E^*\Sigma(X, x)$) such that $U \subseteq F^+(V)$. Hence $x \in e$-$Int(F^+(V))$ (resp. $e^*$-$Int(F^+(V))$). This is a contradiction and hence $F$ not upper e-continuous (resp. upper $e^*$-continuous) at $x$.

Theorem 4.3. Let $F: X \to Y$ be a multifunction. The set of all points $x$ of $X$ is not lower e-continuous (resp. lower $e^*$-continuous) is identical with the union of the e-frontier (resp. $e^*$-frontier) of the lower inverse images of open sets containing $F(x)$.

Proof. The proof is shown similarly as in Theorem (4.2).

Recall in the following $(D, >)$ is directed set, $(F_\lambda)$ is a net of multifunction $F_\lambda: X \to Y$ for every $\lambda \in D$ and $F$ is a multifunction from $X$ into $Y$.

Definitions 4.4. Let $(F_\lambda)_{\lambda \in D}$ be a net of multifunctions from $X$ to $Y$. A multifunction $F^*: X \to Y$ is defined as follows: for each $x \in X$, $F^*(x) = \{y \in Y: \text{for each open neighborhood } V \text{ of } y \text{ and each } \eta \in D, \text{ exists } \lambda \in D \text{ such that } \lambda > \eta \text{ and } V \cap F_\lambda(x) \neq \emptyset\}$ is called the upper topological limit [7] of the net $(F_\lambda)_{\lambda \in D}$.

Definitions 4.5. A net $(F_\lambda)_{\lambda \in D}$ is said to be equally upper e-(resp. equally upper $e^*$-) continuous at $x_0 \in X$ if for every open set $V_\lambda$ containing $F_\lambda(x_0)$, there exists a $U \in E\Sigma(X, x_0)$ (resp. $U \in E^*\Sigma(X, x_0)$) such that $F_\lambda(U) \subseteq V_\lambda$ for all $\lambda \in D$. 
Theorem 4.4. Let \((F_\lambda)_{\lambda \in D}\) be a net of multifunctions from a topological space \((X, T)\) into a compact topological space \((Y, T^*)\). If the following are satisfied:

a) \(\bigcup \{ F_\eta(x): \eta > \lambda \} \) is closed in \(Y\) for each \(\lambda \in D\) and each \(x \in X\),

b) \((F_\lambda)_{\lambda \in D}\) is equally upper e-(resp. equally upper e*-) continuous on \(X\), then \(F^*\) upper e-(resp. upper e*-) continuous on \(X\).

Proof. From Definition (4.4) and part (a), we have \(F^*(x) = \bigcap \{(\bigcup \{ F_\eta(x): \eta > \lambda \}): \lambda \in D\}\). Since the net \((\bigcup \{ F_\eta(x): \eta > \lambda \})_{\lambda \in D}\) is a family of closed sets having the finite intersection property and \(Y\) is compact, it is follow that \(F^*(x) \neq \emptyset\) for each \(x \in X\). Now, let \(x_0 \in X\) and let \(V \in T^*\) such that \(V \neq Y\) and \(F^*(x_0) \subseteq V\). Since \(F^*(x_0) \cap (Y \setminus V) = \emptyset\), \(F^*(x_0) \neq \emptyset\) and \((Y \setminus V) \neq \emptyset\), \(\bigcap \{(\bigcup \{ F_\eta(x_0): \eta > \lambda \}): \lambda \in D\} = \emptyset\). Since \(Y\) is compact and the family \(\{(\bigcup \{ F_\eta(x_0): \eta > \lambda \}): \lambda \in D\}\) is a family of closed sets with the empty intersection, there exists \(\lambda \in D\) such that for each \(\eta \in D\) with \(\eta > \lambda\) we have \(F_\eta(x_0) \cap (Y \setminus V) = \emptyset\); hence \(F_\eta(x_0) \subseteq V\). Since the net \((F_\lambda)_{\lambda \in D}\) is equally upper e-(resp. equally upper e*-) continuous on \(X\), there exists a \(U \in E \Sigma(X, x_0)\) (resp. \(U \in E^* \Sigma(X, x_0)\)) such that \(F_\eta(U) \subseteq V\) for each \(\eta > \lambda\), hence \(F^*(U) \subseteq V\) if \(V = Y\), then it is clear that for each \(U \in E \Sigma(X, x_0)\) (resp. \(U \in E^* \Sigma(X, x_0)\)) we have \(F^*(U) \subseteq V\). Hence \(F^*\) is upper e-(resp. upper e*-) continuous at \(x_0\). Since \(x_0\) is arbitrary, the proof completes.

References


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