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A Hierarchy of the Convergence Tests Related to Cauchy's Test

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Abstract

In this study we apply a general theorem on convergence of numerical positive series in order to construct a hierarchy of the specific tests intimately related with Cauchy's test. This chain of the tests starts with simpler criteria, which have a smaller application area, and extends to more sophisticated tests, which can be used more frequently. We also provide some examples to illustrate how these test works and to clarify their relation with the set of the tests based on Kummer's theorem.

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1. Introduction

The known fact on inexistence of a universal test of convergence/divergence, which would work for all kind of numerical series [6, 7], leads to necessity to establish hierarchies of tests with systematic refinement, in such a way that each subsequent test of the constructed family is applicable to a larger range of series than its predecessor. Following other works on systematization of series tests, like the Kummer approach for positive series [3, 4, 7] or the chain of the Ermakov tests for positive monotone series [4, 7, 8], in this study we consider a hierarchy of the tests closely related to famous Cauchy's test [5, 10, 11]. This hierarchy was studied in [4] where a general approach to generation of the chain of tests starting from Cauchy's test was proposed and two initial tests of this chain were formulated, one of which with a partial proof. In what follows we will refer to this chain of tests as the Cauchy hierarchy of tests. In this report, applying the original idea of [4], we provide the formulation and complete proof of the first three tests of the considered hierarchy. Using this systematization, we clarify the rationale behind the construction of the considered family of tests, the principal points of the proofs, and the form of infinite extension of this chain of tests. Also we perform a comparison of the Cauchy hierarchy with that of Kummer, and show some examples of series whose behavior can be determined by applying the tests of the considered chain.

2. Revision of some basic results

In this section we present a brief list of the results, which we will use in the next sections. These results can be found in classic books of analysis and calculus (e.g., [6, 10, 11]).

Definition. A series $\sum a_n$ is called convergent if there exists a finite limit of the partial sums of this series. Otherwise, a series is called divergent.

Remark. To simplify notation we will use $\sum a_n$ for a series $\sum_{n=p}^{+\infty} a_n$, $p \in \mathbf{N}$.

Necessary condition of convergence (Divergence test). If a series $\sum a_n$ is convergent, then its general term a_n approaches zero as $n \to \infty$.

In this study, only the series of positive terms are considered, that is, the series $\sum a_n$ with $a_n > 0$, $n = p, p+1, \ldots$. We will refer to such series as positive series. The following two results hold for positive series.

Comparison test. Suppose that $\sum a_n$ and $\sum b_n$ are positive series with $0 < a_n \le b_n$, $n = p, p+1, \ldots$. If $\sum b_n$ is convergent then $\sum a_n$ is also convergent. Equivalently, if $\sum a_n$ is divergent, then $\sum b_n$ is also divergent.

Integral test. Suppose that $\sum a_n$ is a positive series. If there exists a function $f(x)$ defined on $[p,+\infty)$ such as it is continuous and decreasing on this interval, and $f(n) = a_n$, $\forall n$, then the series $\sum a_n$ is convergent if, and only if, the integral $\int_p^{+\infty} f(x)dx$ is convergent.

Let us recall also l'Hospital's rule and some properties of the upper and lower limits that will be frequently used.

l'Hospital's rule. Suppose $f(x)$ and $g(x)$ are differentiable functions and $g'(x) \neq 0$ in a deleted neighborhood of the point a. Suppose also that $\lim_{x\to a} f(x) = \lim_{x\to a} g(x) = 0$, or $\lim_{x\to a} f(x) = \pm\infty$ and $\lim_{x\to a} g(x) = \pm\infty$. Under these conditions, if the limit $\lim_{x\to a} \frac{f'(x)}{g'(x)}$ exists (finite or infinite), then the limit $\lim_{x\to a} \frac{f(x)}{g(x)}$ also exists and is equal to the former.

In what follows we use the symbols $\overline{\lim_{n\to\infty}} x_n$ and $\underline{\lim_{n\to\infty}} x_n$ to denote the upper and lower limits of a sequence $\{x_n\}_{n\in\mathbf{N}}$.

Properties of upper and lower limits.

1. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence. Then $\overline{\lim_{n\to\infty}} x_n = a$ if, and only if, $\underline{\lim_{n\to\infty}}(-x_n) = -a$.

2. Let $\{x_n\}_{n\in\mathbf{N}}$ be a sequence with $\overline{\lim_{n\to\infty}} x_n = a$ and $\{y_n\}_{n\in\mathbf{N}}$ be a sequence with $\lim_{n\to\infty} y_n = b$. Then $\overline{\lim_{n\to\infty}}(x_n \pm y_n) = a \pm b$. If additionally $b > 0$, then $\overline{\lim_{n\to\infty}}(x_n \cdot y_n) = a \cdot b$ and $\overline{\lim_{n\to\infty}}(x_n / y_n) = a/b$. Similar properties are also true for lower limits.

Let us introduce the following notations, which will be used in the next sections:

$$D_n = \frac{a_n}{a_{n+1}}; \tag{1}$$

$$R_n = n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right); \tag{2}$$

$$B_n = \ln n \cdot \left(n \cdot \left(\frac{a_n}{a_{n+1}} - 1 \right) - 1 \right); \tag{3}$$

$$I_n = \frac{n}{\ln n} \left(1 - a_n^{\frac{1}{n}} \right); \tag{4}$$

$$J_n = \frac{\ln n}{\ln\ln n}\left[\frac{n}{\ln n}\cdot\left(1-a_n^{\frac{1}{n}}\right)-1\right] = \frac{\ln n}{\ln\ln n}\left[I_n - 1\right]; \tag{5}$$

$$L_n = \frac{\ln\ln n}{\ln\ln\ln n}\left\{\frac{\ln n}{\ln\ln n}\left[\frac{n}{\ln n}\cdot\left(1-a_n^{\frac{1}{n}}\right)-1\right]-1\right\} = \frac{\ln\ln n}{\ln\ln\ln n}\left\{J_n - 1\right\}. \tag{6}$$

The first three quantities can be used to formulate the following tests for positive series.

d'Alembert's test (Ratio test).

1) If $\underline{\lim}_{n\to\infty} D_n > 1$, then $\sum a_n$ converges;

2) If $\overline{\lim}_{n\to\infty} D_n < 1$, then $\sum a_n$ diverges.

Raabe's test.

1) If $\underline{\lim}_{n\to\infty} R_n > 1$, then $\sum a_n$ converges;

2) If $\overline{\lim}_{n\to\infty} R_n < 1$, then $\sum a_n$ diverges.

Bertrand's test.

1) If $\underline{\lim}_{n\to\infty} B_n > 1$, then $\sum a_n$ converges;

2) If $\overline{\lim}_{n\to\infty} B_n < 1$, then $\sum a_n$ diverges.

These tests can be derived from the general result on construction of the tests, first introduced by Kummer [4,7]. For this reason they belong to the Kummer hierarchy of refining tests [3, 4, 7]. General Kummer's theorem is presented below.

Kummer's theorem. Let $\sum a_n$ be a positive series. Consider $K_n = d_n^{-1} \cdot D_n - d_{n+1}^{-1}$, where D_n is defined in (1) and $\sum d_n$ is a divergent positive series. In this case:

1) If $\underline{\lim}_{n\to\infty} K_n > 0$, then $\sum a_n$ converges;

2) If $\overline{\lim}_{n\to\infty} K_n < 0$, then $\sum a_n$ diverges.

There exists an interesting result that under certain condition all the tests of the Kummer hierarchy do not work. This result is presented below (see [3, 4] for details).

Proposition. If $\overline{\lim}_{n\to\infty} D_n > 1 > \underline{\lim}_{n\to\infty} D_n$, where D_n is given in (1), then the tests of the Kummer refining chain, that is, the tests obtained by using $d_n = 1$, $1/n$, $1/(n\ln n)$, $1/(n\ln n\ln\ln n)$, etc. in Kummer's theorem (the first three of these tests are, respectively, d'Alembert's, Raabe's and Bertrand's tests), do not provide any conclusion on the series convergence/divergence.

Remark. In the case $\overline{\lim_{n\to\infty}} R_n > 1 > \underline{\lim_{n\to\infty}} R_n$, or $\overline{\lim_{n\to\infty}} B_n > 1 > \underline{\lim_{n\to\infty}} B_n$, etc., the result of the above Proposition holds.

3. Cauchy's test

In this section we present different forms of famous Cauchy's test, which can be found in textbooks of calculus and analysis (e.g. [6, 7, 10]).

Cauchy's test (Root test) in the upper/lower limit form. Let $\sum a_n$ be a positive series. Denote $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = C$. In this case:

1) If $C < 1$, then $\sum a_n$ converges;

2) If $C > 1$, then $\sum a_n$ diverges.

Remark 1. Note that if $\overline{\lim_{n\to\infty}} \sqrt[n]{a_n} = 1$, then Cauchy's test is inconclusive both for $\underline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} = 1$ and $\underline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} < 1$. In fact, in the first case, for both the series $\sum 1/n$ and $\sum 1/n^2$, one has $\lim_{n\to\infty} \sqrt[n]{a_n} = 1$, however the former series diverges whereas the latter converges. In the case $\underline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} < 1$, we can consider the following two series. The first series is convergent:

$$1 + \frac{1}{2} + \frac{1}{3^2} + \frac{1}{2^2} + \ldots + \frac{1}{(2n-1)^2} + \frac{1}{2^n} + \ldots$$

with $a_{2n-1} = 1/(2n-1)^2$ and $a_{2n} = 1/2^n$, so that:

$$\overline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} = \lim_{n\to\infty} a_{2n-1}^{\frac{1}{2n-1}} = \lim_{n\to\infty} e^{\frac{-2\ln(2n-1)}{2n-1}} = 1;$$

$$\underline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} = \lim_{n\to\infty} a_{2n}^{\frac{1}{2n}} = \frac{1}{\sqrt{2}} < 1.$$

The second series is divergent:

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{2^2} + \ldots + \frac{1}{2n-1} + \frac{1}{2^n} + \ldots$$

with $a_{2n-1} = 1/(2n-1)$ and $a_{2n} = 1/2^n$, so that:

$$\overline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} = \lim_{n\to\infty} a_{2n-1}^{\frac{1}{2n-1}} = \lim_{n\to\infty} e^{\frac{-\ln(2n-1)}{2n-1}} = 1;$$

$$\underline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} = \lim_{n\to\infty} a_{2n}^{\frac{1}{2n}} = \frac{1}{\sqrt{2}} < 1.$$

Remark 2. The most know form of Cauchy's test usually presented in calculus textbooks uses the general limits and represents a particular case of the above formulation.

Remark 3. Another version of Cauchy's test can be formulated in the form without limits as follows: for $a_n > 0$, $\sqrt[n]{a_n} = C_n$,

1) If there exist $n_0 \in \mathbf{N}$ and $q \in \mathbf{R}$, $0 < q < 1$, such that $C_n \le q$, $\forall n > n_0$, $n \in \mathbf{N}$, then $\sum a_n$ converges;

2) If there exists $n_0 \in \mathbf{N}$ such that $C_n \ge 1$, $\forall n > n_0$, $n \in \mathbf{N}$, then $\sum a_n$ diverges.

The proof of this statement is similar to the presented above (e.g. [5, 6, 7]).

4. Relation between Cauchy's test and the Kummer hierarchy

There exists an interesting relation between Cauchy's and d'Alembert's tests. A study of this relation becomes simpler if d'Alembert's test is reformulated in the equivalent form using the ratio inverted with respect to D_n in (1) (see [5, 6, 10]):

d'Alembert's test. Let $\sum a_n$ be a positive series.

1) If $\overline{\lim\limits_{n\to\infty}} \dfrac{a_{n+1}}{a_n} < 1$, then $\sum a_n$ converges;

2) If $\underline{\lim\limits_{n\to\infty}} \dfrac{a_{n+1}}{a_n} > 1$, then $\sum a_n$ diverges.

The following proposition connects the last formulation of d'Alembert's test with Cauchy's test (see [6, 7, 10] for details).

Proposition. Let $\{a_n\}_{n\in\mathbf{N}}$ be a sequence of positive numbers. Then

$$\underline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n} \le \underline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \sqrt[n]{a_n} \le \overline{\lim_{n\to\infty}} \frac{a_{n+1}}{a_n}. \tag{7}$$

It follows from this Proposition that whenever d'Alembert's test is applicable for a chosen series, Cauchy's test is too. In fact, denoting $C = \overline{\lim\limits_{n\to\infty}} \sqrt[n]{a_n}$, we have: if $\overline{\lim\limits_{n\to\infty}} \dfrac{a_{n+1}}{a_n} < 1$, then $C \le \overline{\lim\limits_{n\to\infty}} \dfrac{a_{n+1}}{a_n} < 1$; and if $\underline{\lim\limits_{n\to\infty}} \dfrac{a_{n+1}}{a_n} > 1$, then $1 < \underline{\lim\limits_{n\to\infty}} \dfrac{a_{n+1}}{a_n} \le \underline{\lim\limits_{n\to\infty}} \sqrt[n]{a_n} \le C$.

Remark. Obviously, the same relation between Cauchy's and d'Alembert's tests remains if the latter is formulated using the ratio D_n.

On the other hand, an applicability of Cauchy's test does not imply an applicability of d'Alembert's test, which can be seen using the following examples.

Example 1. Analyze the behavior of the series $\sum\limits_{n=1}^{+\infty} 2^{(-1)^n - n}$.

Notice that $a_{2n} = 2^{1-2n}$ and $a_{2n+1} = 2^{-2-2n}$. Therefore:

$$\underline{\lim}_{n\to\infty} D_n = \lim_{n\to\infty} D_{2n-1} = \frac{1}{2} < 1 < 8 = \lim_{n\to\infty} D_{2n} = \overline{\lim}_{n\to\infty} D_n ,$$

and consequently, d'Alembert's test is inconclusive. However, it is simple to verify convergence by applying Cauchy's test:

$$\overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim}_{n\to\infty} 2^{\frac{(-1)^n - n}{n}} = \overline{\lim}_{n\to\infty} 2^{-1+\frac{(-1)^n}{n}} = \frac{1}{2} < 1 .$$

Example 2. Analyze the behavior of the series $\sum_{n=1}^{+\infty} 2^{n-(-1)^n}$.

Analogously to the previous example, d'Alembert's test does not work:

$$\underline{\lim}_{n\to\infty} D_n = \lim_{n\to\infty} D_{2n} = \frac{1}{8} < 1 < 2 = \lim_{n\to\infty} D_{2n-1} = \overline{\lim}_{n\to\infty} D_n .$$

Nevertheless, Cauchy's test readily reveals divergence of the series:

$$\overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim}_{n\to\infty} 2^{\frac{n-(-1)^n}{n}} = \overline{\lim}_{n\to\infty} 2^{1-\left(\frac{(-1)^n}{n}\right)} = 2 > 1 .$$

Thus, Cauchy's test has a wider scope: whenever d'Alembert's test is working, Cauchy's test is too, but the inverse statement is not true.

Furthermore, for the last two series the following inequality holds

$$\overline{\lim}_{n\to\infty} D_n > 1 > \underline{\lim}_{n\to\infty} D_n , \tag{8}$$

implying, due to the Proposition in section 2, that all the tests of the Kummer hierarchy do not work in this case (see details in [3]). It does not seem to be an evident result, because quite sophisticated tests in the Kummer chain can be constructed using the general terms $1/n$, $1/(n \ln n)$, $1/(n \ln n \ln(\ln n))$, etc., of the divergent series, which at each next step approximate closer and closer the general term $1/n^{\lambda}, \lambda > 1$ of the convergent series. Nevertheless even quite sophisticated tests of the Kummer chain are inconclusive with respect to the series in the last two examples.

In general, the tests of the Kummer hierarchy fail in two cases:

1) when both the upper and lower limit of D_n is equal 1, or when it happens for any subsequent quantity (R_n, B_n, etc.);

2) when evaluation (8) takes place, or when it happens for any subsequent quantity (R_n, B_n, etc.);

Notice that if the condition 1) occurs, then Cauchy's test is also inconclusive, because it follows from inequalities (7) that in this case $\underline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \overline{\lim}_{n\to\infty} \sqrt[n]{a_n} = \lim_{n\to\infty} \sqrt[n]{a_n} = 1$.

5. A generalization of Cauchy's test

In this section we introduce the basic theorem used subsequently to construct a chain of refining tests starting from Cauchy's test.

We start with an auxiliary proposition used to proof the basic theorem. A similar proposition can be found in [4], but our version contains less suppositions still providing the same result, and it is presented below with complete proof.

Proposition. If $\sum F'(n)$ is a divergent series, where $F(x) > 0$, $F'(x) > 0$ and $F'(x)$ is decreasing, then the series $\sum \frac{F'(n)}{[F(n)]^p}$ converges for $p > 1$ and diverges for $p \le 1$.

Proof.

First, notice that if $\sum F'(n)$ diverges, then the integral $\int_a^\infty F'(x)dx$ also diverges by the Integral test, and besides, the condition $F'(x) > 0$ implies in

$$\int_a^x F'(v)dv = F(x) - F(a) \underset{x\to+\infty}{\longrightarrow} +\infty,$$

that is, $F(x) \underset{x\to+\infty}{\longrightarrow} +\infty$. Notice also that the conditions $F'(x) > 0$ and $F(x) > 0$ imply $\frac{F'(x)}{[F(x)]^p} > 0$.

Let us split the proof in two cases: $p < 0$ e $p > 0$, noting that in the case $p = 0$ the statement of the Proposition is trivial.

1) Case $p < 0$. Since $F(x) \underset{x\to+\infty}{\longrightarrow} +\infty$, there exists $n_0 \in \mathbf{N}$ such that $F(n) > 1$, $\forall n > n_0$, that means that $[F(n)]^{-p} > 1$ and $\frac{F'(n)}{[F(n)]^p} > F'(n)$. Since $\sum F'(n)$ is a divergent series, according to the comparison test the series $\sum \frac{F'(n)}{[F(n)]^p}$ is also divergent.

2) Case $p > 0$. Since $F'(x) > 0$, the function $F(x)$ is increasing and, respectively, $[F(x)]^{-p}$ is decreasing. Recalling that $F'(x)$ is decreasing (according to the Proposition conditions), it follows that $F'(x) \cdot \frac{1}{[F(x)]^p}$ is decreasing as a product of two positive decreasing functions. Therefore, we can apply the Integral test for study the behavior of the series $\sum \frac{F'(n)}{[F(n)]^p}$. Indeed,

$$\int_a^{+\infty} \frac{F'(x)}{[F(x)]^p}\,dx = \lim_{x\to+\infty}\int_a^x \frac{F'(v)}{[F(v)]^p}\,dv = \lim_{y\to+\infty}\int_b^y \frac{dt}{t^p} = \lim_{y\to+\infty}\begin{cases}[\ln t]_b^y, & p=1\\ \left[\dfrac{1}{1-p}t^{1-p}\right]_b^y, & p\neq 1, p>0\end{cases}$$

$$= \lim_{y\to+\infty}\begin{cases}\ln y - \ln b, & p=1\\ \dfrac{1}{1-p}\left(y^{1-p}-b^{1-p}\right), & p\neq 1, p>0\end{cases} = \begin{cases}+\infty, & 0<p\le 1\\ \dfrac{b^{1-p}}{p-1}, & p>1\end{cases}.$$

Hence, the series $\sum \frac{F'(n)}{[F(n)]^p}$ converges for $p>1$ and diverges for $0<p\le 1$.

It follows from the results of 1) and 2) that the Proposition is proved.

Using the last Proposition, we can prove the following basic theorem (a similar result presented in [4] contains more suppositions comparing to our version).

Theorem 1. Let $\sum F'(n)$ be a divergent series, where $F(x)>0$, $F'(x)>0$ and $F'(x)$ is decreasing. If $\sum a_n$ is a positive series, then denoting $\frac{\ln[F'(n)/a_n]}{\ln[F(n)]} = W_n$, the following statements hold:

1) If $\underline{\lim}_{n\to\infty} W_n > 1$, then $\sum a_n$ converges;

2) If $\overline{\lim}_{n\to\infty} W_n < 1$, then $\sum a_n$ diverges.

Proof.

1) If $\underline{\lim}_{n\to\infty} W_n > 1$, then one can find $p>1$ and $m\in\mathbf{N}$ such that for $\forall n>m$ it holds $W_n > p > 1$. Considering still $F(n)>1$ (because $F(x)>0$ and $F(x) \underset{x\to+\infty}{\to} +\infty$), we obtain for $\forall n>m$:

$$\ln\left[\frac{F'(n)}{a_n}\right] > p\cdot\ln[F(n)] = \ln[F(n)]^p,$$

that is, $\frac{F'(n)}{a_n} > [F(n)]^p$, which implies $a_n < \frac{F'(n)}{[F(n)]^p}$. Hence, $\sum a_n$ converges due to comparison with the series $\sum \frac{F'(n)}{[F(n)]^p}$, which is convergent for $p>1$ according to the Proposition.

2) In the case $\overline{\lim_{n\to\infty}} W_n < 1$, there exists $m \in \mathbf{N}$, such that $W_n \le 1$, $\forall n > m$, and consequently, $\frac{F'(n)}{a_n} \le F(n)$, that is, $a_n \ge \frac{F'(n)}{F(n)}$. Hence, $\sum a_n$ diverges due to comparison with the divergent series $\sum \frac{F'(n)}{F(n)}$.

Using Theorem 1 it is possible to construct the chain of refining tests (the Cauchy hierarchy of tests) by choosing different functions $F(x)$. In what follows we present the first three tests of this hierarchy.

Using $F(x) = x$ in Theorem 1, we obtain the first test, formulated by Jamet [4, 9]. Different formulations of this test in the form without limits were also studied in [1]. The form of the test presented below is equivalent to those given in [4], but we provide a complete proof of this test, whereas in [4] the proof is given only in the case when $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$.

Test I. Consider a positive series $\sum a_n$ and denote $I_n = \frac{n}{\ln n}\left(1 - a_n^{\frac{1}{n}}\right)$ (see formula (4)). In this case:

1) If $\underline{\lim}_{n\to\infty} I_n > 1$, then $\sum a_n$ converges;

2) If $\overline{\lim}_{n\to\infty} I_n < 1$, then $\sum a_n$ diverges.

Proof.

Primarily, let us consider two simple cases when Cauchy's test is also applicable. If $\lim_{n\to\infty} a_n^{\frac{1}{n}} < 1$, then the series $\sum a_n$ converges according to Cauchy's test, and at the same time, $\lim_{n\to\infty} I_n = +\infty > 1$, that is, the first statement of Test I is applicable. On the other hand, if $\lim_{n\to\infty} a_n^{\frac{1}{n}} > 1$, then the series $\sum a_n$ diverges due to Cauchy's test, and $\lim_{n\to\infty} I_n = -\infty < 1$, which shows that the second statement of Test I holds.

Now let us consider the case when $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$. Under this condition, we will apply Theorem 1. First, notice that $F(x) = x > 0$, $\forall x > 0$; $F'(x) = 1 > 0$ and the derivative can be considered as a (non-strictly) decreasing function. Besides, $\sum F'(n) = \sum 1$ is the divergent series. Under these conditions, it follows from Theorem 1 that

$$W_n = \frac{\ln \frac{F'(n)}{a_n}}{\ln F(n)} = -\frac{\ln a_n}{\ln n}, \qquad (9)$$

and therefore, we have convergence of $\sum a_n$ if $\varliminf_{n\to\infty} W_n > 1$, and divergence if $\varlimsup_{n\to\infty} W_n < 1$.

To connect the last conditions with the statements of Test I, we should show relation between the quantities W_n and I_n. Notice first that the following chain of the inequalities holds:

$$1+\ln x \le x \le \frac{1}{1-\ln x}, \ \forall x \in (0,e). \tag{10}$$

Choosing $x = a_n^{\frac{1}{n}}$, with $a_n \in (0,e^n)$, and applying (10), we obtain

$$1+\frac{\ln a_n}{n} = 1+\ln a_n^{\frac{1}{n}} \le a_n^{\frac{1}{n}} \le \frac{1}{1-\ln a_n^{\frac{1}{n}}} = \frac{1}{1-\frac{\ln a_n}{n}}.$$

After simple transformations the last formula can be rewritten as follows:

$$-\frac{\ln a_n}{\ln n}\cdot\frac{1}{1-\frac{\ln a_n}{n}} \le \frac{n}{\ln n}\left(1-a_n^{\frac{1}{n}}\right) = I_n \le -\frac{\ln a_n}{\ln n}, \tag{11}$$

or, using (9),

$$W_n \cdot \frac{1}{1-\frac{\ln a_n}{n}} \le I_n \le W_n. \tag{12}$$

Since $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$, we have $\lim_{n\to\infty}\frac{\ln a_n}{n} = 0$. In this case, it follows form the Squeeze Theorem that $\varliminf_{n\to\infty} I_n = \varliminf_{n\to\infty} W_n$ and $\varlimsup_{n\to\infty} I_n = \varlimsup_{n\to\infty} W_n$. This concludes the proof for the case $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$.

Notice that there is no loss of generality in considering $a_n \in (0,e^n)$, since for $a_n \ge e^n$, the series $\sum a_n$ diverges by comparison with the series $\sum e^n$, and in this case the upper limit of I_n is equal to $-\infty$, so that we have the result in agreement with Test I.

Finally, let us see what happens when the limit of $a_n^{\frac{1}{n}}$ does not exist. Consider first that $\varlimsup_{n\to\infty} a_n^{\frac{1}{n}} \ne 1$. In this case, the condition $\varlimsup_{n\to\infty} I_n < 1$ implies $\varlimsup_{n\to\infty} a_n^{\frac{1}{n}} > 1$ (for if $\varlimsup_{n\to\infty} a_n^{\frac{1}{n}} < 1$ then $\varlimsup_{n\to\infty} I_n = +\infty$, that contradicts the condition $\varlimsup_{n\to\infty} I_n < 1$), and consequently, the series diverges. Similarly, it follows from the condition $\varliminf_{n\to\infty} I_n > 1$ that $\varlimsup_{n\to\infty} a_n^{\frac{1}{n}} < 1$, and the series converges. Hence, the statements of Test I are true. Consider now that $\overline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} = 1$. In this case, we have $\underline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} < 1$, that implies $\overline{\lim_{n\to\infty}} I_n = +\infty$, and in this situation there is nothing to prove with respect to the second statement of Test I. If, under the last condition, it occurs that $\underline{\lim_{n\to\infty}} I_n > 1$, then it readily follows from the right inequality in (12) that $\underline{\lim_{n\to\infty}} W_n > 1$, which implies the series convergence.

Remark 1. It follows from the above proof that Test I can be applied when $\overline{\lim_{n\to\infty}} a_n^{\frac{1}{n}} = 1$, and, in particular, when $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$, that is, under conditions when Cauchy's test does not work.

Remark 2. Evidently, if Cauchy's test is applicable, there is no necessity to consider more sophisticated tests. Nevertheless, even in such situation, there is no problem in considering also Test I.

Let us choose now the second function $F(x) = \ln x$ to be used in Theorem 1. The respective test was formulated in [4], albeit without a proof.

Test J. Consider a positive series $\sum a_n$, and denote $J_n = \frac{\ln n}{\ln\ln n}\left[I_n - 1\right]$ (see formula (5)). In this case:

1) If $\underline{\lim_{n\to\infty}} J_n > 1$, then $\sum a_n$ converges;

2) If $\overline{\lim_{n\to\infty}} J_n < 1$, then $\sum a_n$ diverges.

Proof.

We show validity of the above statements following the scheme of the proof for Test I.

Primarily, if $\lim_{n\to\infty} I_n > 1$, then the series $\sum a_n$ converges according to Test I and, at the same time, $\lim_{n\to\infty} J_n = +\infty > 1$, that is, the first statement holds. On the other hand, if $\lim_{n\to\infty} I_n < 1$, then $\sum a_n$ diverges by Test I, and $\lim_{n\to\infty} J_n = -\infty < 1$, that shows that the second statement is applicable.

To study the case when $\lim_{n\to\infty} I_n = 1$ we apply Theorem 1. First, notice that $F(x) = \ln x > 0$, $\forall x > 1$, $F'(x) = 1/x > 0$, $\forall x > 0$, and $F'(x)$ is a decreasing function for $x > 0$. Besides, $\sum F'(n) = \sum 1/n$ is a divergent series. Under these conditions, it follows from Theorem 1 that

$$W_n = \frac{\ln \frac{F'(n)}{a_n}}{\ln F(n)} = -\frac{\ln a_n}{\ln\ln n} - \frac{\ln n}{\ln\ln n}, \tag{13}$$

and consequently, we obtain convergence of $\sum a_n$ if $\underline{\lim}_{n\to\infty} W_n > 1$, and divergence if $\overline{\lim}_{n\to\infty} W_n < 1$.

To determine relation between the quantities W_n and J_n, let us consider the following evaluations. Subtracting 1 from each side of (11) and multiplying the result by $\dfrac{\ln n}{\ln\ln n}$, we obtain:

$$-\frac{\ln n}{\ln\ln n}-\frac{\ln a_n}{\ln\ln n}\cdot\frac{1}{1-\frac{\ln a_n}{n}}\le\frac{\ln n}{\ln\ln n}(I_n-1)=J_n\le-\frac{\ln a_n}{\ln\ln n}-\frac{\ln n}{\ln\ln n}. \qquad (14)$$

Using (13), we rewrite (14) in the form:

$$W_n-\frac{\ln a_n}{\ln\ln n}\cdot\frac{\frac{\ln a_n}{n}}{1-\frac{\ln a_n}{n}}\le J_n\le W_n. \qquad (15)$$

Now let us investigate the behavior of the second term in the left-hand side of (15). Since $\lim_{n\to\infty} I_n = 1$, we can represent $I_n = \dfrac{n}{\ln n}\left(1-a_n^{\frac{1}{n}}\right) = 1+\alpha_n$, where $\alpha_n \underset{n\to\infty}{\to} 0$. Therefore, $1-a_n^{\frac{1}{n}} = \dfrac{\ln n}{n}(1+\alpha_n)$, and consequently, $\lim_{n\to\infty}\left(1-a_n^{\frac{1}{n}}\right) = \lim_{n\to\infty}\dfrac{\ln n}{n}(1+\alpha_n) = 0$. Hence, $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$ (which means that Cauchy's test fails), or, equivalently,

$$\lim_{n\to\infty}\frac{\ln a_n}{n}=0. \qquad (16)$$

Denoting $x_n = \dfrac{\ln n}{n}(1+\alpha_n)$, we can rewrite $a_n^{\frac{1}{n}} = 1-x_n$, that is, $\dfrac{\ln a_n}{n} = \ln(1-x_n)$, and according to the above results $\lim_{n\to\infty} x_n = 0$. Hence, $0 < x_n < 1$, $\forall n > n_0$, $n_0 \in \mathbf{N}$. In this way, it is possible to expand $\ln(1-x_n)$ in the Taylor series:

$$\frac{\ln a_n}{n}=\ln(1-x_n)=-\sum_{k=1}^{\infty}\frac{x_n^k}{k}=-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\ln n}{n}(1+\alpha_n)\right)^k.$$

Consequently, we obtain

$$\frac{\ln a_n}{\ln n}=\frac{\ln a_n}{n}\cdot\frac{n}{\ln n}=-\sum_{k=1}^{\infty}\frac{1}{k}\left(\frac{\ln n}{n}\right)^{k-1}(1+\alpha_n)^k.$$

Noting that $\lim_{n\to\infty}\left(\dfrac{\ln n}{n}\right)^{k-1}(1+\alpha_n)^k = 0$, $\forall k > 1$, $k \in \mathbf{N}$, and the last series converges uniformly, we conclude that

$$\lim_{n\to\infty} \frac{\ln a_n}{\ln n} = -1. \tag{17}$$

Additionally, by applying l'Hospital's rule, it is simple to arrive to

$$\lim_{x\to+\infty} \frac{(\ln x)^2}{x \ln \ln x} = 0. \tag{18}$$

Applying now the auxiliary limits (16), (17), (18) for evaluation of the second term in the left-hand side of (15), we obtain:

$$\lim_{n\to\infty} \frac{\ln a_n}{\ln\ln n} \cdot \frac{\frac{\ln a_n}{n}}{1-\frac{\ln a_n}{n}} = \lim_{n\to\infty}\left(\frac{\ln a_n}{\ln n}\right)^2 \cdot \frac{(\ln n)^2}{n \ln\ln n} \cdot \frac{1}{1-\frac{\ln a_n}{n}} = 0. \tag{19}$$

Hence, it follows from (15) that $\underline{\lim}_{n\to\infty} J_n = \underline{\lim}_{n\to\infty} W_n$ and $\overline{\lim}_{n\to\infty} J_n = \overline{\lim}_{n\to\infty} W_n$. This concludes the proof for the case $\lim_{n\to\infty} I_n = 1$.

Finally, let us see what happens when the limit of I_n does not exist. If $\underline{\lim}_{n\to\infty} I_n > 1$, then $\underline{\lim}_{n\to\infty} J_n = +\infty$ and, at the same time, the series converges according to Test I, which confirms the first statement of Test J. Analogously, if $\overline{\lim}_{n\to\infty} I_n < 1$ then we arrive to validity of the second statement. Consider now that $\underline{\lim}_{n\to\infty} I_n \le 1$ and $\overline{\lim}_{n\to\infty} I_n \ge 1$. Notice that if $\underline{\lim}_{n\to\infty} I_n < 1 < \overline{\lim}_{n\to\infty} I_n$, then we have $\underline{\lim}_{n\to\infty} J_n = -\infty$ and $\overline{\lim}_{n\to\infty} J_n = +\infty$, and consequently, there is nothing to prove in Test J. Hence, it remains to analyze two cases: 1) $\overline{\lim}_{n\to\infty} I_n > 1$ and $\underline{\lim}_{n\to\infty} I_n = 1$; and 2) $\overline{\lim}_{n\to\infty} I_n = 1$ and $\underline{\lim}_{n\to\infty} I_n < 1$. In the first case, we have $\overline{\lim}_{n\to\infty} J_n = +\infty$ and the second statement of Test J is out of consideration. On the other hand, if $\underline{\lim}_{n\to\infty} J_n > 1$, then it follows from the right inequality in (15) that $\underline{\lim}_{n\to\infty} W_n > 1$, and consequently, the series converges by Theorem 1.

In the second case, we have $\underline{\lim}_{n\to\infty} J_n = -\infty$, showing that the first statement of Test J is out of consideration. Let us show that the second statement is true. Notice that without loss of generality we can consider $I_n > 0$ starting from some index n, for if it would not be so, then for any n would exist k_n such that $I_{k_n} \le 0$, that is, $a_{k_n}^{\frac{1}{k_n}} \ge 1$, which means that $a_{k_n} \ge 1$, and the last implies divergence of the series $\sum a_n$. Hence, we can consider $I_n > 0$ and together with $\overline{\lim}_{n\to\infty} I_n = 1$ it ensures that for $\forall\varepsilon > 0$, exists $n_0 \in \mathbf{N}$, such that for all $n > n_0$ the following inequality holds: $0 < I_n < 1 + \varepsilon$. Therefore, $1 - \frac{\ln n}{n}(1+\varepsilon) < a_n^{\frac{1}{n}} < 1$, that implies the existence of the limit $\lim_{n\to\infty} a_n^{\frac{1}{n}} = 1$ (the last being equivalent to (16)), and also the validity of the inequality $\ln\left(1-\frac{\ln n}{n}(1+\varepsilon)\right) < \frac{\ln a_n}{n} < 0$. Therefore, $\ln\left[\left(1-\frac{\ln n}{n}(1+\varepsilon)\right)^{\frac{n}{\ln n}\cdot\frac{1}{1+\varepsilon}}\right]^{-(1+\varepsilon)} < \frac{\ln a_n}{\ln n} < 0$, where the first term approaches $-(1+\varepsilon)$ as $n \to \infty$; that is, $\frac{\ln a_n}{\ln n}$ is a limited sequence. Using the last result together with limits (16) and (18), we conclude that formula (19) is true in this case. Hence, it follows from (15) that $\overline{\lim_{n\to\infty}} J_n = \overline{\lim_{n\to\infty}} W_n$, and in the case $\overline{\lim_{n\to\infty}} J_n < 1$, we have divergence of the series by Theorem 1.

Remark. The above proof shows that Test J is a generalization of Test I, that is, if Test I works, then Test J also does, and the latter can be applicable even in situations when Test I does not work (see respective examples in section 7).

Choosing the third function $F(x) = \ln\ln x$ for Theorem 1, we arrive to the following test.

Test L. Consider a positive series $\sum a_n$, and denote $L_n = \frac{\ln\ln n}{\ln\ln\ln n}(J_n - 1)$ (see formula (6)). In this case:

1) If $\underline{\lim}_{n\to\infty} L_n > 1$, then $\sum a_n$ converges;

2) If $\overline{\lim_{n\to\infty}} L_n < 1$, then $\sum a_n$ diverges.

Proof.

We follow below the scheme of the two preceding proofs.

Primarily, if $\lim_{n\to\infty} J_n > 1$, then the series $\sum a_n$ converges by Test J and, at the same time, $\lim_{n\to\infty} L_n = +\infty > 1$, that is, the first statement in Test L is applicable. On the other hand, if $\lim_{n\to\infty} J_n < 1$, then $\sum a_n$ diverges by Test J, and $\lim_{n\to\infty} L_n = -\infty < 1$, showing that the second statement holds.

Consider now the case when $\lim_{n\to\infty} J_n = 1$. Notice that $F(x) = \ln\ln x > 0$, $\forall x > e$, $F'(x) = 1/(x\ln x) > 0$, $\forall x > 1$, and $F'(x)$ is a decreasing function for $x > 1$. Besides, $\sum F'(n) = \sum 1/(n\ln n)$ is a divergent series. Under these conditions, defining

$$W_n = \frac{\ln\frac{F'(n)}{a_n}}{\ln F(n)} = -\frac{\ln a_n}{\ln\ln\ln n} - \frac{\ln n}{\ln\ln\ln n} - \frac{\ln\ln n}{\ln\ln\ln n}, \tag{20}$$

we invoke Theorem 1 to conclude that $\sum a_n$ converges if $\underline{\lim}_{n\to\infty} W_n > 1$, and diverges if $\overline{\lim}_{n\to\infty} W_n < 1$.

As before, we perform some transformations to establish relations between W_n and L_n. Subtracting 1 from each side of (14) and multiplying the result by $(\ln\ln n)/(\ln\ln\ln n)$, and using additionally (20), we obtain

$$W_n - \frac{\ln a_n}{\ln\ln\ln n}\cdot\frac{\frac{\ln a_n}{n}}{1-\frac{\ln a_n}{n}} \le L_n \le W_n . \tag{21}$$

Let us analyze the behavior of the second term in the left-hand side of (21). Since $\lim_{n\to\infty} J_n = 1$, we can represent $J_n = \frac{\ln n}{\ln\ln n}(I_n - 1) = 1+\alpha_n$, where $\alpha_n \underset{n\to\infty}{\to} 0$. Therefore, $I_n = \frac{\ln\ln n}{\ln n}(1+\alpha_n)+1$, and noticing that $\lim_{n\to\infty}\frac{\ln\ln n}{\ln n}(1+\alpha_n) = 0$, we obtain $\lim_{n\to\infty} I_n = 1$. Recalling that in this case formulas (16) and (17) are satisfied, and employing also the following auxiliary limit (which can be readily calculated by applying l'Hospital's rule)

$$\lim_{x\to+\infty}\frac{(\ln x)^2}{x\ln\ln\ln x} = 0 , \tag{22}$$

we arrive to

$$\lim_{n\to\infty}\frac{\ln a_n}{\ln\ln\ln n}\cdot\frac{\frac{\ln a_n}{n}}{1-\frac{\ln a_n}{n}} = \lim_{n\to\infty}\left(\frac{\ln a_n}{\ln n}\right)^2\cdot\frac{(\ln n)^2}{n\ln\ln\ln n}\cdot\frac{1}{1-\frac{\ln a_n}{n}} = 0 . \tag{23}$$

Hence, it follows from (21) that $\underline{\lim}_{n\to\infty} L_n = \underline{\lim}_{n\to\infty} W_n$ and $\overline{\lim}_{n\to\infty} L_n = \overline{\lim}_{n\to\infty} W_n$. This concludes the proof in the case when $\lim_{n\to\infty} J_n = 1$.

Finally, the study of the case when the limit of J_n does not exist can be performed in the same way as in the proof of Test J. If $\underline{\lim}_{n\to\infty} J_n > 1$ then $\underline{\lim}_{n\to\infty} L_n = +\infty$ and, at the same time, the series converges, which agrees with the first statement of Test L. Analogously, if $\overline{\lim}_{n\to\infty} J_n < 1$ then we arrive to the validity of the second statement. If $\underline{\lim}_{n\to\infty} J_n < 1 < \overline{\lim}_{n\to\infty} J_n$, then there is nothing to prove. In situation when $\overline{\lim}_{n\to\infty} J_n > 1$ and $\underline{\lim}_{n\to\infty} J_n = 1$, the second statement is out of consideration, and the first statement follows from the right inequality in (21), noting that if $\underline{\lim}_{n\to\infty} L_n > 1$ then $\underline{\lim}_{n\to\infty} W_n > 1$. In the last situation, when $\overline{\lim}_{n\to\infty} J_n = 1$

and $\underline{\lim}_{n\to\infty} J_n < 1$, the first statement is out of consideration, and the validity of the second can be established by noting that from the condition $\overline{\lim}_{n\to\infty} J_n = 1$ it follows that $\overline{\lim}_{n\to\infty} I_n = 1$, and subsequently, employing the same reasoning as that used in the last part of the proof of Test J, we conclude that $\overline{\lim}_{n\to\infty} L_n = \overline{\lim}_{n\to\infty} W_n$.

Remark 1. The above proof shows that Test L has a wider scope than Test J.

Remark 2. Using a similar approach as in formulation of Tests I, J, and L, other even finer tests can be constructed by choosing the functions $F(x) = \ln\ln\ln x$, $F(x) = \ln\ln\ln\ln x$, etc. In this way, an infinite chain of refining tests related to Cauchy's test will be generated. It is worth to note that an analysis of each test in the Cauchy hierarchy can be related to behavior of certain expression from preceding test, and the proof of subsequent test can be simplified by employing certain considerations in the proof of the preceding test.

6. Restrictions of the constructed hierarchy

In the proofs of Tests I, J and L, we see that the results for the subsequent test were derived by using information on the limit of the respective expression in the preceding test. (applying the behavior of the expression in the preceding test.) So it is natural to ask what happens with subsequent tests if the preceding one fails. Let us consider the situation when $\underline{\lim}_{n\to\infty} I_n < 1 < \overline{\lim}_{n\to\infty} I_n$ (that is, Test I does not work). In this case one can readily note that the limits of the expression J_n will obey the same evaluation: $\underline{\lim}_{n\to\infty} J_n < 1 < \overline{\lim}_{n\to\infty} J_n$, implying that Test J will be also inconclusive. Furthermore, from the last inequalities it follows that for the next expression L_n we have $\underline{\lim}_{n\to\infty} L_n = -\infty < 1$ e $\overline{\lim}_{n\to\infty} L_n = +\infty > 1$, that is, Test L does not work either. And the same is true for all the tests of the considered hierarchy starting from Test I.

It can be shown that similar situation occurs if this kind of inequality will arise in more refined tests. For example, if $\underline{\lim}_{n\to\infty} J_n < 1 < \overline{\lim}_{n\to\infty} J_n$, then all the tests of the Cauchy hierarchy starting from Test J will fail; if $\underline{\lim}_{n\to\infty} L_n < 1 < \overline{\lim}_{n\to\infty} L_n$, then all the tests starting from Test L will fail, etc. Hence, if such kind of inequality will arise for a certain series, then refinement of tests in the Cauchy hierarchy will not help in this situation. However, if it occurs that the general limit of the respective expression in some test is equal to 1 (for example, $\lim_{n\to\infty} I_n = 1$), then it is worth to try an application of finer tests in the Cauchy hierarchy until certain conclusion about series convergence/divergence will be obtained, or until the above undetermined and unsolved (within the considered chain) situation will occur.

We can note that the above behavior of the Cauchy hierarchy is quite similar to the situation described in the Proposition of section 2, where inequality (8) (or a similar one with one of the quantities R_n, B_n, etc.) implies that all the tests of the Kummer hierarchy are inconclusive.

7. Applications to some examples

To illustrate the considered tests let us analyze some examples.

Example 1. Investigate the behavior of the series $\sum_{n=2}^{+\infty} \frac{1}{\ln n}$.

Since $\lim_{x\to+\infty} \frac{\ln(\ln x)}{x} = 0$, we have $\lim_{n\to\infty} a_n^{\frac{1}{n}} = \lim_{n\to\infty} e^{-\ln(\ln n)^{\frac{1}{n}}} = \lim_{n\to\infty} e^{-\frac{\ln(\ln n)}{n}} = 1$, that is, Cauchy's test fails. Let us try Test I:

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty} \frac{n}{\ln n}\left(1-\left(\frac{1}{\ln n}\right)^{\frac{1}{n}}\right) = \lim_{n\to\infty} \frac{e^{-\ln\ln n/n}-1}{-\ln\ln n/n}\cdot \lim_{n\to\infty}\frac{n}{\ln n}\cdot\frac{\ln\ln n}{n} = 1\cdot 0 = 0.$$

Here two simple limits were used: $\lim_{x\to 0}\frac{e^x-1}{x} = 1$ and $\lim_{x\to+\infty}\frac{\ln\ln x}{\ln x} = 0$. Therefore, Test I shows the divergence of the series.

Example 2. Investigate the behavior of the series $\sum_{n=2}^{+\infty}\left(1-\frac{\ln n}{n}-\frac{k\ln\ln n}{n}\right)^n$, $k\in\mathbf{R}$.

For Cauchy's test and Test I we have, respectively:

$$\lim_{n\to\infty} a_n^{1/n} = \lim_{n\to\infty}\left(1-\frac{\ln n}{n}-\frac{k\ln\ln n}{n}\right) = 1$$

and

$$\lim_{n\to\infty} I_n = \lim_{n\to\infty}\frac{n}{\ln n}\left[1-1+\frac{\ln n}{n}+\frac{k\ln\ln n}{n}\right] = \lim_{n\to\infty}\left(1+\frac{k\ln\ln n}{\ln n}\right) = 1.$$

That is, both tests are inconclusive. However, applying Test J, we find:

$$\lim_{n\to\infty} J_n = \lim_{n\to\infty}\frac{\ln n}{\ln\ln n}\left[I_n - 1\right] = \lim_{n\to\infty}\frac{\ln n}{\ln\ln n}\left[1+\frac{k\ln\ln n}{\ln n}-1\right] = \lim_{n\to\infty} k = k.$$

Therefore, if $k>1$ then the series converges, and if $k<1$ then it diverges. For $k=1$ Test J does not work, so let us apply Test L. Since in this case $J_n = 1$, we obtain:

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\frac{\ln\ln n}{\ln\ln\ln n}\left[J_n - 1\right] = \lim_{n\to\infty}\frac{\ln\ln n}{\ln\ln\ln n}\left[1-1\right] = \lim_{n\to\infty} 0 = 0 < 1,$$

that implies divergence of the series.

Remark. The series with the general terms

$$(1-(\ln n)/n-(\ln\ln n)/n-k(\ln\ln\ln n)/n)^n,$$

$$(1-(\ln n)/n-(\ln\ln n)/n-(\ln\ln\ln n)/n-k(\ln\ln\ln\ln n)/n)^n,$$

etc. can be analyzed in a similar way, employing more refined tests of the considered hierarchy.

Example 3. Analyze the behavior of the series $\sum_{n=3}^{+\infty}\left(1-\frac{\ln n}{n}\cdot a^{\frac{\ln\ln n}{\ln n}}\right)^n$, $a>0$.

Since $\lim_{n\to\infty}\frac{\ln\ln n}{\ln n}=0$, both Cauchy's test and Test I fail:

$$\lim_{n\to\infty} a_n^{1/n}=\lim_{n\to\infty}\left(1-\frac{\ln n}{n}\cdot a^{\frac{\ln\ln n}{\ln n}}\right)=1,$$

$$\lim_{n\to\infty} I_n=\lim_{n\to\infty}\frac{n}{\ln n}\cdot\frac{\ln n}{n}\cdot a^{\frac{\ln\ln n}{\ln n}}=\lim_{n\to\infty} a^{\frac{\ln\ln n}{\ln n}}=1.$$

On the other hand, changing the variable $x=\frac{\ln\ln n}{\ln n}\underset{n\to\infty}{\to}0^+$, we obtain:

$$\lim_{n\to\infty} J_n=\lim_{n\to\infty}\frac{\ln n}{\ln\ln n}\left[a^{\frac{\ln\ln n}{\ln n}}-1\right]=\lim_{x\to 0^+}\frac{a^x-1}{x}=\ln a\cdot\lim_{x\to 0^+}a^x=\ln a,$$

that implies, according to Test J, that the series converges for $a>e$, and diverges for $a<e$. To analyze the case $a=e$, we can use Test L. In fact,

$$L_n=\frac{\ln\ln n}{\ln\ln\ln n}\left[\frac{\ln n}{\ln\ln n}\left(e^{\frac{\ln\ln n}{\ln n}}-1\right)-1\right]=\frac{\ln n}{\ln\ln\ln n}\left[e^{\frac{\ln\ln n}{\ln n}}-1-\frac{\ln\ln n}{\ln n}\right].$$

Then, applying limit and changing the variable $x=\ln n\underset{n\to\infty}{\to}+\infty$, we obtain:

$$\lim_{n\to\infty} L_n=\lim_{x\to+\infty}\frac{e^{\frac{\ln x}{x}}-1-\frac{\ln x}{x}}{\frac{\ln\ln x}{x}}=\lim_{x\to+\infty}\frac{\left(e^{\ln x/x}-1\right)\cdot(1-\ln x)\cdot\ln x}{1-\ln x\cdot\ln\ln x}$$

$$=\lim_{x\to+\infty}\frac{e^{\ln x/x}-1}{\ln x/x}\cdot\lim_{x\to+\infty}\frac{\ln^2 x\cdot(1-\ln x)}{x(1-\ln x\cdot\ln\ln x)}=\lim_{x\to+\infty}\frac{\ln^2 x}{x\ln\ln x}\cdot\lim_{x\to+\infty}\frac{1-1/\ln x}{1-1/(\ln x\cdot\ln\ln x)}=0\cdot 1=0.$$

Hence, the series is convergent if $a>e$, and divergent if $a\le e$.

Remark. The series with the general terms

$$\left(1-(\ln n)/n-((\ln\ln n)/n)\cdot a^{(\ln\ln\ln n)/(\ln\ln n)}\right)^n,$$

$$\left(1-(\ln n)/n-(\ln\ln n)/n-((\ln\ln\ln n)/n)\cdot a^{(\ln\ln\ln\ln n)/(\ln\ln\ln n)}\right)^n,$$

etc. can be analyzed in a similar way using more sophisticated tests of the chain.

Example 4. Analyze the behavior of the series

$$\sum_{n=3}^{+\infty}\left(1-\frac{\ln n}{n}-\frac{\ln\ln n}{n}\cos^2\frac{1}{n}\right)^n .$$

It can be seen that Test J does not work in this case:

$$\lim_{n\to\infty} J_n = \lim_{n\to\infty}\frac{\ln n}{\ln\ln n}\left\{1+\frac{\ln\ln n}{\ln n}\cos^2\frac{1}{n}-1\right\} = \lim_{n\to\infty}\cos^2\frac{1}{n} = 1 .$$

However, Test L shows the divergence of the series. Indeed, using auxiliary limit

$$\lim_{x\to\infty}\left(\frac{\ln\ln x}{\ln\ln\ln x}\cdot\frac{1}{x^2}\right) = \lim_{x\to\infty}\frac{1}{\ln\ln\ln x}\cdot\lim_{x\to\infty}\frac{\ln\ln x}{x^2} = \lim_{x\to\infty}\frac{1}{\ln\ln\ln x}\cdot\lim_{x\to\infty}\frac{1}{2x^2\ln x} = 0,$$

we obtain

$$\lim_{n\to\infty} L_n = \lim_{n\to\infty}\frac{\ln\ln n}{\ln\ln\ln n}\left[J_n-1\right] = \lim_{n\to\infty}\frac{\ln\ln n}{\ln\ln\ln n}\left[\cos^2\frac{1}{n}-1\right]$$

$$= -\lim_{n\to\infty}\frac{\ln\ln n}{\ln\ln\ln n}\sin^2\frac{1}{n} = -\lim_{n\to\infty}\left(\frac{\ln\ln n}{\ln\ln\ln n}\cdot\frac{1}{n^2}\right)\cdot\lim_{n\to\infty}\left(\frac{\sin(1/n)}{(1/n)}\right)^2 = 0<1 .$$

Example 5. Analyze the series

$$\sum_{n=3}^{+\infty}\left(1-\frac{\ln n}{n}-\frac{\ln\ln n}{n}\cos^2\frac{1}{n}\cdot\left(a+b(-1)^n\right)\right)^n,\ a,b\ \geq\ 0 .$$

Notice first that

$$J_n = \frac{\ln n}{\ln\ln n}\left\{1+\frac{\ln\ln n}{\ln n}\cos^2\frac{1}{n}\cdot\left(a+b(-1)^n\right)-1\right\} = \cos^2\frac{1}{n}\cdot\left(a+b(-1)^n\right).$$

It leads to conclusion that the general limit of J_n does not exists. Then let us analyze the upper and lower limits of the above expression. Since a and b are not negative, it follows readily that

$$\underline{\lim}_{n\to\infty} J_n = a-b\,;\ \overline{\lim}_{n\to\infty} J_n = a+b .$$

Therefore, the series converges when $a-b>1$, and it diverges when $a+b<1$.

Let us consider some specific values:

1) if $a=3,\ \ b=1$, then $\underline{\lim}_{n\to\infty} J_n = 2>1$, that implies convergence;

2) if $a=1/2,\ b=1/3$, then $\overline{\lim}_{n\to\infty} J_n = 5/6<1$, that is, the series diverges;

3) if $a=1,\ \ b=0$ we have $\lim_{n\to\infty} J_n = 1$, and Test J is inconclusive, so that we should try to apply finer tests of the considered hierarchy; in fact, in this case we come back to the previous example, and consequently, Test L shows divergence;

4) if $a = 1$, $b = 1$, then $\varliminf_{n\to\infty} J_n = 0 < 1 < 2 = \varlimsup_{n\to\infty} J_n$, which means that there is no reason to apply more refined tests of this hierarchy, because all the subsequent tests will be inconclusive.

Example 6. Investigate the series

$$\sum_{n=2}^{+\infty}\left(1-\frac{\ln n}{n}\cdot\left(a\cdot\left(1+\sin^2\sqrt{\frac{\ln\ln n}{\ln n}}\right)+b\cdot\sin\frac{n\pi}{4}\right)\right)^n,\ a,b>0.$$

For this series $\lim_{n\to\infty} a_n^{1/n} = 1$, that is, Cauchy's test does not work. So let us try Test I. Denoting $t_n = \sqrt{\frac{\ln\ln n}{\ln n}}$, we have

$$I_n = \frac{n}{\ln n}\left(\frac{\ln n}{n}\cdot\left(a\cdot\left(1+\sin^2 t_n\right)+b\cdot\sin\frac{n\pi}{4}\right)\right) = a\cdot\left(1+\sin^2 t_n\right)+b\cdot\sin\frac{n\pi}{4},$$

and then, $\varliminf_{n\to\infty} I_n = \lim_{n\to\infty} I_{8n-2} = a - b$ and $\varlimsup_{n\to\infty} I_n = \lim_{n\to\infty} I_{8n-6} = a + b$. Therefore, the series converges when $a - b > 1$, and diverges when $a + b < 1$.

If $a+b=1$, then $\varlimsup_{n\to\infty} I_n = 1$, and if $a-b=1$, then $\varliminf_{n\to\infty} I_n = 1$, and in both cases Test I is inconclusive. For example, choosing, $a = b = 1/2$, we have $\varliminf_{n\to\infty} I_n = 0 < 1$ and $\varlimsup_{n\to\infty} I_n = 1$. On the other hand, when $a = 2$ and $b = 1$, it follows that $\varliminf_{n\to\infty} I_n = 1$ and $\varlimsup_{n\to\infty} I_n = 3 > 1$. Therefore Test I fails in both cases.

Let us pass to Test J. We have

$$J_n = \frac{\ln n}{\ln\ln n}\left(a\cdot\left(1+\sin^2 t_n\right)+b\cdot\sin\frac{n\pi}{4}-1\right). \tag{24}$$

Hence, for $a = b = 1/2$, we obtain

$$J_n = \frac{\ln n}{\ln\ln n}\left(-\frac{1}{2}+\frac{1}{2}\sin^2 t_n+\frac{1}{2}\sin\frac{n\pi}{4}\right) = \frac{\ln n}{\ln\ln n}\left(-\frac{1}{2}+\frac{1}{2}\sin\frac{n\pi}{4}\right)+\frac{1}{2}\frac{\ln n}{\ln\ln n}\frac{\sin^2 t_n}{t_n^2}t_n^2 .$$

Notice that the upper limit of the first term in the right-hand side is zero (by choosing $n = 8k - 6$) and the limit of the second term is $1/2$ (recalling that $t_n^2 = \frac{\ln\ln n}{\ln n}$). Therefore, $\varlimsup_{n\to\infty} J_n = 1/2 < 1$, and Test J shows that the series is divergent when $a = b = 1/2$.

In the case $a = 2$ and $b = 1$, expression (24) assumes the form

$$J_n = \frac{\ln n}{\ln\ln n}\left(2\left(1+\sin^2 t_n\right)+\sin\frac{n\pi}{4}-1\right) = \frac{\ln n}{\ln\ln n}\left(1+\sin\frac{n\pi}{4}\right)+2\frac{\ln n}{\ln\ln n}\frac{\sin^2 t_n}{t_n^2}t_n^2 .$$

The lower limit of the first term in the right-hand side is zero (choosing $n = 8k - 2$) and the limit of the second is 2. Therefore, $\underline{\lim}_{n\to\infty} J_n = 2 > 1$, and by Test J, the series is convergent when $a = 2$ and $b = 1$.

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