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On Q-Fuzzy Bi-Γ-Ideals in Γ-Semigroups

S. Lekkoksung

Rajamangala University of Technology Isan
Khon Kaen Campus, Thailand
Lekkoksung_somsak@hotmail.com

Abstract

In this paper, we consider the Q-fuzzification of bi-Γ-ideals in Γ-semigroups, and investigate some of their related properties.

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1 Introduction

The concept of fuzzy sets was introduced by Lofti Zadeh[13] in his classic paper in 1965. Azirel Rosenfeld[8] used the idea of fuzzy set to introduce the notions of fuzzy subgroups. The idea of fuzzy subsemigroup was also introduced by Kuroki [3, 5, 6]. In [4], Kuroki characterized several classes of semigroups in terms of fuzzy left, fuzzy right and fuzzy bi-ideals. The notion of a Γ-semigroup was introduced by Sen and Saha[12] as a general- ization of semigroups and ternary semigroup. S.K. Sardar and S.K. Majumder [1, 2, 10, 11] have introduced the notion of fuzzification of ideals, prime ide- als, semiprime ideals and ideal extensions of Γ-semigroups and studied them via its operator semigroups. In this paper, we consider a Q-fuzzification of the concept of a bi-Γ-ideal in a Γ-semigroup, and some properties of such bi-Γ-ideals are investigated.

2 Preliminary Notes

In this section we discuss some elementary definitions that we use in the sequel.

Definition 2.1 *Let S and Γ be two non-empty sets. S is called a Γ-semigroup if there exist mapping from $S\times\Gamma\times S$ to S, written as $(a,\alpha,b)\rightarrow a\alpha b$ satisfying the following associative law: $(a\alpha b)\beta c = a\alpha(b\beta c)$ for all $a,b,c\in S$ and for all $\alpha,\beta\in\Gamma$.*

Let S be a Γ-semigroup. A non-empty subset A of S is said to be a sub Γ-semigroup of S if $ASA \subseteq A$.

Definition 2.2 *Let S be a Γ-semigroup. By a left(right) ideal of S we mean a non-empty set A of S such that $S\Gamma A \subseteq A(A\Gamma S \subseteq A)$. By two side ideal or simply an ideal, we mean a non-empty subset of S which is both a left and right ideal of S.*

Definition 2.3 *Let S be a Γ-semigroup. A sub Γ-semigroup A of S is called a bi-Γ-ideal of S if $A\Gamma S\Gamma A \subseteq A$.*

Let Q and X be two non-empty sets. A mapping $\mu : X \times Q \to [0,1]$ is called the *Q-fuzzy subset* of X and the complement of a set μ, denoted by $\mu^{'}$ is the Q-fuzzy subset in X given by $\mu^{'}(x,q) = 1 - \mu(x,q)$ for all $x \in X$ and for all $q \in Q$.

Let μ be a Q-fuzzy subset of a non-empty set X. Then the set $\mu_t = \{x \in X \mid \mu(x,q) \geq t, \forall q \in Q\}$ for $t \in [0,1]$, is called the *level subset* or *t-level subset* of μ.

Let A be a non-empty subset of X. Then $\mu_A : X \times Q \to [0,1]$ is called a *characteristic function* which is defined by

$$\mu_A(x,q) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A, \end{cases}$$

for all $x \in S$ and for all $q \in Q$.

Definition 2.4 *Let μ be a Q-fuzzy subset of a Γ-semigroup S. A Q-fuzzy subset μ is called a Q-fuzzy sub Γ-semigroup of S if $\mu(x\gamma y, q) \geq min\{\mu(x,q), \mu(y,q)\}$ for all $x, y \in S, \gamma \in \Gamma$ and $q \in Q$.*

3 Main Results

In what follow, S will denote a Γ-semigroup unless otherwise specified. Now, we introduce a notion of Q-fuzzy bi-Γ-ideal of S.

Definition 3.1 *A Q-fuzzy subset μ of S is called a Q-fuzzy bi-Γ-ideal of S if*
(i) $(\forall x, y \in S, \gamma \in \Gamma, q \in Q)(\mu(x\gamma y, q) \geq min\{\mu(x,q), \mu(y,q)\})$,
(ii) $(\forall x, y, z \in S, \alpha, \beta \in \Gamma, q \in Q)(\mu(x\alpha y\beta z, q) \geq min\{\mu(x,q), \mu(z,q)\})$.

Lemma 3.2 *If B is a bi-Γ-ideal of S then for any $0 < t < 1$, there exists a Q-fuzzy bi-Γ-ideal μ of S such that $\mu_t = B$*

Proof. Let $\mu : S \times Q \to [0,1]$ be defined by

$$\mu(x,q) = \begin{cases} t & \text{if } x \in B \\ 0 & \text{if } x \notin B \end{cases}$$

for all $x \in S$ and for all $q \in Q$, where t is a fixed number in $(0,1)$. Then, clearly $\mu_t = B$.

Now suppose that B is a bi-Γ-ideal of S. For all $x, y \in S, q \in Q$ and $\gamma \in \Gamma$ such that $x\gamma y \in B$, we have

$$\mu(x\gamma y, q) \geq t = \min\{\mu(x,q), \mu(y,q)\}.$$

Also, for all $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$ such that $x\alpha y\beta z \in B$, we have

$$\mu(x\alpha y\beta z, q) \geq t = \min\{\mu(x,q), \mu(y,q)\}.$$

Thus μ is a Q-fuzzy bi-Γ-ideal of S. ■

Lemma 3.3 *Let B be a non-empty subset of S. Then B is a bi-Γ-ideal of S if and only if χ_B is a Q-fuzzy bi-Γ-ideal of S.*

Proof. Let $x, y \in S, q \in Q$ and $\gamma \in \Gamma$. From the hypothesis, $x\gamma y \in B$.

(i) If $x, y \in B$, then $\chi_B(x,q) = 1$ and $\chi_B(y,q) = 1$. In this case $\chi_B(x\gamma y, q) = 1 \geq \min\{\chi_B(x,q), \chi_B(y,q)\}$.

(ii) If $x \in B, y \notin B$, then $\chi_B(x,q) = 1$ and $\chi(y,q) = 0$. Thus, $\chi_B(x\gamma y, q) = 1 \geq \min\{\chi_B(x,q), \chi_B(y,q)\}$.

(iii) If $x \notin B, y \in B$, then, similarly with case (ii).

(iv) If $x \notin B, y \notin B$, then $\chi_B(x,q) = 0$ and $\chi(y,q) = 0$. Thus, $\chi_B(x\gamma y, q) = 1 \geq \min\{\chi_B(x,q), \chi_B(y,q)\}$.

Let $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$. From the hypothesis, $x\alpha\ y\beta z \in B$.

(i) If $x, z \in B$, then $\chi_B(x,q) = 1$ and $\chi_B(z,q) = 1$. Thus $\chi_B(x\alpha y\beta z, q) = 1 \geq \min\{\chi_B(x,q), \chi_B(z,q)\}$.

(ii) If $x \in B$ and $z \notin B$, then $\chi_B(x,q) = 1$ and $\chi_B(z,q) = 0$. Thus $\chi_B(x\alpha y\beta z, q) = 1 \geq \min\{\chi_B(x,q), \chi_B(z,q)\}$.

(ii) If $x \notin B$ and $z \in B$, then $\chi_B(x,q) = 0$ and $\chi_B(z,q) = 1$. Thus $\chi_B(x\alpha y\beta z, q) = 1 \geq \min\{\chi_B(x,q), \chi_B(z,q)\}$.

(ii) If $x \notin B$ and $z \notin B$, then $\chi_B(x,q) = 0$ and $\chi_B(z,q) = 0$. Thus $\chi_B(x\alpha y\beta z, q) = 1 \geq \min\{\chi_B(x,q), \chi_B(z,q)\}$.

Conversely, suppose χ_B is a Q-fuzzy bi-Γ-ideal of S. Then by Lemma 3.2, χ_B is two-value. Hence B is a bi-Γ-ideal of S. This completes the proof. ■

The following theorem proves that an intersection of Q-fuzzy bi-Γ-ideals is also a Q-fuzzy bi-Q-ideal.

Theorem 3.4 *If $\{A_i\}_{i\in\lambda}$ is a family of Q-fuzzy bi-Γ-ideals of S then $\cap A_i$ is a Q-fuzzy bi-Γ-ideal of S, where $\cap A_i = \{\wedge\mu_i\}$ and $\wedge\mu_i(x,q) = \min\{\mu_i(x,q) \mid i \in \lambda\}$ for every $x \in S$ and $q \in Q$.*

Proof. Let $x, y \in S, q \in Q$. Then we have

$$\begin{aligned}
\text{(i)} \quad \wedge\,\mu_{\mathrm{i}}(\mathrm{x}\gamma\mathrm{y},\mathrm{q}) &= \inf\{\min\{\mu_i(x,q),\mu_i(y,q)\} \mid i \in \lambda\} \\
&= \min\{\{\inf(\mu_i(x,q)),\inf(\mu_i(y,q))\} \mid i \in \lambda\} \\
&= \min\{\{\inf(\mu_i(x,q)) \mid i \in \lambda\},\{\inf(\mu_i(y,q)) \mid i \in \lambda\}\} \\
&= \min\{\wedge\mu_i(x,q),\wedge\mu_i(y,q)\}.
\end{aligned}$$

Let $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$.

$$\begin{aligned}
\text{(ii)} \wedge\,\mu_{\mathrm{i}}(\mathrm{x}\alpha\mathrm{y}\beta\mathrm{z},\mathrm{q}) &= \inf\{\min\{\mu_i(x,q),\mu_i(z,q)\} \mid i \in \lambda\} \\
&= \min\{\{\inf(\mu_i(x,q)),\inf(\mu_i(z,q))\} \mid i \in \lambda\} \\
&= \min\{\{\inf(\mu_i(x,q)) \mid i \in \lambda\},\{\inf(\mu_i(z,q)) \mid i \in \lambda\}\} \\
&= \min\{\wedge\mu_i(x,q),\wedge\mu_i(z,q)\}.
\end{aligned}$$

Hence, $\cap A_i$ is a Q-fuzzy bi-Γ-ideal of S. ■

Theorem 3.5 *If μ is a Q-fuzzy bi-Γ-ideal of S then $\mu^{'}$ is also a Q-fuzzy bi-Γ-ideal of M.*

Proof. (i) Let $x, y \in S, q \in Q$ and $\gamma \in \Gamma$. We have:

$$\begin{aligned}
\mu^{'}(x\gamma y,q) &= 1-\mu(x\gamma y,q) \\
&= 1-\min\{\mu(x,q),\mu(y,q)\} \\
&= \min\{1-\mu(x,q),1-\mu(y,q)\} \\
&= \min\{\mu^{'}(x,q),\mu^{'}(y,q)\}.
\end{aligned}$$

Let $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$. We have:

$$\begin{aligned}
\mu^{'}(x\alpha y\beta z,q) &= 1-\mu(x\alpha y\beta z,q) \\
&= 1-\min\{\mu(x,q),\mu(z,q)\} \\
&= \min\{1-\mu(x,q),1-\mu(z,q)\} \\
&= \min\{\mu^{'}(x,q),\mu^{'}(z,q)\}.
\end{aligned}$$

Therefore, $\mu^{'}$ is also a Q-fuzzy bi-Γ-ideal of S. ■

The following therorem gives the relation between Q-fuzzy bi-Γ-ideal and bi-Γ-ideal.

Theorem 3.6 *A Q-fuzzy subset μ in a Γ-ideal of S is a Q-fuzzy bi-Γ-ideal of S if and only if the level set $U(\mu;t) = \{x \in S \mid \mu(x,q) \geq t, \forall q \in Q\}$ is a bi-Γ-ideal of S when it is non-empty.*

Proof. Let μ be a Q-fuzzy bi-Γ-ideal of S and $x, y \in S, q \in Q$. Then $\mu(x\gamma y,q) \geq \min\{\mu(x,q),\mu(y,q)\}$.

$$\begin{aligned} x, y \in U(\mu; t), q \in Q, \gamma \in \Gamma \quad &\Rightarrow \quad \mu(x, q) \geq t, \mu(y, q) \geq t \\ &\quad\quad\; \mu(x\gamma y, q) \geq \min\{\mu(x, q), \mu(y, q)\} \\ &\quad\quad\; \mu(x\gamma y, q) \geq t \\ &\Rightarrow \quad x\gamma y \in U(\mu; t). \end{aligned}$$

Also, let $x, y, z \in S, q \in Q$ and $\alpha, \beta \in \Gamma$. Then

$$\mu(x\alpha y\beta z) \geq \min\{\mu(x, q), \mu(z, q)\}.$$

$$\begin{aligned} x, y, z \in U(\mu; t), q \in Q, \alpha, \beta \in \Gamma \quad &\Rightarrow \quad \mu(x, q) \geq t, \mu(y, q) \geq t, \mu(z, q) \geq t \\ &\quad\quad\; \mu(x\alpha y\beta z, q) \geq \min\{\mu(x, q), \mu(z, q)\} \\ &\quad\quad\; \mu(x\alpha y\beta z, q) \geq t \\ &\Rightarrow \quad x\alpha y\beta z \in U(\mu; t). \end{aligned}$$

Thus, $U(\mu; t)$ is a bi-Γ-ideal of S.

Conversely, assume that $U(\mu; t)$ is a bi-Γ-ideal of S.

Let $t = \min\{\mu(x, q), \mu(y, q)\}$. Then

$$\begin{aligned} x, y \in U(\mu; t), q \in Q, \gamma \in \Gamma \quad &\Rightarrow \quad x\gamma y \in U(\mu; t) \\ &\Rightarrow \quad \mu(x\gamma y, q) \geq t \\ &\Rightarrow \quad \mu(x\gamma y, q) \geq \min\{\mu(x, q), \mu(y, q)\}. \end{aligned}$$

Next, define $t = \min\{\mu(x, q), \mu(z, q)\}$. Then

$$\begin{aligned} x, y, z \in U(\mu; t), q \in Q, \alpha, \beta \in \Gamma \quad &\Rightarrow \quad x\alpha y\beta z \in U(\mu; t) \\ &\Rightarrow \quad \mu(x\alpha y\beta z, q) \geq t \\ &\Rightarrow \quad \mu(x\alpha y\beta z, q) \geq \min\{\mu(x, q), \mu(z, q)\}. \end{aligned}$$

Hence μ is a Q-fuzzy bi-Γ-ideal of S. ■

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