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On a New Weighted Hilbert's Type Inequality

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Abstract

A new refinement of the Hilbert inequality is established by means of the weight function method. Some particular results are given.

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1 Introduction

If $a_n, b_n > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1$, and $0 < \sum_{n=1}^{\infty} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} b_n^q < \infty$, then the famous Hilbert's inequality(see Hardy et al. [1]) is given by

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left(\sum_{n=1}^{\infty} a_n^p \right)^{1/p} \left(\sum_{n=1}^{\infty} b_n^q \right)^{1/q}, \tag{1}$$

where the constant factor $\frac{\pi}{\sin(\pi/p)}$ is the best possible.

When $p = 2$, Hilbert's inequality (1) can be written in the form

$$\sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} < \pi \left(\sum_{n=1}^{\infty} a_n^2 \right)^{1/2} \left(\sum_{n=1}^{\infty} b_n^2 \right)^{1/2} \tag{2}$$

Inequality (1) and (2) have been proved in many different ways and they have varied extensions and applications (see Hardy et al. [1] and reference cited in [1]).

Recently, He et al. [3] introduced the weight function

$$\omega(n) = \frac{\sqrt{n}}{n+1} \left(\frac{\sqrt{n} - 1}{\sqrt{n} + 1} - \frac{\ln n}{\pi} \right) \tag{3}$$

and gave an extension of (2) as

$$\left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\right)^4 < \pi^4\left\{\left(\sum_{n=1}^{\infty}a_n^2\right)^2-\left(\sum_{n=1}^{\infty}\omega(n)a_n^2\right)^2\right\}\left\{\left(\sum_{n=1}^{\infty}b_n^2\right)^2-\left(\sum_{n=1}^{\infty}\omega(n)b_n^2\right)^2\right\}. \quad (4)$$

In this work, it is interesting to consider the inequality (1) and build a new inequality by using certain weight functions.

We first give two lemmas:

Lemma 1.1 *Let n be a positive integer and $p > 1$, then*

$$\int_0^{\infty}\frac{x^{-1/p}}{(1+x)}dx = \frac{\pi}{\sin(\pi/p)}$$

and

$$\int_0^{\infty}\frac{x^{-1/p}}{(1+x)(1+nx)}dx = \frac{\pi}{\sin(\pi/p)}\frac{n^{1/p}-1}{n-1}.$$

Proof. Please see page 114, Example 2.12 in [2].

Lemma 1.2 *If $f(x) = \frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{p}}\left(1-\left(\frac{x}{1+x}-\frac{n}{1+n}\right)\right)$ and $g(x) = \frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{q}}\left(1+\left(\frac{x}{1+x}-\frac{n}{1+n}\right)\right)$, where $p, q > 1$, $n \in N$ and $x \in (0,\infty)$, then $f(x)$ and $g(x)$ are monotonously decreasing in $(0,\infty)$, also*

$$\int_0^{\infty} f(x)dx = \frac{\pi}{\sin(\pi/p)}(1+\omega(p,n)), \qquad \int_0^{\infty} g(x)dx = \frac{\pi}{\sin(\pi/q)}(1-\omega(q,n)) \quad (5)$$

where $\omega(p,n)$ and $\omega(q,n)$ are defined by

$$\omega(r,n) = -\frac{1}{n+1}+\frac{n^{1/r}-1}{n-1}. \quad (6)$$

Proof. Clearly, $f(x)$ and $g(x)$ are monotonously decreasing in $(0,\infty)$. Compute the integral.

$$\begin{aligned}\int_0^{\infty} f(x)dx &= \int_0^{\infty}\frac{1}{x+n}\left(\frac{n}{x}\right)^{\frac{1}{p}}\left(\frac{1}{1+x}+\frac{n}{1+n}\right)dx\\ &= \int_0^{\infty}\frac{x^{-1/p}}{1+x}\left(\frac{1}{1+nx}+\frac{n}{1+n}\right)dx\\ &= \int_0^{\infty}\frac{x^{-1/p}}{(1+x)(1+nx)}dx+\frac{n}{1+n}\int_0^{\infty}\frac{x^{-1/p}}{1+x}dx\end{aligned}$$

Here, the second equality comes from substituting new x for $\frac{x}{n}$.

By Lemma 1.1, we obtain the first integral of (5) and the second integral can be gotten similarly.

2 Main Results

Our main results can be stated as follows:

Theorem 2.1 *If $a_n, b_n > 0, p > 1, \frac{1}{p} + \frac{1}{q} = 1,$ and $0 < \sum_{n=1}^{\infty} a_n^p < \infty, 0 < \sum_{n=1}^{\infty} b_n^q < \infty,$ then*

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n} < \frac{\pi}{\sin(\pi/p)}\left\{\sum_{m=1}^{\infty}(1-\omega(q,m))a_m^p\right\}^{\frac{1}{2p}}\left\{\sum_{n=1}^{\infty}(1+\omega(p,n))a_n^q\right\}^{\frac{1}{2q}}$$
$$\cdot \left\{\sum_{m=1}^{\infty}(1-\omega(q,m))b_m^p\right\}^{\frac{1}{2p}}\left\{\sum_{n=1}^{\infty}(1+\omega(p,n))b_n^q\right\}^{\frac{1}{2q}}$$

where $\omega(p,n)$ and $\omega(q,n)$ are defined by

$$\omega(r,n) = -\frac{1}{n+1} + \frac{n^{1/r}-1}{n-1}. \tag{7}$$

Proof. Let $c(x)$ be a real function with $1 - c(n) + c(m) \geq 0 \quad (n, m \in N)$. When $b_n = a_n$, we have

$$\begin{aligned}
&\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m b_n}{m+n}\\
=\ &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m a_n}{m+n}(1-c(m)+c(n))\\
=\ &\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m}{(m+n)^{\frac{1}{p}}}\left(\frac{m}{n}\right)^{\frac{1}{pq}}(1-c(m)+c(n))^{\frac{1}{p}}\frac{a_n}{(m+n)^{\frac{1}{q}}}\left(\frac{n}{m}\right)^{\frac{1}{pq}}(1-c(m)+c(n))^{\frac{1}{q}}\\
\leq\ & J_1^{\frac{1}{p}} J_2^{\frac{1}{q}},
\end{aligned}$$

where

$$J_1 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}(1-c(m)+c(n))$$

and

$$J_2 = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_n^q}{m+n}\left(\frac{n}{m}\right)^{\frac{1}{p}}(1-c(m)+c(n)).$$

Let $c(x) = \frac{x}{1+x}$. By Lemma 1.2, since $g(x)$ is monotonously decreasing on $(0, \infty)$, we have

$$\begin{aligned}
J_1 &= \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_m^p}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}\left(1-\frac{m}{1+m}+\frac{n}{1+n}\right)\\
&= \sum_{m=1}^{\infty}\left(\sum_{n=1}^{\infty}\frac{1}{m+n}\left(\frac{m}{n}\right)^{\frac{1}{q}}\left(1+\left(\frac{n}{1+n}-\frac{m}{1+m}\right)\right)\right)a_m^p\\
&< \sum_{m=1}^{\infty}\left\{\int_0^{\infty} g(x)dx\right\}a_m^p
\end{aligned}$$

Using (5), we get

$$J_1 < \frac{\pi}{\sin(\pi/q)} \sum_{m=1}^{\infty} (1 - \omega(q, m)) a_m^p. \tag{8}$$

Similarly, we can obtain

$$J_2 < \frac{\pi}{\sin(\pi/p)} \sum_{n=1}^{\infty} (1 + \omega(p, n)) a_n^q \tag{9}$$

where $\omega(p, n)$ and $\omega(q, n)$ are defined by (7).

Therefore,

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m a_n}{m+n} < \left\{ \frac{\pi}{\sin(\pi/q)} \sum_{m=1}^{\infty} (1 - \omega(q, m)) a_m^p \right\}^{\frac{1}{p}} \left\{ \frac{\pi}{\sin(\pi/p)} \sum_{n=1}^{\infty} (1 + \omega(p, n)) a_n^q \right\}^{\frac{1}{q}}$$

Since $\frac{1}{p} + \frac{1}{q} = 1$ and $\sin(\pi/p) = \sin(\pi/q)$, it follows that

$$\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m a_n}{m+n} < \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{m=1}^{\infty} (1 - \omega(q, m)) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} (1 + \omega(p, n)) a_n^q \right\}^{\frac{1}{q}} \tag{10}$$

Next, consider the case for $b_n \neq a_n$.

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} &= \int_0^1 \left(\sum_{m=1}^{\infty} a_m t^{m-1/2} \right) \left(\sum_{n=1}^{\infty} b_n t^{n-1/2} \right) dt \\ &\leq \left\{ \int_0^1 \left(\sum_{m=1}^{\infty} a_m t^{m-1/2} \right)^2 dt \int_0^1 \left(\sum_{n=1}^{\infty} b_n t^{n-1/2} \right)^2 dt \right\}^{\frac{1}{2}} \\ &= \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m a_n}{m+n} \right)^{\frac{1}{2}} \left(\sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{b_m b_n}{m+n} \right)^{\frac{1}{2}} \end{aligned}$$

So,

$$\begin{aligned} \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} \frac{a_m b_n}{m+n} &< \left\{ \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{m=1}^{\infty} (1 - \omega(q, m)) a_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} (1 + \omega(p, n)) a_n^q \right\}^{\frac{1}{q}} \right\}^{\frac{1}{2}} \\ &\cdot \left\{ \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{m=1}^{\infty} (1 - \omega(q, m)) b_m^p \right\}^{\frac{1}{p}} \left\{ \sum_{n=1}^{\infty} (1 + \omega(p, n)) b_n^q \right\}^{\frac{1}{q}} \right\}^{\frac{1}{2}} \\ &= \frac{\pi}{\sin(\pi/p)} \left\{ \sum_{m=1}^{\infty} (1 - \omega(q, m)) a_m^p \right\}^{\frac{1}{2p}} \left\{ \sum_{n=1}^{\infty} (1 + \omega(p, n)) a_n^q \right\}^{\frac{1}{2q}} \\ &\cdot \left\{ \sum_{m=1}^{\infty} (1 - \omega(q, m)) b_m^p \right\}^{\frac{1}{2p}} \left\{ \sum_{n=1}^{\infty} (1 + \omega(p, n)) b_n^q \right\}^{\frac{1}{2q}} \end{aligned}$$

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