Fixed Point Theorems and $\nabla^*$-Distance

in Partially Ordered $D^*$-Metric Spaces

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Abstract

In this paper, we introduce a new concept on a complete partially ordered $D^*$-
metric space by using the concept of $D^*$-metric space it's called $\nabla^*$-distance
which is a generalized of the concept of a $w$-distance. The main result of our paper
is prove some fixed point theorems in complete partially ordered $D^*$-metric space.

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1. Introduction and preliminaries

The study of metric fixed point theory has been researched extensively in the
past decades, since fixed point theory plays a fundamental role in mathematics
and applied sciences, such as optimization, mathematical models, and economic
theories. The Banach fixed point theorem for contraction mappings has been
generalized and extended in many directions [1-4,6-9,12,14]. Recently, S. Shaban
et al. [17] introduced the concept of $D^*$-metric space which as a probable
modification of the definition of D-metric introduced by Dhage [5]. Afterwards,
many authors [20,21,11] proved some fixed point theorems in these spaces. In
recent times, fixed point theory has developed rapidly in partially ordered metric
spaces such as J.J. Nieto, R.R. Lopez [12, 13], Ran and Reurings [16] and Petruşel and Rus [15] presented some new results for contractions in partially ordered metric spaces. The main idea in [12, 13, 16] involves combining the ideas of an iterative technique in the contraction mapping principle with those in the monotone technique. In this paper, using the concept of D*-metric space, we define a $\nabla^*$-distance on a complete D* -metric space which is a generalization of the concept of $w$-distance due to O. Kada et al. [10]. Using the concept of $\nabla^*$-distance, we prove some fixed point theorems in complete partially ordered D* -metric space, which is the main result of our paper.

**Definition 1.1.** [17] Let X be a non empty set. A D* -metric on X is a function, $D^*: X^3 \rightarrow [0, \infty)$, that satisfies the following conditions for all $x, y, z, a \in X$:

a) $D^*(x, y, z) \geq 0$,
b) $D^*(x, y, z) = 0$ if and only if $x = y = z$,
c) $D^*(x, y, z) = D^*(p\{x, y, z\})$ (symmetry) where $p$ is a permutation function,
d) $D^*(x, y, z) \leq D^*(x, y, a) + D^*(a, z, z)$.

Then the function $D^*$ is called a D*-metric and the pair $(X, D^*)$ is called a D* -metric space.

**Remark 1.1.** [17] In a D*-metric space, for all $x, y, z \in X$, $D^*(x, x, y) = D^*(x, y, y)$.

**Definition 1.2.** [17] Let $(X, D^*)$ be a D*-metric space then:

a) A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that, for each $m, n \geq n_0$, $D^*(x_n, x_n, x_m) < \varepsilon$.
b) If every Cauchy sequence in $(X, D^*)$ is convergent, then $(X, D^*)$ is said to be complete D*-metric.
c) A sequence $\{x_n\}$ in X is said to be converges to point $x \in X$ if and only if for each $\varepsilon > 0$, there exists a positive integer $n_0$ such that for all $m, n \geq n_0$, $D^*(x, x_n, x_m) < \varepsilon$.

For more information on D*-metrics, we refer the reader to [20,21,11,18].

Now, we introduce the concept of $\nabla^*$-distance on a D* -metric space $(X, D^*)$, which is a generalization of $w$-distance due to O. Kada et al. [10].

**Definition 1.3.** Let $(X, D^*)$ be a D*-metric space. Then a function, $\nabla^*: X \times X \times X \rightarrow [0, \infty)$ is called an $\nabla^*$-distance on X if the following conditions are satisfied:

a) $\nabla^*(x, y, z) \leq \nabla^*(x, y, a) + \nabla^*(a, z, z)$ for all $x, y, z, a \in X$,
b) for any $x, y \in X, \nabla^*(x, y, \cdot), \nabla^*(x, \cdot, y):X \rightarrow [0, \infty)$ are lower semi-continuous,
c) For each $\varepsilon > 0$, there exists a $\lambda > 0$ such that $\nabla^*(x, y, a) \leq \lambda$ and $\nabla^*(a, z, z) \leq \lambda$ imply $D^*(x, y, z) \leq \varepsilon$.

**Example 1.1.** Let $(X, d)$ be a metric space and $D^*: X^3 \rightarrow [0, \infty)$ defined by: $D^*(x, y, z) = \max\{d(x, y), d(y, z), d(z, x)\}$ for all $x, y, z \in X$ (see [17]). Then $\nabla^* = D^*$ is an $\nabla^*$-distance on X.
Proof. (a) And (b) are obvious. We show (c). Let $\varepsilon > 0$ be given and put $\lambda = \frac{\varepsilon}{2}$. If $D^*(x, y, a) \leq \lambda$ and $D^*(a, z, z) \leq \lambda$, we have, $d(x, a) \leq \lambda$, $d(y, a) \leq \lambda$, $d(a, z) \leq \lambda$ and $d(y, z) \leq \lambda$, which imply that $D^*(x, y, z) \leq 2 \lambda = \varepsilon$.

**Example 1.2.** Let $(X, D^*)$ be a $D^*$-metric space. Then the function $\nabla^*: X \times X \times X \to [0, \infty)$ defined by $\nabla^*(x, y, z) = t$, for all $x, y, z \in X$ is an $\nabla^*$-distance on $X$, where $t$ is a positive real number.

**Proof.** The proofs of (a) and (b) are immediate. To show (c), for any $\varepsilon > 0$, put $\lambda = \frac{\varepsilon}{2}$. Then $\nabla^*(x, y, a) \leq \lambda$ and $\nabla^*(a, z, z) \leq \lambda$ imply that $D^*(x, y, z) \leq \varepsilon$.

**Example 1.3.** Let $X = R$ be a metric space with the metric $D^*$ defined by:

$D^*(x, y, z) = |x - y| + |y - z| + |z - x|$, for all $x, y, z \in R$ (see [17]). Then the function, $\nabla^*: R^3 \to [0, \infty)$ defined by:

$\nabla^*(x, y, z) = |z - x| + |x - y|$ for all $x, y, z \in R$ is an $\nabla^*$-distance on $R$.

**Proof.** The proofs of (a) and (b) are immediate. We show (c), let $\varepsilon > 0$ be given and put $\lambda = \frac{\varepsilon}{3}$. If $\nabla^*(x, y, a) \leq \lambda$ and $\nabla^*(a, z, z) \leq \lambda$, we have, $|x - a| \leq \lambda$, $|a - z| \leq \lambda$ and $|y - a| \leq \lambda$, which imply that $D^*(x, y, z) \leq \lambda + \lambda + \lambda = 3\lambda = \varepsilon$.

**Lemma 1.1.** Let $X$ be a metric space with metric $D^*$ and $\nabla^*$ be an $\nabla^*$-distance on $X$. Let $\{x_n\}$, $\{y_n\}$ be a sequences in $X$, $\{\delta_n\}$ and $\{\beta_n\}$ be a sequences in $[0, \infty)$ converging to zero and let $x, y, z, a \in X$. Then we have the following:

a) If $\nabla^*(y, y, x_n) \leq \delta_n$ and $\nabla^*(x_n, z, z) \leq \beta_n$ for $n \in N$, then $D^*(y, y, z) < \varepsilon$ and hence $y = z$.

b) If $\nabla^*(y_m, y_m, x_n) \leq \delta_n$ and $\nabla^*(x_n, z, z) \leq \beta_n$ for any $m > n \in N$, then $D^*(y_m, y_m, z)$ converges to 0 and hence $\{y_n\}$ converges to $z$.

c) If $\nabla^*(x_n, x_n, x_n) \leq \delta_n$ for any $n, m \in N$ with $n \leq m$, then $\{x_n\}$ is a $D^*$-Cauchy sequence in $X$.

d) If $\nabla^*(x_n, x_n, a) \leq \delta_n$ for any $n \in N$, then $\{x_n\}$ is a $D^*$-Cauchy sequence in $X$.

**Proof.** First, we prove (b). Let $\varepsilon > 0$ be given. From the definition of $\nabla^*$-distance, there exists $\lambda > 0$ such that $\nabla^*(u, v, a) \leq \lambda$ and $\nabla^*(a, z, z) \leq \lambda$ imply $D^*(u, v, z) \leq \varepsilon$. Choose $n_0 \in N$ such that $\delta_n \leq \lambda$ and $\beta_n \leq \lambda$ for every $n \geq n_0$. Then, for every $m > n \geq n_0$, we have, $\nabla^*(y_m, y_m, x_n) \leq \delta_n \leq \lambda$, $\nabla^*(x_n, z, z) \leq \beta_n \leq \lambda$, and hence $D^*(y_m, y_m, z) \leq \varepsilon$ therefore we obtain $\{y_n\}$ converges to $z$. Proof (a) follows from (b). Now we will prove (c). Let $\varepsilon > 0$ be given. As in the proof of (b), choose $\lambda > 0$ and then $n_0 \in N$. Then, for any $m > n \geq n_0$, $\nabla^*(x_n, x_n, x_{n+1}) \leq \delta_n \leq \lambda$, $\nabla^*(x_{n+1}, x_m, x_m) \leq \delta_{n+1} \leq \lambda$, therefore $D^*(x_n, x_n, x_m) \leq \varepsilon$. This implies that $\{x_n\}$ is a $D^*$-Cauchy sequence. Condition (d) is a special case of (c). So as in the proof of (c), we can prove (d). This completes the proof.
S. Shaban et al. In [17] showed that the function $D^*(x_n, y_n, z_n)$ is continuous on $X^3$ in $D^*$-metric space. Also, $X$ is said to be $\nabla^*$-bounded if there is a constant $M > 0$ such that $\nabla^*(x, y, z) \leq M$ for all $x, y, z \in X$.

### 2. Some fixed point theorems on $\nabla^*$-distance

In this section, we prove some fixed point theorems by using $\nabla^*$-distance in partially ordered $D^*$-metric space.

**Definition 2.1.** [19] Suppose $(X, \preceq)$ is a partially ordered set and $F: X \to X$ is a mapping of $X$ into itself. We say that $F$ is non-decreasing if for $x, y \in X$, $x \preceq y$ implies $F(x) \preceq F(y)$.

**Theorem 2.1.** Let $(X, \preceq)$ be a partially ordered set and suppose that there exists a $D^*$-metric on $X$ such that $(X, D^*)$ is a complete $D^*$-metric space and $\nabla^*$ is an $\nabla^*$-distance on $X$ and $F: X \to X$ be a non-decreasing mapping with respect to $\preceq$. Let $X$ be $\nabla^*$-bounded. Suppose that for all $x \preceq Fx$ and $w \in X$ there exists $r \in [0, 1)$ such that, $\nabla^*(Fx, F^2x, Fw) \leq r \nabla^*(x, Fx, w)$,

Also inf $\{ \nabla^*(x, y, x) + \nabla^*(y, Fx) + \nabla^*(x, F^2x, y): x \leq Fx \} > 0$ for every $x, y \in X$ with $y \neq Fy$. If there exists $x_0 \in X$ with $x_0 \preceq Fx_0$, then $F$ has a fixed point. Furthermore, if $v = Fv$, then $\nabla^*(v, v, v) = 0$.

**Proof.** We will discuss two cases (a) $Fx_0 = x_0$, (b) $Fx_0 \neq x_0$.

(a) If $Fx_0 = x_0$ then the proof is finished.

(b) Suppose that $Fx_0 \neq x_0$. Since $x_0 \preceq Fx_0$ and $F$ is non-decreasing mapping, we obtain $x_0 \preceq Fx_0 \preceq F^2x_0 \preceq \ldots \preceq F^{m+1}x_0 \preceq \ldots$.

For all $n \in \mathbb{N}$ and $t \geq 0$, $\nabla^*(F^{m}x_0, F^{m+1}x_0, F^{m+1}x_0) \leq r \nabla^*(F^{m-1}x_0, F^{m}x_0, F^{m+1}x_0) \leq \ldots$.

Thus, for any $m > n$ in which $m = n + k$, $(k \in \mathbb{N})$, we have:

$$\nabla^*(F^{m}x_0, F^{m+1}x_0) \leq r^n \nabla^*(F^{n}x_0, F^{n+1}x_0) \leq \ldots \leq \sum_{j=n}^{m-1} r^j M = \frac{r^n}{1-r} M.$$  

By part (c) of Lemma (1.1), $\{F^n x_0\}$ is a $D^*$-Cauchy sequence. Since $X$ is $D^*$-complete, then, there exists a point $z \in X$ such that $\{F^n x_0\}$ converges to $z$. Let $n \in \mathbb{N}$ be fixed point. Then, by lower semi-continuity of $\nabla^*$, for $m > n$ we have:

$$\nabla^*(F^m x_0, z) \leq \liminf_{m \to \infty} \nabla^*(F^m x_0, F^m x_0, F^m x_0) \leq \frac{r^n}{1-r} M ,$$

and

$$\nabla^*(F^m x_0, z) \leq \liminf_{m \to \infty} \nabla^*(F^m x_0, F^m x_0, F^m x_0) \leq \frac{r^n}{1-r} M .$$
Suppose that $Fz \neq z$. Since $F^n x_0 \leq F^{n+1} x_0$. Then, by hypothesis, we have:

$$0 \leq \inf \{ \nabla^* (F^n x_0, z, F^n x_0) + \nabla^* (F^n x_0, z, F^{n+1} x_0) + \nabla^* (F^n x_0, F^{n+2} x_0, z) : n \in \mathbb{N} \}$$

$$\leq \inf \{ \left( \frac{r^n}{1-r} \right) M + \left( \frac{r^n}{1-r} \right) M + (\frac{r^n}{1-r}) M : n \in \mathbb{N} \}$$

$$= \inf \{ (\frac{r^n}{1-r}) M : n \in \mathbb{N} \} = 0,$$

This is a contradiction. Therefore, we have $Fz = z$.

Now, if $Fv = v$, then we have:

$$\nabla^* (v, v, v) = \nabla^* (Fv, F^2 v, F^3 v) \leq r \nabla^* (v, Fv, F^2 v) = r \nabla^* (v, v, v),$$

so $\nabla^* (v, v, v) = 0$.

**Theorem 2.2.** Let $(X, \leq)$ be a partially ordered set and suppose that there exists a $D^*$-metric in $X$ such that $(X, D^*)$ is a complete $D^*$-metric space and $w \in X$. Hence, by part (a) of Lemma (1.1), we have $D^* (Fv, F^2 v, F^3 v) = 0$. Further suppose that there exists $v \in X$ such that $\forall n \in X, D^* (Fv, F^n v, x) = 0$. Then, by hypothesis, we have:

(a), (b) If $Fy \neq y$, then $\forall x \in X, \forall \in X$, and $\forall v \in X, Fv = v$, then $\forall x \in X, D^* (Fv, F^n v, x) = 0$. Further suppose that there exists $v \in X$ such that $\forall n \in X, D^* (Fv, F^n v, x) = 0$. Then, by hypothesis, we have:

$$\nabla^* (y, y, x) = \nabla^* (x, F^n v, F^2 x, y) = 0,$$

Thus we have:

$$\lim_{n \to \infty} \nabla^* (x_n, y, x_n) + \nabla^* (y, x_n, Fx_n) + \nabla^* (x_n, F^2 x_n, y) = 0.$$

Hence, by part (a) of Lemma (1.1), we have $\lim_{n \to \infty} D^* (y, F^n v, x) = 0$. And by Using the continuity of the $D^*$-metric, $\lim_{n \to \infty} Fx_n = \lim_{n \to \infty} F^2 x_n = y$.

Also we have:

$$\lim_{n \to \infty} \nabla^* (Fv, Fy, Fx_n) \leq r \lim_{n \to \infty} \nabla^* (y, y, x_n) = 0,$$

$$\lim_{n \to \infty} \nabla^* (Fx_n, Fy) \leq r \lim_{n \to \infty} \nabla^* (Fv, F^2 x_n, Fy)$$

$$\leq r \lim_{n \to \infty} \nabla^* (x_n, Fx_n, y)$$

$$\leq r \lim_{n \to \infty} \nabla^* (x_n, F^2 x_n, y) = 0.$$

Hence, by part (a) of Lemma (1.1), we have $D^* (Fy, y, Fy) = 0$ and hence $Fy = y$, and (a) $\Rightarrow$ (b). Next, we show that (c) $\Rightarrow$ (b).

Let $F$ be continuous mapping. Further suppose that $\{x_n\}$ and $\{Fx_n\}$ converge to $y$.
Then we have $Fy = F(\lim_{n \to \infty} x_n) = \lim_{n \to \infty} Fx_n = y$.

Next, we recall the following example to shows the validity of Theorem (2.1).

**Example 2.1.** Consider Example (1.3) Define a mapping $F: [-1, 1] \to [-1, 1]$ as follows:

$$Fx = \frac{x}{2} \text{ for all } x \in \mathbb{R}.$$ 

Then $F$ is continuous and non-decreasing mapping with respect to $\leq$. Then we have:

$$\nabla^*(Fx, F^2x, Fw) = (|F^2x - Fx| + |Fx - Fw|)
= (|\frac{x}{4} - \frac{x}{2}| + |\frac{x}{2} - \frac{w}{2}|)
= \frac{1}{2} (|\frac{x}{2} - x| + |x - w|)
= \frac{1}{2} \nabla^*(x, Fx, w)$$

For all $x, w \in [-1, 1]$ with $y \neq Fy$ implies $y \neq 0$, Then we have:

$$\inf \{ \nabla^*(x, y, x) + \nabla^*(x, y, Fx) + \nabla^*(x, F^2x, y); x \leq Fx \} > 0.$$ 

Therefore, all the conditions of Theorem (2.1) are satisfied and then $F$ has a fixed point $x = 0$. Moreover, $\nabla^*(0, 0, 0) = 0$.

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**References**


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