Cyclicity of the Multiplication Operator on Some Function Spaces

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Abstract

This paper characterizes some sufficient conditions for a vector in a Hilbert space, with special reproducing kernel, to be cyclic for the multiplication operator.

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1 Introduction

Let $H$ be a Hilbert space of complex-valued analytic functions on the open unit disc $D$ such that point evaluations are bounded linear functionals on $H$. Then for every $w \in D$ there exists a function $k_w$ in $H$ such that $f(w) = \langle f, k_w \rangle$ for all $f \in H$. Now if we define $K : D \times D \rightarrow \mathbb{C}$ by

$$K(z, w) = k_w(z),$$

then $K$ is a positive definite function with the reproducing property

$$f(w) = \langle f(\cdot), K(\cdot, w) \rangle$$

for every $w \in D$ and $f \in H$. The function $K$ is called the reproducing kernel for $H$.

Note that if $K : D \times D \rightarrow \mathbb{C}$ is positive definite then

$$\{ \sum_{j=1}^{n} a_j K(\cdot, w_j) : a_1, \ldots, a_n \in \mathbb{C} \text{ and } w_1, \ldots, w_n \in D \}$$
is dense in a Hilbert space $H(K)$ of functions with

$$\| \sum_{j=1}^{n} a_j K(\cdot, w_j) \|^2 = \sum_{j,k=0}^{n} a_j \bar{a}_k K(w_j, w_k)$$

and

$$f(w) = \langle f(\cdot), K(\cdot, w) \rangle$$

for every $w$ in $D$ and $f$ in $H(K)$. Thus evaluation at $w$ is a bounded linear functional for each $w$ in $D$. Note also that convergence in $H(K)$ implies uniform convergence on compact subsets of $D$. Some sources on spaces of analytic functions are [1–18].

Throughout this paper $H$ is a Hilbert space of analytic functions on $D$ such that $1 \in H$, $zH \subset H$ and point evaluations are bounded for every $w \in D$.

We say that a vector $f$ in a Hilbert space $H$ is a cyclic vector of a bounded operator $A$ on $H$ if

$$H = \text{span}\{ A^n f : n = 0, 1, 2, \ldots \}.$$ 

Here $\text{span}\{\cdot\}$ is the closed linear span of the set $\{\cdot\}$.

## 2 Main Result

In the main theorems of this paper we give sufficient conditions for a vector $f$ in $H(K)$ to be cyclic for the multiplication operator, $M_z$, acting on $H(K)$.

**Theorem 2.1** Let $H = H(K)$ have a reproducing kernel of the form

$$K(z, w) = \sum_{n=0}^{\infty} a_n (zw)^n,$$

where $a_n > 0$ and there exists a number $c > 0$ such that

$$\sup_{k,n} \frac{a_{k+1}a_{n+1}}{a_{k+n+1}} \leq c,$$

and

$$\left\{ \frac{a_n}{a_{n+1}} \right\}_n \in \ell^1.$$

If $f$ is a vector in $H$ with $f(0) \neq 0$, then $f$ is a cyclic vector for the operator $M_z$. 

Proof. Put
\[ M = \text{span}\{M^n_z f : n = 0, 1, 2, \ldots\} \]
and let
\[ g = \sum_{n=0}^{\infty} g_n z^n \]
be any vector in \( M^\perp \). It is sufficient to show that \( g \equiv 0 \). Note that
\[
0 = \langle M^n_z f, g \rangle = \langle \sum_{k=0}^{\infty} f_k z^{n+k}, g \rangle = \sum_{k=n}^{\infty} f_k - n \bar{g}_k \| z^k \|^2 = f_0 \bar{g}_n \| z^n \|^2 + \sum_{k=n+1}^{\infty} f_k - n \bar{g}_k \| z^k \|^2.
\]
Hence
\[
|\bar{g}_n| \| z^n \| \leq \frac{1}{|f_0|} \| z^n \| \sum_{k=n+1}^{\infty} |f_k - n| \bar{g}_k \| z^k \|^2.
\]
\[
= \frac{1}{|f_0|} \sum_{k=n+1}^{\infty} \left( |f_k - n| \| z^{k-n} \| \| z^k \| \| \bar{g}_k \| \| z^k \| \right) \frac{\| z^k \|}{\| z^{k-n} \| \| z^n \|}.
\]
Note that
\[
\frac{\| z^k \|}{\| z^{k-n} \| \| z^n \|} = \left( \frac{a_{n+1} a_{k-n}}{a_k} \right)^{\frac{1}{2}} \leq c^{\frac{1}{2}}.
\]
Now by the Hölder inequality, we obtain
\[
|\bar{g}_n| \| z^n \| \leq \frac{c^{\frac{1}{2}}}{|f_0|} \left( \frac{a_n}{a_{n+1}} \right)^{\frac{1}{2}} \| f \| \| g \|.
\]
For all \( n \geq 0 \), define
\[
\gamma_n = \frac{c^{\frac{1}{2}}}{|f_0|} \| f \| \left( \frac{a_n}{a_{n+1}} \right)^{\frac{1}{2}}.
\]
Hence, \( \{\gamma_n\}_n \in \ell^2 \) and for all \( n \geq 0 \) and all
\[
\| g \| = \sum_{n=0}^{\infty} g_n z^n
\]
in \( M^\perp \), we get
\[
|g_n| \| z^n \| \leq \gamma_n \| g \|.
\]
Note that for a given $\varepsilon > 0$, there exists an integer $N \geq 0$ such that
\[
\sum_{n=N+1}^{\infty} \gamma_n^2 < \frac{\varepsilon^2}{2^2}.
\]
Let $M_N$ be the closed linear span of the set
\[
\{ z^n : n = 0, 1, \ldots, N \}
\]
in $H$. Then for each vector
\[
h(z) = \sum_{n=0}^{\infty} h_n z^n
\]
in ball($M^\perp$), the vector
\[
h^{(N)}(z) = \sum_{n=0}^{N} h_n z^n
\]
is in ball($M_N$). Now we have
\[
\|h^{(N)} - h\| = \left\| \sum_{n=N+1}^{\infty} h_n z^n \right\|
\]
\[
= \left( \sum_{n=N+1}^{\infty} |h_n|^2 \|z^n\|^2 \right)^{\frac{1}{2}}
\]
\[
\leq \|h\| \left( \sum_{n=N+1}^{\infty} \gamma_n^2 \right)^{\frac{1}{2}}
\]
\[
< \frac{\varepsilon}{2}.
\]
Since $M_N$ is finite-dimensional, ball($M_N$) is compact. Thus it follows from the inequality
\[
\|h^{(N)} - h\| < \frac{\varepsilon}{2}
\]
that ball($M^\perp$) is totally bounded. Also, ball($M^\perp$) is complete, thus indeed ball($M^\perp$) is compact. By the F. Riesz theorem, $M^\perp$ is finite-dimensional and so there exists a least integer $m$ such that $M^{*m}|_{M^\perp} = 0$. Thus
\[
0 = M_z^{*m} g = M_z^{*m}(\sum_{k=0}^{\infty} g_k z^k)
\]
\[
= \sum_{k=0}^{\infty} g_{k+m} z^k.
\]
This implies that $g_k = 0$ for all $k \geq m$ and consequently for
\[
h = \sum_{k=0}^{\infty} h_k z^k
\]
in $M$, $h_k = 0$ for all $k < m$. Since $f \in M$ with $f_0 \neq 0$, it follows that $m = 0$. Thus $g_k = 0$ for all $k$ and so $g \equiv 0$. This completes the proof.
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