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Approximation and Decomposability in the Space of Pettis Integrable Functions

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Abstract. A characterization of closed decomposable subsets of the space of Pettis integrable functions is given. This characterization is based on an approximation with respect to Pettis-norm we get for Pettis integrable selections of a multifunction. As an application we obtain the existence of the multivalued conditional expectation for Pettis integrable multifunction.

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1. Introduction

Aumann [1] and Debreu [10] marked the beginning of a systematic study of the multivalued integral, which was a very useful tool in areas like optimization and mathematical economics. The set-valued Pettis integral theory, which goes back to the monograph by Castaing and Valadier [7], has attracted recently the attention of several authors, see e.g. [3, 5, 11, 16, 33, 34] witch deal with the multifunctions whose values are bounded subsets (in particular weakly compact) of a Banach space X and the paper by Elamri and Hess [14] in which they introduce the concept of the Aumann-Pettis integral for multifunctions not necessarily bounded.

The notion of decomposability in the space of Bochner integrable functions was introduced by Hiai and Umegaki in [18]. In this context, they showed that any decomposable closed subset of L^1_X is the set of Bochner integrable selections of some multifunction. In this paper, we show that this characterization of decomposable, can be extended to the space of Pettis-integrable functions P^1_X equipped with the Pettis norm $\|.\|_{Pe}$. The proof is based on an approximation result of Pettis integrable selections that we show and also uses a theorem of existence of the essential supremum of the family of measurable multifunctions due to Valadier [28].

The main result of Section 3 is Theorem 3.1 in which we show that if F is a multifunction Aumann-Pettis integrable and if there exists a countable family (f_n) in P^1_X such that for each $\omega \in \Omega : F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$ then any Pettis-integrable selection of F can be approximated by simple selections with respect to the sequence $(f_n)_{n\geq 1}$. The approximation is given with respect to Pettis-norm. Then we give some results similar to those obtained by Hiai and Umegaki in ([18], section 1) for integrable multifunctions. We will show, in particular in Proposition 3.6, Corollary 3.12 and Corollary 3.11 that under certain conditions, the set of Pettis integrable selections of a multifunction is closed with respect to Pettis-norm.

In section 4, we show in theorem 4.1 that if D is a decomposable and closed subset of P^1_X then there is a measurable multifunction F which is Aumann-Pettis integrable such that the set of Pettis integrable selections of F is equal to D. This result will allow us to demonstrate in the sequel, an integral representation theorem for convex weakly-compact valued multimeasures with σ-bounded variation. This will be done in Theorem 4.8. Finally, we give an application to the existence of the Pettis conditional expectation for convex weakly-compact valued multifunctions.

2. Notations and preliminaries

Let $(X, \|.\|)$ be a separable Banach space whose dual is X^* and B^* is the unit ball of X^*. We denote by $\mathrm{cl}(X)$ the set of all closed nonempty subsets of X. The following families will be considered.

$\mathrm{cc}(X)$ the set of all convex closed nonempty subsets of X.

$\mathrm{wk}(X)$ the set of all weakly compact nonempty subsets of X.

$\mathrm{cwk}(X)$ the set of all convex weakly compact nonempty subsets of X.

For each $A \subset X$, $\overline{A}$ (resp. $\overline{\mathrm{co}}A$) denotes the norm-closure (resp. the closed convex hull) of A. The support function $\delta^*(., A)$ is defined respectively by

$$\delta^*(x^*, A) = \sup\left\{\langle x^*, x\rangle : x \in A\right\},\ x^* \in X^*$$

we denote also

$$h(A) = \sup\left\{\|x\| : x \in A\right\}.$$

This is the Hausdorff distance between A and the singleton $\{0\}$. In the sequel $(\Omega, \mathcal{A}, P)$ denotes a probability space and L^1_X is the space of all Bochner integrable functions from Ω into X up to equality a.s.. Next, we say that $f : (\Omega, \mathcal{A}, P) \to X$ is Pettis integrable whenever it is scalarly integrable and for each A in $\mathcal{A}$, there is $x_A \in X$ such that

$$\langle x^*, x_A\rangle = \int_A \langle x^*, f(\omega)\rangle\, dP, \quad \forall x^* \in X^*.$$

x_A is called the Pettis integral of f over A and will be denoted by $\int_A f\,dP$. In what follows, denote by $\mathrm{P}^1_X(\Omega, \mathcal{A}, P)$, or simply by P^1_X the space of all Pettis integrable functions f from $(\Omega, \mathcal{A}, P)$ into X up to the equivalence $\equiv$ defined by

$$f \equiv g \Longleftrightarrow \forall x^* \in X^*, \langle x^*, f\rangle = \langle x^*, g\rangle \text{ p.s.} \tag{2.1}$$

The space P^1_X is endowed with the Pettis norm defined by

$$\|f\|_{Pe} = \sup_{x^* \in B^*} \int_\Omega |\langle x^*, f(\omega)\rangle|\,dP. \tag{2.2}$$

Since X is assumed to be separable, the Pettis Measurability Theorem (see [12], Theorem 2, p. 42) implies that measurability and scalar measurability are equivalent. Then P^1_X is the space of all Pettis integrable functions up to the equality a.s.. Indeed, two measurable functions are equal almost surely if and only if they are $\equiv$-equivalent. We may refer to ([12], Corollary 7, p. 48).
A map $F : \Omega \longrightarrow \mathrm{cl}(X)$ is called a multifunction. It is said to be scalarly measurable if for every $x^* \in X^*$, the map $\delta^*(x^*, F(.))$ is measurable. The multifunction F is said to be Effros measurable (or shortly measurable) if for any open set U of X, the subset F^-U defined by

$$F^-U = \{\omega \in \Omega : X(\omega) \cap U \neq \emptyset\}$$

is in $\mathcal{A}$. The sub-σ-field $\mathcal{A}_F$ of $\mathcal{A}$ is the smallest sub-σ-field making measurable F. Recall that $f : (\Omega, \mathcal{A}) \longrightarrow X$ is called a selection of F, if $f(\omega) \in F(\omega)$ for every $\omega \in \Omega$. It is known that every measurable multifunction with closed nonempty values admits at least one measurable selection (see [6], Theorem III.6). Furthermore, a multifunction F with nonempty closed values in X is measurable if and only if there is a countable family of measurable selections (f_n) such that for each $\omega \in \Omega : F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$ (see [6], Theorem III.9).
Given a measurable multifunction $F : (\Omega, \mathcal{A}, P) \longrightarrow \mathrm{cl}(X)$. We denote by $S^{Pe}_F(\mathcal{A})$, or simply by S^{Pe}_F, the subset of $\mathrm{P}^1_X(\mathcal{A})$ defined by

$$S^{Pe}_F(\mathcal{A}) = \left\{f \in \mathrm{P}^1_X(\mathcal{A}) : f(\omega) \in F(\omega) \text{ a.s.}\right\}.$$

$\mathcal{A}$ can optionally be replaced by a sub-σ-field $\mathcal{B}$ of $\mathcal{A}$.
The multifunction F is said to be Aumann-Pettis integrable if $S^{Pe}_F \neq \emptyset$. The Aumann-Pettis integral of F over $A \in \mathcal{A}$ is defined and denoted by

$$\int_A F\,dP = \left\{\int_A f\,dP : f \in S^{Pe}_F\right\}. \tag{2.3}$$

Also, besides $\int_A F\,dP$ taken on $(\Omega, \mathcal{A}, P)$ by (2.3), the Aumann-Pettis integral of F on $(\Omega, \mathcal{A}, P)$ is defined and denoted by

$$\int_A^{(\mathcal{A})} F\,dP = \left\{\int_A f\,dP : f \in S^{Pe}_F(\mathcal{A})\right\}. \tag{2.4}$$

Remark 2.1. Let $F : \Omega \longrightarrow \mathrm{cl}(X)$ be a measurable multifunction. Then F is Aumann-Pettis integrable if and only if F admits at least one selection Pettis integrable. Indeed, let $f \in \mathrm{P}^1_X$ such that $f(\omega) \in F(\omega)$ a.s. and let $(f_n)_{n\geq 1}$ be a sequence of measurable selections of F such that $F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$. we define the set A as follows

$$A = \{\omega \in \Omega : f(\omega) \in F(\omega)\}.$$

Then

$$A = \bigcap_{m\geq 1} \bigcup_{n\geq 1} \left\{\omega \in \Omega : \|f_n(\omega) - f(\omega)\| < \frac{1}{m}\right\}.$$

It follows that $A \in \mathcal{A}$ and therefore $P(A) = 1$. Let $h = \mathbf{1}_A f + \mathbf{1}_{\Omega\setminus A} f_1$. The function h is a Pettis integrable selection of F.

When F is $\mathrm{cwk}(X)$-valued multifonction, we say that F is Pettis integrable if it is scalarly integrable and for each $A \in \mathcal{A}$, there is $X_A \in \mathrm{cwk}(X)$ such that

$$\forall x^* \in X^*, \quad \delta^*(x^*, X_A) = \int_A \delta^*(x^*, X)\, dP.$$

The space of all Pettis integrable multifunctions F from $(\Omega, \mathcal{A}, P)$ into $\mathrm{cwk}(X)$ up to the equality a.s. is denoted by $\mathcal{P}^1_{\mathrm{cwk}(E)}(\mathcal{A})$ or simply by $\mathcal{P}^1_{\mathrm{cwk}(E)}$. Recall in this context, the result of the author (see [33], Theorem 3.2 and [34], Theorem 1),which provides a caracterisation of Pettis integrability.

Theorem 2.2. *Let F be a scalarly integrable* $\mathrm{cwk}(X)$*-valued multifunction, then F is Pettis integrable if and only if the set $\{\delta^*(x^*, F) : x^* \in B^*\}$ is uniformly integrable.*

To end this section include the important result of valadier [28] showing the existence of the essential supremum of a family of measurable multifunctions.

Lemma 2.3. [28] *Let X be a separable metrizable space and $\mathfrak{A}$ the set of all* $\mathrm{cl}(X)$*-valued measurable multifunctions Γ. Then, for any subset $\mathfrak{S}$ of $\mathfrak{A}$, there exists a countable subset $\mathfrak{S}_1$ of $\mathfrak{S}$ such that the multifunction Φ defined by* $\Phi(\omega) = \overline{\cup\{\Gamma(\omega) : \Gamma \in \mathfrak{S}_1\}}$ *for all $\omega \in \Omega$, is the essential supremum of $\mathfrak{S}$ in $\mathfrak{A}$: i.e. Φ is the smallest element of $\mathfrak{A}$ (in the sense of inclusion a.s.) such that $\forall \Gamma \in \mathfrak{S}$, $\Gamma \subset \Phi$, a.s.*

3. Approximation of Pettis integrable selections of a measurable multifunction

Given a multifunction F. Let $(f_n)_{n\geq 1}$ be a sequence of measurable selections of F, and $\{A_1, \cdots, A_m\}$ a finite partition of Ω in $\mathcal{A}$. The function g defined by

$$g = \sum_{i=1}^{m} \mathbf{1}_{A_i} f_i$$

is called simple selection of F with respect to (f_n). The following theorem is our first main result of this paper. It allows to approximate each Pettis integrable selection of F by simple selections.

Theorem 3.1. *Let $F : \Omega \to \mathrm{cl}(X)$ such that $S_F^{Pe} \neq \emptyset$. Assume that there exists a sequence $(f_n)_{n\geq 1}$ in P_X^1 satisfying $F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$ for all $\omega \in \Omega$. Then for any $f \in S_F^{Pe}$ and any $\varepsilon > 0$, there exists a finite partition $\{A_1 \cdots, A_m\}$ of Ω in $\mathcal{A}$, such that*

$$\left\| f - \sum_{i=1}^{m} \mathbf{1}_{A_i} f_i \right\|_{Pe} < \varepsilon.$$

Proof. Let $\varepsilon > 0$ and let $f \in S_F^{Pe}$. Since $F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$ for all $\omega \in \Omega$, we have

$$\Omega = \bigcup_{n\geq 1} \left\{ \omega \in \Omega : \|f(\omega) - f_n(\omega)\| < \frac{\varepsilon}{2} \right\}.$$

One can thus obtain a countable partition $(Q_n)_{n\geq 1}$ of Ω in $\mathcal{A}$ such that:

$$\|f(\omega) - f_n(\omega)\| < \frac{\varepsilon}{2} \tag{3.1}$$

holds for all $\omega \in Q_n$ and all $n \geq 1$. Since $f - f_1$ is Pettis-integrable, the set $\{\langle x^*, f - f_1\rangle : x^* \in B^*\}$ is uniformly integrable (see [24], Theorem 5.2). There is therefore an integer $m \geq 1$ such that

$$\sup_{x^*\in B^*} \sum_{i=m+1}^{+\infty} \int_{Q_i} |\, \langle x^*, f - f_i\rangle \,|\, dP < \frac{\varepsilon}{2} \tag{3.2}$$

holds. We define the finite partition $\{A_1, \cdots, A_m\}$ of Ω by $Q_1 = A_1 \cup (\cup_{j\geq m+1} Q_j)$ and $B_i = Q_i$ for $i = 2, \cdots, m$. For all $x^* \in B^*$, we have

$$\begin{aligned} \int_{\Omega} |\langle x^*, (f - \sum_{i=1}^{m} \mathbf{1}_{A_i} f_i)\rangle|\, dP &\leq \sum_{i=1}^{m} \int_{B_i} |\langle x^*, (f - f_i)\rangle|\, dP \\ &= \sum_{i=1}^{m} \int_{Q_i} |\langle x^*, (f - f_i)\rangle|\, dP \\ &\quad + \sum_{i=m+1}^{+\infty} \int_{Q_i} |\langle x^*, (f - f_1)\rangle|\, dP. \end{aligned}$$

Hence, taking the supremum on B^* in the right side, and using inequalities (3.1) and (3.2), we get for all $x^* \in B^*$

$$\begin{aligned} \int_{\Omega} |\langle x^*, (f - \sum_{i=1}^{m} \mathbf{1}_{A_i} f_i)\rangle|\, dP &\leq \frac{\varepsilon}{2} + \sup_{y^*\in B^*} \sum_{i=m+1}^{+\infty} \int_{Q_i} |\, \langle y^*, f - f_1\rangle \,|\, dP \\ &< \varepsilon. \end{aligned} \tag{3.3}$$

Taking the supremum over B^* in the left side of (3.3) yields the desired result. □

Here is a lemma inspired by Lemma 1.1 of Hiai and Umegaki [18], which allows to represent a multifunction by a sequence of Pettis-integrable selections.

Lemma 3.2. *Let* $F : \Omega \to \mathrm{cl}(X)$ *be a measurable and Aumann-Pettis integrable multifunction. Then there exists a sequence* $(f_n)_{n\geq 1}$ *in* P^1_X *such that* $F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$ *for all* $\omega \in \Omega$.

Proof. As in Remark 2.1, let g be a Pettis integrable selection of F. Let $(g_n)_{n\geq 1}$ a sequence of measurable functions satisfying $F(\omega) = \overline{\{g_n(\omega) : n \geq 1\}}$ for all $\omega \in \Omega$. Such a sequence exists since the multifunction F is measurable. For each integer $n \geq 1$ and each integer $m \geq 1$, we set

$$A_n^m = \{\omega \in \Omega : \|g_m(\omega)\| \leq n\} \tag{3.4}$$

and

$$f_n^m = \mathbf{1}_{A_n^m}\, g_m + \mathbf{1}_{\Omega\setminus A_n^m}\, g. \tag{3.5}$$

For each $n, m \geq 1$ the function f_n^m is thus a measurable selection of F. Moreover, the function $\mathbf{1}_{A_n^m}\, g_m$ is bounded, therefore Pettis integrable. It follows that each function f_n^m belongs to P^1_X. One checks easily that

$$F(\omega) = \overline{\{f_n^m(\omega) : n \geq 1, m \geq 1\}}$$

holds.

□

Remark 3.3. By Lemma 3.2, if $F_1, F_2 : \Omega \to \mathrm{cl}(X)$ are two measurable and Aumann-Pettis integrable multifunctions such that $S_{F_1}^{Pe} = S_{F_2}^{Pe}$, then $F_1 = F_2$ a.s.

Proposition 3.4. *Let* $F_1, F_2 : \Omega \to \mathrm{cl}(X)$ *two measurable and Aumann-Pettis integrable multifunctions. Consider the multifunction* F *defined on* Ω *by* $F(\omega) = \overline{F_1(\omega) + F_2(\omega)}$, *then we have*

$$\overline{S_F^{Pe}}^{\|\cdot\|_{Pe}} = \overline{S_{F_1}^{Pe} + S_{F_2}^{Pe}}^{\|\cdot\|_{Pe}} \quad a.s.$$

Proof. By Lemma 3.2, there exist two sequences $(f_n^1)_{n\geq 1}$ and $(f_n^2)_{n\geq 1}$ respectively in $S_{F_1}^{Pe}$ and $S_{F_2}^{Pe}$ such that

$$F_1(\omega) = \overline{\{f_n^1(\omega) : n \geq 1\}} \text{ and } F_2(\omega) = \overline{\{f_n^2(\omega) : n \geq 1\}}$$

hold for all $\omega \in \Omega$. It follows that $F(\omega) = \overline{\left\{f_i^1(\omega) + f_j^2(\omega) : i \geq 1, j \geq 1\right\}}$. By Theorem 3.1, for every $\varepsilon > 0$ and every $f \in S_F^{Pe}$ there exists a finite partition $\{A_1 \cdots, A_m\}$ of Ω in $\mathcal{A}$ and two finite sequences of integers $i_1, \cdots, i_m$ and $j_1 \cdots, j_m$ such that

$$\left\| f - \sum_{k=1}^{m} \mathbf{1}_{A_k}(f_{i_k}^1 + f_{j_k}^2) \right\|_{Pe} < \varepsilon.$$

Then $S_F^{Pe} \subset \overline{S_{F_1}^{Pe} + S_{F_2}^{Pe}}^{\|\cdot\|_{Pe}}$. Conversely, it is easy to see that $S_{F_1}^{Pe} + S_{F_2}^{Pe} \subset \overline{S_F^{Pe}}^{\|\cdot\|_{Pe}}$, which leads to the conclusion. □

Proposition 3.5. *Let $F : \Omega \to \mathrm{cl}(X)$ be a measurable and Aumann-Pettis integrable multifunction, then we have*

$$\overline{S_{\overline{\mathrm{co}}\,F}^{Pe}}^{\|\cdot\|_{Pe}} = \overline{\mathrm{co}}^{\|\cdot\|_{Pe}}\, S_F^{Pe}.$$

Proof. We are given according to Lemma 3.2 a sequence $(f_n)_{n\geq 1}$ in P_X^1 such that

$$F(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}, \quad \forall \omega \in \Omega,$$

and we define the set V with

$$V = \left\{ g : g = \sum_{i=1}^{m} \alpha_i f_i, \alpha_i \in \mathbb{Q}^+, \sum_{i=1}^{m} \alpha_i = 1, m \geq 1 \right\}.$$

V is a countable subset of $S_{\overline{\mathrm{co}}\,F}^{Pe}$ and $\overline{\mathrm{co}}\,F(\omega) = \overline{\{g(\omega) : g \in V\}}$ for all $\omega \in \Omega$. By Theorem 3.1, for all $f \in S_{\overline{\mathrm{co}}\,F}^{Pe}$ and all $\varepsilon > 0$, there exists a finite partition $\{A_1, \cdots, A_n\}$ of Ω in $\mathcal{A}$ and $g_1, \cdots, g_n$ in V such that

$$\left\| f - \sum_{i=1}^{m} \mathbf{1}_{A_i}\, g_i \right\|_{Pe} < \varepsilon.$$

We can verify without major difficulty, that the function $\sum_{i=1}^{m} \mathbf{1}_{A_i}\, g_i$ is a convex combination of elements of S_F^{Pe}. This implies that $f \in \overline{\mathrm{co}}^{\|\cdot\|_{Pe}}\, S_F^{Pe}$ and consequently $\overline{S_{\overline{\mathrm{co}}\,F}^{Pe}}^{\|\cdot\|_{Pe}} \subset \overline{\mathrm{co}}^{\|\cdot\|_{Pe}}\, S_F^{Pe}$. Conversely, $\overline{\mathrm{co}}^{\|\cdot\|_{Pe}}\, S_F^{Pe} \subset \overline{S_{\overline{\mathrm{co}}\,F}^{Pe}}^{\|\cdot\|_{Pe}}$ follows from $\mathrm{co}\, S_F^{Pe} \subset S_{\overline{\mathrm{co}}\,F}^{Pe}$. □

In general S_F^{Pe} is not closed for $\|.\|_{Pe}$. The following proposition is due to Ziat (see [32], Proposition 3.8, p. 44) in which the author shows that if F is weakly compact values, then S_F^{Pe} is $\|.\|_{Pe}$-closed.

Proposition 3.6. *Let $F : \Omega \to \mathrm{wk}(X)$ be a multifunction. Then S_F^{Pe} is $\|\cdot\|_{Pe}$-closed.*

Proof. If $S_F^{Pe} = \emptyset$ the result is true. Otherwise, let $(f_n)_{n\geq 1}$ a sequence in S_F^{Pe} converging to f in P_X^1 with respect to the norm $\|.\|_{Pe}$. For all $x^* \in X^*$ we have

$$\lim_{n\to\infty} \langle x^*, f_n \rangle = \langle x^*, f \rangle \quad \text{in } \mathrm{L}_{\mathbb{R}}^1. \tag{3.6}$$

Since X is separable, there exists a dense sequence $(x_k^*)_{k\geq 1}$ in X^* for the Mackey topology $\tau(X^*, X)$ (see e.g. [6], Lemma III.32).
For $k = 1$, there is a subsequence (f_n^1) of (f_n) and a negligible $N_1 \subset \Omega$ satisfying

$$\forall \omega \in \Omega \setminus N_1, \lim_{n\to\infty} \left\langle x_1^*, f_n^1(\omega) \right\rangle = \langle x_1^*, f(\omega) \rangle .$$

For $k=2$, there is a subsequence (f_n^2) of (f_n^1) and a negligible $N_2 \subset \Omega$ such that

$$\forall \omega \in \Omega \setminus N_2, \lim_{n\to\infty} \left\langle x_2^*, f_n^2(\omega)\right\rangle = \left\langle x_2^*, f(\omega)\right\rangle .$$

By successive applications of this process for all $k \geq 2$, we obtain the existence of a subsequence $(f_n^k)_n$ of $(f_n^{k-1})_n$ and a negligible N_k of Ω satisfying

$$\forall \omega \in \Omega \setminus N_k, \lim_{n\to\infty} \left\langle x_k^*, f_n^k(\omega)\right\rangle = \left\langle x_k^*, f(\omega)\right\rangle .$$

Considering the diagonal subsequence $(f_n^n)_n$ and the negligible subset $N = \cup_{k\geq 1} N_k$, we get

$$\forall k \geq 1, \forall \omega \in \Omega \setminus N, \lim_{n\to\infty} \left\langle x_k^*, f_n^n(\omega)\right\rangle = \left\langle x_k^*, f(\omega)\right\rangle .$$

Now, since $f_n^n(\omega)$ remains in the weakly compact subset $X(\omega)$ outside a negligible N' of Ω, it follows that for all ω fixed in $\Omega \setminus (N \cup N')$, the sequence of $\left(x^* \longmapsto \langle x^*, f_n^n(\omega)\rangle\right)_n$ is uniformly continuous for the Mackey topology $\tau(X^*, X)$. Therefore for all $x^* \in X^*$ and all $\omega \in \Omega \setminus (N \cup N')$ we obtain

$$\lim_{n\to\infty} \left\langle x^*, f_n^n(\omega)\right\rangle = \left\langle x^*, f(\omega)\right\rangle . \tag{3.7}$$

Since $F(\omega)$ is weakly closed, we conclude that $f(\omega) \in F(\omega)$ a.s.. □

Remark 3.7. (i) Proposition 3.6 does not require the measurability of F.

(ii) Let τ_{Pe} be the topology defined on the space of scalarly integrable functions with the scalar strong convergence follows

$$\tau_{\mathrm{Pe}}\text{-}\lim f_n = f \Longleftrightarrow \|.\|_{\mathrm{L}^1}\text{-}\lim \langle x^*, f_n\rangle = \langle x^*, f\rangle, \ \forall x^* \in B^*.$$

Obviously τ_{Pe} is weaker than the $\|.\|_{Pe}$-topology. However, from the technical proof of Proposition 3.6 we can say that if $(f_n)_n$ is a sequence of S_F^{Pe} converging to f in the sense of τ_{Pe}, then $f \in S_F^{Pe}$. Indeed, only the scalar strong convergence in (3.6) is required.

Recall a property on the family of all convex closed subsets which contain no line (See Lemma III.33 and Lemma III.34 of [6], Proposition 2.9 of [2]). For this, let $C \in \mathrm{cc}(E)$. We say that C contains no line, if C belongs to the class $\mathcal{C}$ as follows:

$$\mathcal{C} = \{C \in \mathrm{cc}(X) : \exists \text{ a closed half-space } H \text{ in } X\,;\ H \cap C \text{ is bounded}\}.$$

Note that $C \in \mathcal{C}$ if and only if there exists $y \in X^*$ such that $\delta^*(.,C)$ is finite and strongly continuous at y (see [22], Corollary 8.e).
We similarly define $\mathcal{L}$ as the family of all members of $\mathrm{cc}(X)$ that are weakly locally compact and contain no line. The family $\mathcal{L}$ satisfies the following property : $C \in \mathcal{L}$ if and only if there exists $y \in X^*$ such that $\delta^*(.,C)$ is finite and continuous at y in the Mackey topology $\tau(X^*, X)$ (see [6], Corollary I.15).

Lemma 3.8. ([6], Lemma III.34) Let $(x_n^*)_{n\geq 1}$ be a dense sequence in X^* for the Mackey topology $\tau(X^*, X)$ and let $C \in \mathcal{L}$. Then:

$$\mathcal{C} = \bigcap_{n\geq 1} \{x \in X : \langle x_n^*, x\rangle \leq \delta^*(x_n^*, C)\}$$

Lemma 3.9. ([2], Proposition 2.9) Let $(x_n^*)_{n\geq 1}$ be a strongly dense sequence in X^* and let $C \in \mathcal{C}$. Then:

$$\mathcal{C} = \bigcap_{n\geq 1} \{x \in X : \langle x_n^*, x\rangle \leq \delta^*(x_n^*, C)\}$$

Proposition 3.10. *Let $F : \Omega \to \mathrm{cc}(X)$ be a multifunction. we assume that there exists a sequence $(x_n^*)_{n\geq 1}$ in X^* such that:*

$$F(\omega) = \bigcap_{n\geq 1} \{x \in X : \langle x^*, x\rangle \leq \delta^*(x_n^*, F(\omega))\} \quad a.s.. \tag{3.8}$$

Then S_F^{Pe} is $\|.\|_{Pe}$-closed.

Proof. We can assume that $S_F^{Pe} \neq \emptyset$. Let $(f_n)_{n\geq 1}$ a sequence in S_F^{Pe} converging to f in P_X^1 with respect to the norm $\|.\|_{Pe}$. The Pettis norm $\|.\|_{Pe}$ is equivalent to the norm defined by

$$f \longrightarrow \sup_{A\in\mathcal{A}} \left\| \int_A f\, dP \right\|, \quad \text{(see [24], p.198)}, \tag{3.9}$$

it follows that

$$\lim_{n\to\infty} \int_A f_n\, dP = \int_A f\, dP \in \overline{\int_A F\, dP},\ \forall A \in \mathcal{A}. \tag{3.10}$$

If $f \notin S_F^{Pe}$, then there is a $B \in \mathcal{A}$ with $P(B) > 0$ such that $f(\omega) \notin F(\omega)$ for all $\omega \in \Omega$. For $\omega \in B$, by (3.8) there exists an integer $n \geq 1$ such that

$$\langle x_n^*, f(\omega)\rangle > \delta^*(x_n^*, F(\omega)). \tag{3.11}$$

For each n let B_n be the set of $\omega \in B$ such that the above inequality holds for x_n^*. Then we have $B = \cup_{n\geq 1} B_n$. Since $0 < P(B) \leq \sum_{n=1}^{\infty} P(B_n)$, there is an integer $n \geq 1$ with $P(B_n) > 0$. Thus, taking into account (3.11) and Strassen type theorem (see, [14], Theorem 3.9, or also [18, 29, 30]), we have

$$\begin{aligned}\left\langle x_n^*, \int_{B_n} f\, dP \right\rangle &= \int_{B_n} \langle x_n^*, f(\omega)\rangle\, dP \\ &> \int_{B_n} \delta^*(x_n^*, F(\omega))\, dP \\ &= \delta^*\left(x_n^*, \int_{B_n} F\, dP\right)\end{aligned}$$

So that $\int_{B_n} f\, dP \notin \overline{\int_{B_n} F\, dP}$. This contradicts (3.10). □

Using Lemma 3.8 (resp. Lemma 3.9) and Proposition 3.10, we immediately obtain the following corollaries.

Corollary 3.11. *Let $F : \Omega \to \mathcal{L}(X)$ be a multifunction. Then S_F^{Pe} is $\|.\|_{Pe}$-closed.*

Corollary 3.12. *Let $F : \Omega \to \mathcal{C}(X)$ be a multifunction. Assume that X^* is separable. Then S_F^{Pe} is $\|.\|_{Pe}$-closed.*

4. Decomposability in the space of Pettis integrable functions

The notion of decomposability of subsets of measurable functions was introduced by Hiai and Umegaki in [18] as follows: Let D be a set of measurable functions from $(\Omega, \mathcal{A})$ into X, we say that D is decomposable (with respect to $\mathcal{A}$) if $f_1, f_2 \in D$ and $A \in \mathcal{A}$ imply $\mathbf{1}_A f_1 + \mathbf{1}_{\Omega \setminus A} f_2 \in D$. The following result allows to characterize the closed decomposable sets in P_X^1, as subsets of the form S_F^{Pe}.

Theorem 4.1. *Let D be a $\|.\|_{Pe}$-closed nonempty subset of P_X^1. Suppose that D is decomposable with respect to $\mathcal{A}$. Then*

(i) For each sequence $\sigma = (f_n)_{n\geq 1}$ in D, we have $S_{\Gamma_\sigma}^{Pe} \subset D$, where Γ_σ is defined for all $\omega \in \Omega$ by $\Gamma_\sigma(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$.

(ii) There exists a sequence σ_0 in D such that $S_{\Gamma_{\sigma_0}}^{Pe} = D$.

Proof. (i) Let $\sigma = (f_n)_{n\geq 1}$ be a sequence in D and $f \in S_{\Gamma_\sigma}^{Pe}$. By Theorem 3.1, f is $\|.\|_{Pe}$-limit of a sequence of simple selections of F with respect to $(f_n)_{n\geq 1}$ and therefore f belongs to D, which is decomposable and $\|.\|_{Pe}$-closed.

(ii) we consider $\mathfrak{S} = \{\Gamma_\sigma : \sigma \subset D\}$. Then $\mathfrak{S}$ is a subset of the set $\mathfrak{A}$ of all $\mathrm{cl}(X)$-valued measurable multifunctions. By Lemma 2.3, there is a countable subset of $\mathfrak{A}$ denoted by

$$\mathfrak{S}_1 = \{\Gamma_{\sigma_m} : \sigma_m \subset D, m \geq 1\}$$

such that the multifunction Φ defined for all ω in Ω by $\Phi(\omega) = \overline{\cup_{m\geq 1}\Gamma_{\sigma_m}(\omega)}$ be the essential supremum of $\mathfrak{S}$ in $\mathfrak{A}$. The multifunction Φ belongs to $\mathfrak{S}$. Indeed, let for any integer $m \geq 1$, $\sigma_m = (f_n^m)_{n\geq 1} \subset D$. For all $\omega \in \Omega$ we have

$$\Phi(\omega) = \overline{\bigcup_{m\geq 1} \Gamma_{\sigma_m}(\omega)} = \overline{\bigcup_{m\geq 1} \overline{\{f_n^m(\omega) : n \geq 1\}}} = \overline{\{f_n^m(\omega) : n, m \geq 1\}}.$$

Let us show that $S_\Phi^{Pe} = D$.
Since $\Phi \in \mathfrak{S}$, from (i) we have $S_\Phi^{Pe} \subset D$.
Conversely, let $f \in D$, the multifunction Γ_f defined for all $\omega \in \Omega$ by $\Gamma_f(\omega) = \{f(\omega)\}$ belongs to $\mathfrak{S}$. Since Φ is the essential supremum of $\mathfrak{S}$, $\Gamma_f \subset \Phi$ a.s., hence

$$f \in S_\Phi^{Pe} \quad \text{and} \quad S_\Phi^{Pe} = D.$$

□

A variant of Theorem 4.1 is given in the following theorem, which assumes that time, the decomposable D is not necessarily closed for the Pettis norm $\|.\|_{Pe}$.

Theorem 4.2. *Let D be a nonempty subset of* P^1_X*. Suppose that D is decomposable with respect to $\mathcal{A}$. Then*

(i) For each sequence $\sigma = (f_n)_{n\geq 1}$ in D, we have $S^{Pe}_{\Gamma_\sigma} \subset \overline{D}^{\|\cdot\|_{Pe}}$, where Γ_σ is defined for all $\omega \in \Omega$ by $\Gamma_\sigma(\omega) = \overline{\{f_n(\omega) : n \geq 1\}}$.

(ii) There exists a sequence σ_0 in D such that $\overline{S^{Pe}_{\Gamma_{\sigma_0}}}^{\|\cdot\|_{Pe}} = \overline{D}^{\|\cdot\|_{Pe}}$.

Proof. (i) Let $\sigma = (f_n)_{n\geq 1}$ be a sequence in D and $f \in S^{Pe}_{\Gamma_\sigma}$. Since D is decomposable, it is the same for $\overline{D}^{\|\cdot\|_{Pe}}$. By Theorem 3.1, f belongs to $\overline{D}^{\|\cdot\|_{Pe}}$.

(ii) Let us adapt the proof of Theorem 4.1(ii). Consider $\mathfrak{S} = \{\Gamma_\sigma : \sigma \subset D\}$ and let Φ be the essential supremum of $\mathfrak{S}$ in the set $\mathfrak{A}$ of all $\mathrm{cl}(X)$-valued measurable multifunctions, defined for all ω in Ω, by

$$\Phi(\omega) = \overline{\{f^m_n(\omega) : n, m \geq 1\}}$$

Since $\Phi \in \mathfrak{S}$, from (i) we have $S^{Pe}_\Phi \subset \overline{D}^{\|\cdot\|_{Pe}}$, so $\overline{S^{Pe}_\Phi}^{\|\cdot\|_{Pe}} \subset \overline{D}^{\|\cdot\|_{Pe}}$. Conversely, let $f \in D$, the multifunction Γ_f defined for all $\omega \in \Omega$ by $\Gamma_f(\omega) = \{f(\omega)\}$ belongs to $\mathfrak{S}$. Since Φ is the essential supremum of $\mathfrak{S}$, $\Gamma_f \subset \Phi$ p.s.. Hence

$$f \in S^{Pe}_\Phi \quad \text{et} \quad D \subset S^{Pe}_\Phi,$$

and therefore $\overline{D}^{\|\cdot\|_{Pe}} \subset \overline{S^{Pe}_\Phi}^{\|\cdot\|_{Pe}}$ □

Remark 4.3. Theorem 4.2 has utility when the set $S^{Pe}_{\Gamma_{\sigma_0}}$ is closed. Its importance lies in the fact that the σ_0 representation of Γ_{σ_0} is in D.

Corollary 4.4. *Let $F : \Omega \to \mathrm{cl}(X)$ be a measurable multifunction such that $S^{Pe}_F \neq \emptyset$ and let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. Assume that there exists $f \in S^{Pe}_F$ such that the Pettis conditional expectation $\mathbb{E}(f|\mathcal{B})$ exists in P^1_X. Then there exists a unique (a.s.) $\mathcal{B}$-measurable, $\mathrm{cl}(E)$-valued Aumann-Pettis integrable multifunction G such that*

$$S^{Pe}_G(\mathcal{B}) = \overline{\left\{\mathbb{E}(f|\mathcal{B}) : f \in S^{Pe}_F \textit{ and } \mathbb{E}(f|\mathcal{B}) \textit{ exists in } \mathrm{P}^1_X \right\}}^{\|\cdot\|_{Pe}}.$$

Proof. We define the non-empty subset D of $\mathrm{P}^1_X(\mathcal{B})$ as follows

$$D = \left\{\mathbb{E}(f|\mathcal{B}) : f \in S^{Pe}_F \text{ and } \mathbb{E}(f|\mathcal{B}) \text{ exists in } \mathrm{P}^1_X\right\}.$$

Since the set D is decomposable with respect to the sub-σ-field $\mathcal{B}$, it is the same for $\overline{D}^{\|\cdot\|_{Pe}}$. By virtue of Theorem 4.1 and Remark 3.3 there exists a unique (a.s.) $\mathcal{B}$-measurable, $\mathrm{cl}(E)$-valued Aumann-Pettis integrable multifunction G such that $S^{Pe}_G(\mathcal{B}) = \overline{D}^{\|\cdot\|_{Pe}}$. □

It must be emphasized that the Pettis conditional expectation of a Pettis integrable function does not always exist. However, following a remark by Musial [23], Egghe (see [13], Proposition 2.1.2) gave a necessary and sufficient condition for the existence of conditional expectation Pettis, when the space X has the weak Radon-Nikodym property (WRNP). Recall in this context that since X is separable then X has (WRNP) implies that X has (RNP) (see [23], p. 154).

Definition 4.5. Let $m : \mathcal{A} \to X$ be a measure. We say that m is of σ-bounded variation if there is a countable partition $(A_n)_{n\geq 1}$ of Ω in $\mathcal{A}$ such that the restriction $m_{|A_n}$ to A_n is a measure of bounded variation, for each $n \geq 1$.

Lemma 4.6. ([13], Proposition 2.1.2.) *Suppose that X is not necessarily separable and has the weak Radon-Nikodym property* (WRNP). *Let $f \in \mathrm{P}^1_X$ and $m : \mathcal{A} \to X$ be given by the Pettis integral*

$$m(A) = \int_A f\, dP$$

for each $A \in \mathcal{A}$. Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$. The restriction $m_{|\mathcal{B}}$ to $\mathcal{B}$ is of σ-bounded variation if and only if $\mathbb{E}(f|\mathcal{B})$ exists in P^1_X.

The following result gives a sufficient condition for the existence of the multivalued conditional expectation for a multifunction Aumann-Pettis integrable.

Proposition 4.7. *Let $F : \Omega \to \mathrm{cl}(X)$ be a measurable multifunction and Aumann-Pettis integrable. Assume that the following conditions (i) and (ii) hold.*

(i) X has (RNP).

(ii) There is a countable partition $(B_i)_{i\geq 1}$ of Ω in $\mathcal{B}$ satisfying

$$\int_{B_i} h(F)\, dP < +\infty, \quad \forall i \geq 1. \tag{4.1}$$

Then there exists a unique (a.s.) $\mathcal{B}$-measurable, $\mathrm{cl}(E)$-valued Aumann-Pettis integrable multifunction denoted $\mathbb{E}(F|\mathcal{B})$ wich enjoys the following property

a)

$$S^{Pe}_{\mathbb{E}(F|\mathcal{B})}(\mathcal{B}) = \overline{\{\mathbb{E}(f|\mathcal{B}) : f \in S^{Pe}_F\}}$$

b)

$$\overline{\int_B^{(\mathcal{B})} \mathbb{E}(F|\mathcal{B})\, dP} = \overline{\int_B F\, dP},\ \forall B \in \mathcal{B}.$$

$\mathbb{E}(F|\mathcal{B})$ is called the Pettis multivalued conditional expectation of F.

Proof. take $f \in S^{Pe}_F$ and let m be the function defined on $\mathcal{A}$ by $m(A) = \int_A f\, dP$. According to Pettis Theorem (see e.g. [12], Theorem 5, p.53), m is a measure on $\mathcal{A}$. And as

$$\int_{B_i} \|f\|\, dP \leq \int_{B_i} h(F)\, dP < +\infty,$$

then the restriction $f_{|B_i}$ to B_i is Bochner integrable, for each $i \geq 1$. It follows from ([12], Theorem 4(iv), p.46) that the variation of m denoted by $|m|$ verifies $|m|(B_i) = \int_{B_i} \|f\|\, dP < +\infty$ for each $i \geq 1$. So using Lemma 4.6, the Pettis conditional expectation $\mathbb{E}(f|\mathcal{B})$ exists in P^1_X.

Now, by Corollary 4.4, there exists a unique (a.s.) $\mathcal{B}$-measurable multifunction with nonempty closed values, denoted by $\mathbb{E}(F|\mathcal{B})$ such that

$$S^{Pe}_{\mathbb{E}(F|\mathcal{B})}(\mathcal{B}) = \overline{\{\mathbb{E}(f|\mathcal{B}) : f \in S^{Pe}_F\}}^{\|\cdot\|_{Pe}}.$$

Let us show the second point. If $g \in S^{Pe}_{\mathbb{E}(F|\mathcal{B})}(\mathcal{B})$, then there exists a sequence (f_n) in S^{Pe}_F such that

$$\lim_{n\to\infty} \|g - \mathbb{E}(f_n|\mathcal{B})\|_{Pe} = 0.$$

It follows from (3.9) that

$$\int_B g\,dP = \lim_{n\to\infty}\int_B \mathbb{E}(f_n|\mathcal{B})\,dP = \lim_{n\to\infty}\int_B f_n\,dP \in \overline{\int_B F\,dP},\ \forall B \in \mathcal{B}. \tag{4.2}$$

Conversely, if $f \in S^{Pe}_X$, then $\mathbb{E}(f|\mathcal{B})$ is in $\mathrm{P}^1_X(\mathcal{B})$ and satisfies $\int_B f\,dP = \int_B \mathbb{E}(f|\mathcal{B})\,dP$, for all $B \in \mathcal{B}$. Hence

$$\int_B f\,dP \in \int_B^{(\mathcal{B})} \mathbb{E}(F|\mathcal{B})\,dP \quad \forall B \in \mathcal{B}. \tag{4.3}$$

From (4.2) and (4.3), we obtain

$$\overline{\int_B^{(\mathcal{B})} \mathbb{E}(F|\mathcal{B})\,dP} = \overline{\int_B F\,dP} \quad \forall B \in \mathcal{B}.$$

□

Given a sequence $(C_n)_{n\geq 1}$ in cwk(X), the series $\sum_n C_n$ is said to be unconditionally convergent provided that for every choice $x_n \in C_n$, $n \geq 1$, the series $\sum_n x_n$ is unconditionally convergent in X. In this case, the set $\sum_n C_n$ defined by

$$\sum_n C_n = \left\{\sum_n x_n : x_n \in C_n \text{ for all } n \geq 1\right\}$$

belongs to cwk(X), (see [5], Lemma 2.2).
Given a map $M : \mathcal{A} \longrightarrow \mathrm{cwk}(X)$.

(i) M is said to be a finitely additive if $M(A_1 \cup A_2) = M(A_1) + M(A_2)$ whenever $A_1, A_2 \in \mathcal{A}$ are disjoint.
(ii) M is said to be scalarly σ-additive if for every $x' \in X'$, the map $A \longrightarrow \int_A \delta^*(x', F)\,dP$ is σ-additive.
(iii) M is a multimeasure if for every $(A_n)_{n\geq 1}$ in $\mathcal{A}$ such that $A_i \cap A_j = \emptyset$ for $i \neq j$, the series $\sum_n M(A_n)$ is unconditionally convergent and $M(\cup_n A_n) = \sum_n M(A_n)$.

Interestingly, by a result of Costé ([9], p. III.4), M is a multimeasure if and only if M is scalarly σ-additive and $S_M \neq \emptyset$, where S_M denotes the set of all measures $m : \mathcal{A} \to X$ such that $m(A) \in M(A)$ for every $A \in \mathcal{A}$. In this case

$M(A) = \{m(A) : m \in S_M\}$ (see [15], chap. 2, Th. 6, p. 67). The variation of the cwk(X)-valued multimeasure M is denoted by $|M|$, and defined by

$$|M|(A) = \sup \sum_{j=1}^{n} h(M(A_j)), \quad \forall A \in \mathcal{A},$$

The supremum being taken over all finite partitions $\{A_1, \cdots, A_n\}$ of A in $\mathcal{A}$. The multimeasure M is said to be P-absolutely continuous if $P(A) = 0$ implies $M(A) = \{0\}$ for every $A \in \mathcal{A}$.

The following result is a Radon-Nikodym type theorem for cwk(X)-valued multimeasures with σ-bounded variation.

Theorem 4.8. *Assume that X and X^* have* (RNP). *Let M be a multimeasure of σ-bounded variation and absolutely continuous with respect to P, then there exists a unique (a.s.) $\mathcal{A}$-measurable multifunction Γ in $\mathcal{P}^1_{\text{cwk}(X)}(\mathcal{A})$ such that*

$$M(A) = \int_A \Gamma \, dP, \quad \forall A \in \mathcal{A}.$$

Proof. The proof will be achieved in two steps.
Claim 1. In this step, we intend to show that there exists a unique (a.s.) measurable cc(E)-valued scalarly integrable multifunction Γ such that

$$M(A) = \int_A \Gamma \, dP, \quad \forall A \in \mathcal{A},$$

where $\int_A \Gamma \, dP$ denotes the Aumann-Pettis integral of Γ on A.
Let S_M be the set of X-valued vector measures m defined on $\mathcal{A}$, such that $m(A) \in M(A)$ for all $A \in \mathcal{A}$. It follows from theorem Costé (see [9], p. III.4 and [15], Ch.2 th. 6, p. 67) that S is not empty and that we have

$$M(A) = \{m(A) : m \in S_M\}, \quad \forall A \in \mathcal{A}.$$

Since M is of σ-bounded variation, there is a countable partition $(B_i)_{i \geq 1}$ of Ω in $\mathcal{A}$ such that for all $i \geq 1$, $|M|(B_i) < +\infty$. For every $m \in S_M$ and every $i \geq 1$, we have

$$|m|(B_i) \leq |M|(B_i) < +\infty,$$

which shows that the measure m is of σ-bounded variation. In addition, m is absolutely continuous with respect to P follows from the absolute continuity of M with respect to P. Since X has (RNP), m admits a Pettis integrable density with respect to P, denoted f_m. Consider the subset D of P^1_X defined by

$$D = \left\{ f_m \in \mathrm{P}^1_X : m \in S_m \right\}.$$

D is decomposable, because if f_1 and $F_2 \in D$ and $A \in \mathcal{A}$ then for any $C \in \mathcal{A}$, we have

$$\begin{aligned}\int_C \mathbf{1}_A f_1 + \mathbf{1}_{\Omega\setminus A} f_2 \, dP &= m_1 (A \cap C) + m_2 ((\Omega \setminus A) \cap C) \\ &\in M_1 (A \cap C) + M_2 ((\Omega \setminus A) \cap C) \\ &= M(C).\end{aligned}$$

We deduce

$$\mathbf{1}_A f_1 + \mathbf{1}_{\Omega\setminus A} f_2 \in D.$$

Note also that D is $\|.\|_{Pe}$-closed in P^1_X. Indeed, let (f_n) be a sequence in D, $\|.\|_{Pe}$-converging to f. It follows from (3.9) that

$$\int_A f \, dP = \lim_{n\to\infty} \int_A f_n \, dP$$

holds for every $A \in \mathcal{A}$. Then, the first member of the equality belongs to $M(A)$ since $M(A)$ is closed. Thus the function $m : A \longrightarrow \int_A f \, dP$ belongs to S_M and hence $f \in D$.

By Theorem 4.1 and Remark 3.3, there exists a unique (a.s.) cl(X)-valued measurable multifunction Γ such that

$$S^{Pe}_{\Gamma} = D = \{f_m : m \in S_m\}. \tag{4.4}$$

Since M is convex-valued, D is convex, and consequently, Γ is a cc(X)-valued multifunction.

On the other hand, according to Srassen theorem, we obtain

$$\int_\Omega \delta^*(x', \Gamma) \, dP = \delta^* \left(x', \int_\Omega \Gamma \, dP \right), \quad \forall x' \in X'. \tag{4.5}$$

It follows that

$$\int_A \delta^*(x', \Gamma) \, dP = \delta^*(x', M(A)) < +\infty, \quad \forall x' \in X', \tag{4.6}$$

and so Γ is scalarly integrable.

Claim 2. Now, let us show to conclude that Γ is weakly compact values. For all $m \in S_M$ and any integer $i \geq 1$, we have $|m|(B_i) = \int_{B_i} \|f_m\| \, dP < +\infty$. This shows that the restriction of f_m to B_i, denoted $f_{m|B_i}$ is Bochner integrable. We deduce from (4.4) that the set of all Bochner integrable selections of $\Gamma_{|B_i}$ is equal to $\left\{f_{m|B_i} : m \in S_m\right\}$ which we denote $S^1_{\Gamma_{|B_i}}$. Moreover, $S^1_{\Gamma_{|B_i}}$ is uniformly integrable follows from

$$\sup_{m\in S_m} \int_{A\cap B_i} \|f_m\| \, dP \leq |M|(A \cap B_i), \quad \forall A \in \mathcal{A}.$$

Since the set $\left\{\int_{A\cap B_i} f \, dP : f \in D\right\}$ is weakly compact, it follows from Dunford-Pettis theorem (see e.g. [12], Theorem 1, p. 101) that $S^1_{\Gamma_{|B_i}}$ is relatively weakly compact in $\mathrm{L}^1_X(B_i)$. Under these conditions, Klei theorem (see [21], Theorem

6) implies that $\Gamma_{|B_i}$ is a relatively weakly compact valued multifunction. And since Γ is cc(X)-valued multifunction, then Γ is a cwk(X)-multifunction. □

As an application of Theorem 4.8, we have the following result which gives a sufficient condition for the existence of the Pettis conditional expectation for cwk(X)-valued measurable and Pettis integrable multifunctions.

Theorem 4.9. *Assume that X and X^* have* (RNP). *Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ and let $F \in \mathcal{P}^1_{\mathrm{cwk}(E)}$. Assume that there exists a countable partition $(B_i)_{i\geq 1}$ of Ω in $\mathcal{B}$ such that*

$$\int_{B_i} h(F)\, dP < +\infty. \tag{4.7}$$

holds for every $i \geq 1$. Then there exists a unique (a.s.) multifunction $\mathbb{E}(F|\mathcal{B})$ belonging to $\mathcal{P}^1_{cwk(E)}(\mathcal{B})$ and satisfying

(i)
$$\int_B \mathbb{E}(F|\mathcal{B})\, dP = \int_B F\, dP, \quad \forall B \in \mathcal{B}.$$

(ii)
$$S^{Pe}_{\mathbb{E}(F|\mathcal{B})} = \left\{\mathbb{E}(f|\mathcal{B}) : f \in S^{Pe}_F\right\}.$$

Proof. Let M be the multifunction defined on $\mathcal{A}$ by

$$M(A) = \int_A F\, dP = \left\{\int_A f\, dP : f \in S^{Pe}_F\right\}, \quad \forall A \in \mathcal{A}.$$

It follows from the Pettis integrability of F that M is a cwk(X)-valued multimeasure. Let $B \in \mathcal{B}$. According to a corollary of Hahn-Banach Theorem, we have

$$h(M(B \cap B_i)) = \sup_{x' \in B_{X'}} \delta^*(x', M(B \cap B_i)) \leq \int_{B \cap B_i} h(F)\, dP.$$

This ensures the following inequality

$$|M|(B_i) \leq \int_{B_i} h(F)\, dP < +\infty.$$

This proves that M is of σ-bounded variation on $\mathcal{B}$. Finally, it is clear that M is absolutely continuous with respect to P, and using Theorem 4.8, we show that there exists a unique (a.s.) multifunction $\mathbb{E}(F|\mathcal{B})$ belonging to $\mathcal{P}^1_{\mathrm{cwk}(E)}(\mathcal{B})$ such that

$$\int_B \mathbb{E}(F|\mathcal{B})\, dP = \int_B F\, dP, \tag{4.8}$$

holds for all $B \in \mathcal{B}$. Let us show (ii). Let $g \in S^{Pe}_{\mathbb{E}(F|\mathcal{B})}$. Then for any B in $\mathcal{B}$, we have

$$\int_B g\, dP \in \int_B \mathbb{E}(F|\mathcal{B})\, dP = \int_B F\, dP.$$

So there exists $f \in S_F^{Pe}$ such that

$$\int_B g\,dP = \int_B f\,dP$$

holds for all $B \in \mathcal{B}$. This implies that g is the Pettis conditional expectation of f and hence $g \in \left\{\mathbb{E}(f|\mathcal{B}) : f \in S_F^{Pe}\right\}$. Conversely, from (4.7), for all $f \in S_F^{Pe}$, $\mathbb{E}(f|\mathcal{B})$ belongs to P^1_X and therefore

$$\int_B \mathbb{E}(f|\mathcal{B})\,dP = \int_B f\,dP \in \int_B F\,dP = \int_B \mathbb{E}(F|\mathcal{B})\,dP, \quad \forall B \in \mathcal{B}.$$

□

Here is an obvious corollary

Corollary 4.10. *Assume that X and X^* have* (RNP)*. Let $\mathcal{B}$ be a sub-σ-field of $\mathcal{A}$ and let $F \in \mathcal{P}^1_{\mathrm{cwk}(E)}$. Assume that the conditional expectation $\mathbb{E}(h(F)|\mathcal{B}) \in [0,+\infty[$, then there exists a unique (a.s.) multifunction $\mathbb{E}(F|\mathcal{B})$ belonging to $\mathcal{P}^1_{\mathrm{cwk}(E)}(\mathcal{B})$ and satisfying*

(i)

$$\int_B \mathbb{E}(F|\mathcal{B})\,dP = \int_B F\,dP, \quad \forall B \in \mathcal{B}.$$

(ii)

$$S^{Pe}_{\mathbb{E}(F|\mathcal{B})} = \left\{\mathbb{E}(f|\mathcal{B}) : f \in S_F^{Pe}\right\}.$$

Proof. Let (B_i) be the countable partition of Ω in $\mathcal{B}$, defined by

$$B_i = \{\omega \in \Omega : i-1 \le \mathbb{E}(h(F)|\mathcal{B}) < i\}, \quad i \ge 1.$$

It is clear that

$$\int_{B_i} h(F)\,dP = \int_{B_i} \mathbb{E}(h(F)|\mathcal{B})\,dP < +\infty.$$

Theorem 4.9 allows to conclude. □

References

[1] R. J. Aumann, *Integrals of set-valued functions*, J. Math. Anal. Appl. 12 (1965) 1–12.

[2] A. Barbati and C. Hess, *The largest class of closed convex valued multifunctions for which Effros measurability and scalar measurability coincide*, Set-Valued Anal. 6 (1998), 209–236.

[3] B. Cascales, V. Kadets and J. Rodriguez, *The Pettis integral for multi-valued functions via single-valued ones* J. Math. Anal. Appl. 332 (1) (2007), 1–10.

[4] B. Cascales, V. Kadets and J. Rodriguez, *Measurable selectors and set-valued Pettis integral in non-separable Banach spaces* J. Funct. Anal. 256 (3) (2009), 673–699.

[5] B. Cascales and J. Rodr?guez, *Birkhoff integral for multi-valued functions*, J. Math. Anal. Appl. 297 (2004), no. 2, 540–560.

[6] C. Castaing and M. Valadier, *Convex analysis and measurable multifunctions*, Lectures notes in Math. **580**, Springer-Verlag, 1977.

[7] A. Costé, *Sur les multimesures à valeurs fermées bornées d'un espace de Banach*, C. R. Acad. Sci. Paris Sér. A-B 280 (1975), A567–A570.

[8] A. Costé, *La propriété de Radon-Nikodým en intégration multivoque*, C. R. Acad. Sci. Paris Sér. A-B 280 (1975), no. 22, Aii, A1515–A1518.

[9] A. Costé: *Contribution à la théorie de l'intégration multivoque*, Thèse d'Etat, Université Pierre et Marie Curie, Paris, 1977.

[10] G. Debreu, *Integration of correspondences*, Proc. Fifth Berkeley Sympos. Math. Statist. and Probability (Berkeley, Calif., 1965/66), Vol. II: Contributions to Probability Theory, Part 1, Univ. California Press, Berkeley, Calif., 1967, pp. 351-372.

[11] L. Di Piazza and K. Musial, *A decomposition theorem for compact-valued Henstock integral*, Monatsh. Math. 148 (2006), no. 2, 119–126.

[12] J. Diestel and J. J. Jr. Uhl, *Vector measures*, Mathematical Surveys, No. 15. American Mathematical Society, Providence, R.I., 1977.

[13] L. Egghe, *Convergence of adapted sequences of Pettis-integrable functions*, Pacific J. Math. 114 (1984), no. 2, 345–366.

[14] K. El Amri and C. Hess, *On the Pettis integral of closed valued multifunctions*, Set-Valued Anal. 8 (2000), no. 4, 329–360.

[15] C. Godet-Thobie, *Multimesures et multimesures de transition*, Thèse d'Etat, Montpellier 1975.

[16] C. Hess and H. Ziat, *Théorème de Komlós pour des multifonctions intégrables au sens de Pettis et applications*, Ann. Sci. Math. Québec 26 (2002), no. 2, 181–198.

[17] F. Hiai, *Radon-Nikodým theorems for set-valued measures*, J. Multivariate Anal. 8 (1) (1978), 96–118.

[18] F. Hiai et H. Umegaki, *Integrals, conditional expectations, and martingales of multivalued functions*, J. Multivariate Anal. 7 (1) (1977), 149–182.

[19] C. J. Himmelberg, *Measurable relations*, Fund. Math. 87 (1975), 53–72.

[20] R. Huff, *Remarks on Pettis integrability*, Proc. Amer. Math. Soc. 96 (1986), no. 3, 402–404.

[21] H.-A. Klei, *A compactness criterion in $L^1(E)$ and Radon-Nikodym theorems for multimeasures*, Bull. Sci. Math. (2) 112 (1988), no. 3, 305–324.

[22] J.J. Moreau, *Fonctionnelles convexes*, Séminaire sur les équations aux dérivées partielles, Collège de France 1996.

[23] K. Musial, *The weak Radon-Nikodým property in Banach spaces*, Studia Math. 64 (2) (1979), 151–173.

[24] K. Musial, *Topics in the theory of Pettis integration*, School on Measure Theory and Real Analysis (Grado, 1991), Rend. Istit. Mat. Univ. Trieste 23 (1991), no. 1, 177–262 (1993).

[25] C. Olech, *Decomposability as a substitute for convexity*, Multifunctions and integrands (Catania, 1983), 193–205, Lecture Notes in Math., 1091, Springer, Berlin, 1984.

[26] M. Talagrand, *Pettis integral and measure theory*, Mem. Amer. Math. Soc. 51 (307) (1984).

[27] M. Tsukada, *Convergence of best approximations in a smooth Banach space*, J. Approx. Theory 40 (4) (1984), 301–309.

[28] M. Valadier, *Contribution à l'analyse convexe*, Thèse de Doctorat d'État, Paris, 1970, Publication No. 92 Université de Montpellier, Montpellier 1970 iii+83 pp.

[29] M. Valadier, *Sur le théorème de Strassen*, C. R. Acad. Sci. Paris Sér. A 278 (1974), 1021–1024.

[30] M. Valadier, *On the Strassen theorem in : "Analyse convexe et ses applications"* Lecture Notes in Econom. and Math. Systems, Vol. 102, Springer, Berlin, 1974.

[31] M. Valadier, *Sur l'espérance conditionnelle multivoque non convexe*, Ann. Inst. H. Poincaré Sect. B (N.S.) 16 (1980), no. 2, 109–116.

[32] H. Ziat, *Convergence des suites adaptées multivoques, Application à la loi forte des grands nombres multivoque*, Thèse de Doctorat, Montpellier 1993.

[33] H. Ziat, *Convergence theorems for Pettis integrable multifunctions*, Bull. Polish Acad. Sci. Math. 45 (2) (1997), 123–137

[34] H. Ziat, *On a characterization of Pettis integrable multifunctions*, Bull. Polish Acad. Sci. Math. 48 (3) (2000), 227–230

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