Estimates for Univalent Functions

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Abstract

For normalized analytic functions $f$ in the open unit disk $U := \{ z : |z| < 1 \}$, we consider the subclass

$$S^*(\alpha) = \left\{ f : \frac{zf'(z)}{f(z)} < \frac{\alpha + z}{\alpha - z} \right\}, \quad z \in U,$$

where $0 \leq \alpha < 1$. In this note, we determined an upper bound of the pre-Schwarzian norm for functions $f \in S^*(\alpha)$. Furthermore, by using Jack's Lemma, we pose the sufficient conditions for normalized function $f$ to be in $S^*(\alpha)$. Subordination relations are introduced.

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1 Introduction

Let $U := \{ z : |z| < 1 \}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $\mathcal{H}$ denote the space of all analytic functions on $U$. Here we suppose that $\mathcal{H}$ as a topological vector space endowed with the topology of uniform convergence over compact subsets of $U$. Also for $a \in \mathbb{C}$ and $n \in \mathbb{N}$, let $\mathcal{H}[a,n]$ be the subspace of $\mathcal{H}$ consisting of functions of the form

$$f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots .$$

Further, let $\mathcal{A} := \{ f \in \mathcal{H} : f(0) = f'(0) - 1 = 0 \}$ and $\mathcal{S}$ denote the class of all univalent functions in $\mathcal{A}$. A function $f \in \mathcal{A}$ is called starlike if $f(U)$ is a
starlike domain with respect to the origin, and the class of univalent starlike functions is denoted by $S^*$. It is called convex $C$, if $f(U)$ is a convex domain. Each univalent starlike function $f$ is characterized by the analytic condition

$$\Re\left(\frac{zf'(z)}{f(z)}\right) > 0, \quad z \in U.$$ 

Also, it is known that $zf'(z)$ is starlike if and only if $f$ is convex which characterized by the analytic condition

$$\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > 0, \quad z \in U.$$ 

Let $f \in \mathcal{H}$, and $g$ be a univalent function in $U$, with $f(0) = g(0)$. Then, we say that $f$ is subordinated to $g$ (or $g$ is superordinated to $f$), denoted by $f(z) \prec g(z)$, if $f(U) \subset g(U)$.

The pre-Schwarzian derivative $T_f$ of $f$ is defined by

$$T_f(z) = \frac{f''(z)}{f'(z)},$$

with the norm

$$\|f\| = \sup_{z \in U}|T_f|(1 - |z|^2).$$

It is known that $\|f\| < \infty$ if and only if $f$ is uniformly locally univalent i.e. there exists a constant $r = r(f) > 0$ such that $f$ is univalent in each disk of hyperbolic radius $r$ in $U$. It is also known that $\|f\| \leq 6$ for $f \in S$ and that $\|f\| \leq 4$ for $f \in C$. Moreover, it showed that when $\|f\| \leq 3.05$ then $f$ is univalent in $U$. And when $\|f\| \leq 2.28329$ then $f$ is starlike in $U$ (see [4]). Recently, the sharp norm estimate for well known integral operators are determined (see[1,2,6]).

In this paper, we consider the subclass of $A$ defined by

$$S^*(\alpha) = \left\{ f : \frac{zf'(z)}{f(z)} \prec \frac{\alpha + z}{\alpha - z} \right\}, \quad z \in U,$$

where $0 \leq \alpha < 1$. Our aim is compute the sharp norm estimate for functions $f \in S^*(\alpha)$. Moreover, the sufficient conditions for functions $f \in A$ to be in the class $S^*(\alpha)$ are introduced. We need the following result which can be found in [5]:

**Lemma 1**[3] Let $w$ be analytic in $U$ with $w(0) = 0$. If $|w(z)|$ attains its maximum value on the circle $|z| = r < 1$ at a point $z_0$, then

$$z_0w'(z_0) = kw(z_0),$$

where $k$ is a real number and $k \geq 1$. 
2 Main Results

First our result is in the following form:

**Theorem 1** Let $0 \leq \alpha < 1$. If $f \in S^*(\alpha)$, then

$$\|f\| \leq \Omega(\alpha) + 2,$$

where

$$\Omega(\alpha) = \frac{(1 + \alpha)(2\sigma - 1 + \alpha)(\sigma - 1)}{(\sigma - \alpha)(\sigma + 1)},$$

and $\sigma$ is the unique solution of the following equation in $\tau \in [1, \infty)$:

$$(2\tau - 1 + \alpha)(\tau - 1) = 0.$$  

**Proof.** Let $f \in S^*(\alpha)$. Define the functions

$$p(z) := P_f = \frac{zf'(z)}{f(z)} \quad \text{and} \quad q(z) := \frac{\alpha + z}{1 - z}.$$

Assume that there exists an analytic function $w : U \to U$ with $w(0) = 0$ and

$$p = q \circ w = \frac{\alpha + w}{1 - w}, \quad w \neq 1.$$  

Define a function $F \in A$ such that $P_F = q$, i.e.

$$F(z) = zexp\left(\int_0^z \frac{q(t) - 1}{t} dt\right).$$

We proceed to determine the quantities $T_F(|z|)$ and $T_f(z)$. First we observe that

$$T_F(|z|) = \frac{F''(|z|)}{F'(|z|)} = \frac{q(|z| - 1)}{|z|} + \frac{q'(|z|)}{q(|z|)} = \frac{q(|z| - 1)}{|z|} + \frac{2}{1 - |z|^2}.$$  

And the logarithmic differentiation of (4) gives

$$T_f(z) = \frac{f''(z)}{f'(z)} = \frac{2w'(z)}{1 - w^2(z)} + \frac{q(w) - 1}{z}, \quad z \neq 0.$$
Thus by triangle inequality and Schwarz-Pick lemma, we obtain

\begin{align*}
|T_f(z)| &\leq \frac{2|w'(z)|}{1 - |w^2(z)|} + |q(w) - 1| + \frac{|q(w) - 1|}{|z|} \\
&\leq \frac{2(1 - |w(z)|^2)}{(1 - |z|^2)(1 - w^2(z))} + |q(w) - 1| + \frac{|q(w) - 1|}{|z|} \\
&\leq \frac{2}{(1 - |z|^2)} + \frac{|q(w) - 1|}{|z|}.
\end{align*}

By using the facts that $|q(z) - 1| \leq q(|z|) - 1$ and $|w| < |z|$, yield

\begin{align*}
|T_f(z)| &\leq \frac{2}{(1 - |z|^2)} + \frac{|q(w) - 1|}{|z|} \\
&\leq \frac{2}{(1 - |z|^2)} + \frac{q(|w|) - 1}{|z|} \\
&\leq \frac{2}{1 - |z|^2} + \frac{q(|z|) - 1}{|z|} \\
&= T_F(|z|).
\end{align*}

Consequently we have

\begin{equation*}
(1 - |z|^2)|T_f(z)| = (1 - |z|^2)T_F(|z|),
\end{equation*}

and so $\|f\| \leq \|F\|$. Thus to determine the upper estimate of $f \in S^*(\alpha)$, it is enough to compute $\|F\|$. Now we need to evaluate

\begin{equation*}
\Omega(\alpha) = \sup_{0 < t < 1} \frac{1 - t^2}{t} (q(t) - 1).
\end{equation*}

Changing the variable by $\tau = \frac{\alpha + t}{1 - t}$, we pose

\begin{equation*}
\Omega(\alpha) := \sup_{\tau > 0} g(\tau) = \sup_{\tau > 0} \frac{(1 + \alpha)(2\tau - 1 + \alpha)(\tau - 1)}{(\tau - \alpha)(\tau + 1)}, \quad \tau \leq \frac{1 - \alpha}{2}
\end{equation*}

such that

\begin{equation*}
\lim_{\tau \to 1} g(\tau) = 0.
\end{equation*}

Let

\begin{equation*}
h(\tau) := (2\tau - 1 + \alpha)(\tau - 1);
\end{equation*}

it is clear that $h$ has a unique solution in $[1, \infty)$. Thus $g(\tau)$ assumes its maximum at this solution; hence we have (2). Since $\sigma$ is the unique zero of $h(\tau)$ in
[1, ∞), this implies the equation (3). Thus we establish (1). This completes the proof.

Next by using Jack’s Lemma, we illustrate the sufficient condition that allows functions \( f \in \mathcal{A} \) to belong in the class \( S^*(\alpha) \).

**Theorem 2** Let \( 0 \leq \alpha < 1 \). If \( f \in \mathcal{A} \), satisfies

\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} < \frac{\alpha + 3}{\alpha + 1}, \quad z \in U
\]

then \( f \in S^*(\alpha) \).

**Proof.** Define the function \( w(z) \) by

\[
\frac{zf'(z)}{f(z)} = \frac{\alpha + w(z)}{\alpha - w(z)}
\]

where \( w(z) \) ia analytic in \( U \) and satisfies \( w(0) = 0 \) and

\[
\Re \left\{ 1 + \frac{zf''(z)}{f'(z)} \right\} = \Re \left\{ \frac{2zw'(z)}{\alpha^2 - w^2(z)} + \frac{\alpha + w(z)}{\alpha - w(z)} \right\} < \frac{\alpha + 3}{\alpha + 1}.
\]

Suppose that there exists a point \( z_0 \in U \) such that

\[
\max_{|z| \leq |z_0|} |w(z)| = |w(z_0)| = 1.
\]

Then, using Lemma 1 and letting \( w(z_0) = e^{i\theta} \) and \( z_0w'(z_0) = ke^{i\theta}, \ k \geq 1 \) yields

\[
\Re \left\{ \frac{zf''(z_0)}{f'(z_0)} + 1 \right\} = \Re \left\{ \frac{2ke^{i\theta} + \alpha^2 + 2\alpha e^{i\theta} + e^{2i\theta}}{\alpha^2 - e^{2i\theta}} \right\}
\]

\[
= \frac{2k + \alpha^2 + 2\alpha}{\alpha^2 - 1}, \quad e^{i\theta} \to 1
\]

\[
> \frac{\alpha + 3}{\alpha - 1},
\]

which contradicts the hypothesis (5). Therefore, we conclude that \( |w(z)| < 1 \) for all \( z \in U \) that is \( f \in S^*(\alpha) \).

Furthermore, we have the following results:

**Theorem 3** Suppose that \( \Re \left( \frac{\alpha + \frac{1}{z}}{\alpha - z} \right) > 0 \) and \( \Re \left( \frac{f(z)}{zf'(z)} \right) > 0 \). If

\[
1 + \frac{zf''(z)}{f'(z)} < \frac{\alpha + z}{\alpha - z}, \quad \alpha \neq z,
\]

\( \Rightarrow \) the equation (3) holds.
then \( f \in S^*(\alpha) \).

**Proof.** It is clear that \( h(z) := \frac{\alpha + z}{\alpha - z} \) is convex in \( U \). Thus according to [5 Theorem 3.1a], yields \( f \in S^*(\alpha) \).

**Theorem 4** Suppose \( B, D \) are analytic in \( U \) with \( D(0) = 0 \) and \( \Re(B(z)) \geq \frac{2|D(z)|}{\alpha}, \alpha \neq 0 \). If \( \Re(\frac{\alpha + \frac{1}{2} - z}{\alpha - z}) > 0 \) and

\[
\frac{zf''(z)}{f'(z)} [B(z)(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}) + 1] + D(z) < \frac{\alpha + z}{\alpha - z}
\]

then \( f \in S^*(\alpha) \).

**Proof.** Since \( \Re(\frac{\alpha + \frac{1}{2} - z}{\alpha - z}) > 0 \) this implies that \( h(z) := \frac{\alpha + z}{\alpha - z} \) is convex. Indeed, we have \( B(0) = 0 \); hence in view of [5 Theorem 4.1f], yields \( f \in S^*(\alpha) \).

**Theorem 5** Suppose that \( \Re(\frac{\alpha + \frac{1}{2} - z}{\alpha - z}) > 0 \) and \( \Re(\frac{\alpha + z}{\alpha - z}) > 0 \). If

\[
1 + \frac{zf''(z)}{f'(z)} < \frac{\alpha + z}{\alpha - z}, \quad \alpha \neq z,
\]

then \( f \in S^*(\alpha) \).

**Proof.** It is clear that \( h(z) := \frac{\alpha + z}{\alpha - z} \) is convex in \( U \). Assume that \( p(z) := \frac{zf'(z)}{f(z)} \); thus \( p(0) = h(0) \) and

\[
p(z) + \frac{zp'(z)}{p(z)} = 1 + \frac{zf''(z)}{f'(z)} < \frac{\alpha + z}{\alpha - z}
\]

Moreover, define a function \( \phi(\omega) := \frac{1}{\omega} \); we obvious that \( \phi \) is analytic in \( D = \mathbb{C} \setminus \{0\} \) such that \( p(U), h(U) \subset D \). Hence in virtue of [6 Theorem 3.4a,b], we pose \( f \in S^*(\alpha) \).

**Theorem 6** Suppose that \( \Re(\frac{\alpha + \frac{1}{2} - z}{\alpha - z}) > 0 \) and \( \Re(\frac{zf'(z)}{f(z)}) > 0 \). If

\[
1 + \frac{zf''(z)}{f'(z)} < \frac{\alpha + z}{\alpha - z}, \quad \alpha \neq z,
\]

then \( f \in S^*(\alpha) \).

**Proof.** It is clear that \( h(z) := \frac{\alpha + z}{\alpha - z} \) is convex in \( U \). Thus according to [5 Corollary 4.1h.1], yields \( f \in S^*(\alpha) \).
Theorem 7 Let \( f \in A \), such that
\[
\Re\left\{ \frac{(\alpha + z)((\alpha + z)(\alpha - z) + 4\alpha z)}{(\alpha - z)^3} \right\} > 0
\]  
and
\[
1 + \frac{zf''(z)}{f'(z)} < \frac{\alpha + z}{\alpha - z}.
\]
Then \( f \in S^*(\alpha) \).

Proof Assume that
\[
p(z) := \frac{\alpha + z}{\alpha - z}, \quad q(z) = \frac{zf'(z)}{f(z)}.
\]
Computations give
\[
\Re\{p^2(z) + 2p(z)zp'(z)\} = \Re\left\{ \frac{(\alpha + z)((\alpha + z)(\alpha - z) + 4\alpha z)}{(\alpha - z)^3} \right\} > 0;
\]
thus in virtue of [6 Corollary 3.1e.1], we have \( \Re(p(z)) > \sqrt{2\beta(1)} - 1 \), where \( \beta(\frac{1}{n}) = n \int_0^1 \frac{du}{1+uw} \). Indeed, we obtain
\[
q(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{zf''(z)}{f'(z)} < \frac{\alpha + z}{\alpha - z};
\]
hence in view of [5 Theorem 3.2a] we have the desired result.

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References


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