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Fractional Integral Operator and Olsen Inequality in the Non-Homogeneous Classic Morrey Space

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Abstract

We establish the necessary and sufficient condition for the boundedness of the fractional integral operator I_α in the non-homogeneous classic Morrey space. In addition, we also derive Olsen type inequalities involving I_α. In this paper, we use the measure of order which more general than the previous studies. Consequently, the results of previous studies is a particular form of the results of this study.

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1 Introduction

Let R^d be equipped with a metric $|.|$ and a Borel measure μ. We say that a measure μ satisfies the doubling condition ($\mu \in DC$) if there exists a constant $C > 0$ such that for all balls $B(a, r)$, $\mu(B(a, 2r)) \leq C\mu(B(a, r))$. If $\mu \in DC$, then R^d is called homogeneous space. If μ doesn't satisfy the doubling condition, then R^d is called a non-homogeneous space. The results of research on the boundedness of fractional integral operators in homogen spaces can be found in [1, 2, 3, 5, 9, 10]. Researchers have found that some results are still valid even if μ doesn't satisfy the assumption of doubling condition. These results can be seen in [4, 6, 7, 8, 11].

Our main object of study in this paper is the fractional integral operator on non-homogeneous space R^d , I_α, defined for by the formula

$$I_\alpha f(x) := \int_{\Re^d} \frac{f(y)}{|x-y|^{n-\alpha}} dy, 0 < \alpha < n \leq d$$

.

In this paper, we assume that μ satisfies the growth condition of order $s > 0$. We say that a measure μ satisfies the growth condition ($\mu \in GC(s)$), if there exists a constant $C > 0$ such that for all balls $B(a, r)$, $\mu(B(a, r)) \leq Cr^s$. In [4, 6, 7, 8, 11], researchers assume that $\mu \in GC(n)$ with n as in the definition of I_α.

In this paper, we will give the necessary and sufficient condition for the boundedness of I_α in non-homogeneous Lebesgue space $L^p(\mu)$ and in non-homogeneous classic Morrey space $L^{p,\lambda}(\mu)$. In, addition, we shall also derive Olsen type inequalities involving I_α.

We define a classic Morrey space as the set of all $f \in L^p_{loc}(\mu)$ such that

$$\|f : L^{p,\lambda}(\mu)\| := sup_{B:=B(a,r)} \left(\frac{1}{r^\lambda} \int_B |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}.$$

If $\lambda = 0$, then $L^{p,\lambda}(\mu) = L^p(\mu)$.

2 The boundedness of I_α in non-homogeneous Lebesgue space

Let us begin with some assumptions and relevant fact that follow. As customary the letter C denote constants, which are not necessarily the same from line to line.

To prove the boundedness of I_α, we need the boundedness of maximal operator M^s in $L^p(\mu)$. The maximum operator M^s, defined for $s > 0$, by formula

$$M^s f(x) := sup_{r>0} \frac{1}{r^s} \int_{B(x,r)} |f(y)| d\mu(y)$$

where $f \in L^p_{loc}(\mu)$

.

Theorem 2.1 *The maximal operator M^s is boundedness in $L^p(\mu)$.*

Proof. Proof this theorem is similar to proving of the boundedness of M^n in $L^p(\mu)$ (see in [8]).

In the next theorem, we will give the necessary and sufficient condition for the boundedness I_α from $L^p(\mu)$ to $L^q(\mu)$.

Theorem 2.2 *Let $1 < p < q < \infty$. The Operator I_α is bounded from $L^p(\mu)$ to $L^q(\mu)$ if and only if $s = \frac{pq(n-\alpha)}{pq+p-q}$.*

Proof. *Necessity*. Assume that I_α is bounded from $L^p(\mu)$ to $L^q(\mu)$ and $B := B(a, r)$ is be an arbitrary ball in R^d. Since $\|\chi_B : L^p(\mu)\| = \mu(B)^{1/p}$, we get $\chi_B \in L^p(\mu)$. Therefore $\|I_\alpha \chi_B : L^q(\mu)\| \leq C\|\chi_B : L^p(\mu)\| = \mu(B)^{\frac{1}{p}}$. If $x, y \in B$, then $r^{\alpha-n} \leq C I_\alpha \chi_B(y)$. Therefore

$$r^{\alpha-n}\mu(B)^{1+1/q} = C\left(\int_B (r^{\alpha-n}\mu(B))^q d\mu(x)\right)^{1/q} \leq C\left(\int_B |I_\alpha \chi_B(y)|^q d\mu(x)\right)^{1/q}$$

$$\leq \|I_\alpha \chi_B : L^q(\mu)\| \leq C\|\chi_B : L^p(\mu)\| = C\mu(B)^{1/p},$$

so

$$\mu(B) \leq C r^{\frac{pq(n-\alpha)}{pq+p-q}}.$$

Since $\mu \in GC(s)$, we get $s = \frac{pq(n-\alpha)}{pq+p-q}$.

Sufficiency. Let $B := B(x, r)$ is ball in R^d and $f \in L^p(\mu)$. Suppose that $I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x)$ where $f_1(x) = f\chi_B$ and $f_2(x) = f\chi_{B^c}$. For f_1, we have the following estimate:

$$|I_\alpha f_1(x)| \leq \sum_{k=-\infty}^{k=-1} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y)$$

$$\leq C \sum_{k=-\infty}^{k=-1} (2^k r)^{\alpha+s-n} \frac{1}{(2^{k+1} r)^s} \int_{B(a,2^{k+1} r)} |f(y)| d\mu(y)$$

$$\leq Cr^{\alpha+s-n}M^s f(x)\sum_{k=-\infty}^{k=-1}(2^k)^{\alpha+s-n} = Cr^{\alpha+s-n}M^s f(x), \alpha+s-n>0.$$

By Hölder's inequality and the fact that $\mu \in GC(s)$ where $s = \frac{pq(n-\alpha)}{pq+p-q}$, we have the following estimate:

$$|I_\alpha f_2(x)| \leq \sum_{k=0}^{k=\infty}\int_{2^k r\leq|x-y|<2^{k+1}r}\frac{|f(y)|}{|x-y|^{n-\alpha}}d\mu(y)$$

$$\leq C\sum_{k=0}^{k=\infty}(2^k r)^{\alpha-n}\left(\int_{B(a,2^{k+1}r)}|f(y)|^p d\mu(y)\right)^{1/p}\left(\int_{B(a,2^{k+1}r)}d\mu(y)\right)^{\frac{p-1}{p}}$$

$$\leq Cr^{\alpha-n+\frac{s(p-1)}{p}}\|f:L^p(\mu)\|\sum_{k=0}^{k=\infty}(2^k)^{\alpha-n+\frac{s(p-1)}{p}}$$

$$= Cr^{\alpha-n+\frac{s(p-1)}{p}}\|f:L^p(\mu)\|, \alpha-n+\frac{s(p-1)}{p} = -\frac{s}{q} < 0.$$

Combining the two estimates, we get

$$|I_\alpha f(x)| \leq Cr^{\alpha+s-n}(M^s f(x) + r^{-\frac{s}{p}}\|f:L^p(\mu)).$$

Assuming that $f \neq 0$ a.e., we choose $r = \left(\frac{M^s f(x)}{\|f:L^p(\mu)\|}\right)^{-\frac{p}{s}}$. Then we have

$$|I_\alpha f(x)| \leq CM^s f(x)^{1-\frac{p(\alpha-n+s)}{s}}\|f:L^p(\mu)\|^{\frac{p(\alpha-n+s)}{s}}$$

$$\leq CM^s f(x)^{\frac{p}{q}}\|f:L^p(\mu)\|^{1-\frac{p}{q}}$$

By using the boundedness of M^s on $L^p(\mu)$, Theorem 2.2. is completely proved.

If we choose $s = n$, then we will get the following result which can be viewed as Hardy-Littlewood-Sobolev type for non-homogeneous space.

Corollary 2.3 *Let $1 < p < \frac{n}{\alpha}$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$. The operator I_α is bounded from $L^p(\mu)$ to $L^q(\mu)$ if and only if $\mu \in GC(n)$.*

3 The boundedness of I_α in non-homogeneous classic Morrey space

Before we present the boundedness of I_α in the classic Morrey space, we have the following lemma which shows particularly that the space $L^{p,\lambda}(\mu)$ is not empty. The lemma will also be useful later when we prove the necessary condition for the boundedness of I_α in the classic Morrey space.

Lemma 3.1 *If $B_o := B(a_o, r_o)$, then $\chi_{B_o} \in L^{p,\frac{s\lambda}{n}}(\mu)$ where χ_{B_o} is the characteristic function of the ball B_o. Moreover, there exists a constant $C > 0$ such that*

$$\|\chi_{B_o} : L^{p,\frac{s\lambda}{n}}(\mu)\| \leq C r_o^{\frac{s(n-\lambda)}{np}}.$$

Proof. Let $B_o := B(a_o, r_o)$ be an arbitrary ball in R^d. It is easy to see that

$$\|\chi_{B_o} : L^{p,\frac{s\lambda}{n}}(\mu)\| = sup_{B:=B(a,r)} \left(\frac{\mu(B \cap B_o)}{r^{\frac{s\lambda}{n}}}\right)^{\frac{1}{p}}.$$

We may suppose that $B \cap B_o \neq \emptyset$. If $r \leq r_o$,we use fact that $\mu \in GC(s)$, then $\frac{\mu(B\cap B_o)}{r^{\frac{s\lambda}{n}}} \leq \frac{\mu(B)}{r^{\frac{s\lambda}{n}}} \leq C r_o^{\frac{s(n-\lambda)}{n}}, n - \lambda > 0$. On the other hand, if $r > r_o$, then $\frac{\mu(B\cap B_o)}{r^{\frac{s\lambda}{n}}} \leq \frac{\mu(B_o)}{r_o^{\frac{s\lambda}{n}}} \leq C r_o^{\frac{s(n-\lambda)}{n}}$.
This completes the proof of the lemma.

In the next theorem, we will give the necessary and sufficient condition for the boundedness I_α from $L^{p,\frac{s\lambda_1}{n}}(\mu)$ to $L^{q,\lambda_2}(\mu)$.

Theorem 3.2 *Let $\mu \in GC(s)$ with $s = \frac{pq(n-\alpha)}{pq+p-q}$, $1 < p < q < \infty$ and $0 < \lambda_1 < \frac{np}{q}$. The operator I_α is bounded from $L^{p,\frac{s\lambda_1}{n}}(\mu)$ to $L^{q,\lambda_2}(\mu)$ if and only if $\frac{s\lambda_1}{np} = \frac{\lambda_2}{q}$.*

Proof. *Necessity.* Assume that I_α is bounded from $L^{p,\frac{s\lambda_1}{n}}(\mu)$ to $L^{q,\lambda_2}(\mu)$ and $B_o := B(a_o, r_o)$ is ball in R^d. If $x, y \in B_o$, then $r^{\alpha-n}\mu(B_o) \leq C I_\alpha \chi_{B_o}(y)$. Base on Lemma 3.1., we have

$$r_o^{-\frac{\lambda_2}{q}+\alpha-n}\mu(B)^{1+1/q} = C\left(\frac{1}{r_o^{\lambda_2}}\int_{B_o}(r_o^{\alpha-n}\mu(B_o))^q d\mu(x)\right)^{1/q}$$

$$\leq C\left(\frac{1}{r_o^{\lambda_2}}\int_{B_o}|I_\alpha\chi_{B_o}(y)|^q d\mu(x)\right)^{1/q} \leq \|I_\alpha\chi_{B_o} : L^{q,\lambda_2}(\mu)\|$$

$$\leq C\|\chi_{B_o} : L^{p,\frac{s\lambda_1}{n}}(\mu)\| \leq C r_o^{\frac{s(n-\lambda_1)}{np}},$$

so

$$\mu(B_o)^{1+\frac{1}{q}} \leq C(r_o^s)^{\frac{n-\lambda_1}{np}+\frac{\lambda_2}{qs}+\frac{n-\alpha}{s}}.$$

Since $\mu \in GC(s)$ where $s = \frac{pq(n-\alpha)}{pq+p-q}$, we have $1 + \frac{1}{q} = \frac{n-\lambda_1}{np} + \frac{\lambda_2}{qs} + \frac{n-\alpha}{s}$. Thus, $\frac{s\lambda_1}{np} = \frac{\lambda_2}{q}$.
Sufficiency. For $a \in R^d$ and $r > 0$, let $B := B(a, r)$, $\tilde{B} := B(a, 2r)$, and $f \in L^{p,\frac{s\lambda_1}{n}}(\mu)$. Suppose that $I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x)$ where $f_1(x) = f\chi_{\tilde{B}}$

and $f_2(x) = f\chi_{\tilde{B}^c}$.
Since

$$\|f_1 : L^p(\mu)\| = (2r)^{\frac{s\lambda_1}{n}} \left(\frac{1}{(2r)^{\frac{s\lambda_1}{n}}} \int_{\tilde{B}} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} < \infty,$$

we get $f_1 \in L^p(\mu)$. Base on Theorem 2.2 and the fact that $\frac{s\lambda_1}{np} = \frac{\lambda_2}{q}$ we have

$$\left(\frac{1}{r^{\lambda_2}} \int_{B(a,r)} |I_\alpha f_1(y)|^q d\mu(y) \right)^{\frac{1}{q}} \leq r^{-\frac{\lambda_2}{q}} \left(\int_{R^d} |I_\alpha f_1(y)|^q d\mu(y) \right)^{\frac{1}{q}}$$

$$\leq Cr^{-\frac{\lambda_2}{q} + \frac{s\lambda_1}{np}} \left(\frac{1}{(2r)^{\frac{s\lambda_1}{n}}} \int_{\tilde{B}} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \leq C\|f : L^{p,\frac{s\lambda_1}{n}}(\mu)\|.$$

Now we observe that if $x \in B(a,r)$ and $y \in \tilde{B}^c$ then $|x-y| > r$. Hence Hölder's inequality and the fact that $\mu \in GC(s)$ with $s = \frac{pq(n-\alpha)}{pq+p-q}$ and $0 < \lambda_1 < \frac{np}{q}$ yield

$$|I_\alpha f_2(x)| \leq \int_{\tilde{B}^c} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y) \leq \int_{|x-y|>r} \frac{f(y)}{|x-y|^{n-\alpha}} d\mu(y)$$

$$\leq \sum_{k=0}^{k=\infty} (2^k r)^{\alpha-n} \int_{2^k r \leq |x-y| < 2^{k+1} r} |f(y)| d\mu(y)$$

$$\leq C \sum_{k=0}^{k=\infty} (2^k r)^{\alpha-n} \left(\int_{B(x,2^{k+1}r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}} \left(\int_{B(x,2^{k+1}r)} d\mu(y) \right)^{\frac{p-1}{p}}$$

$$\leq C \sum_{k=0}^{k=\infty} (2^k r)^{\alpha-n+\frac{s\lambda_1}{np}+\frac{s(p-1)}{p}} \left(\frac{1}{(2^{k+1}r)^{\frac{s\lambda_1}{n}}} \int_{B(x,2^{k+1}r)} |f(y)|^p d\mu(y) \right)^{\frac{1}{p}}$$

$$\leq Cr^{\alpha-n+\frac{s\lambda_1}{np}+\frac{s(p-1)}{p}} \|f : L^{p,\frac{s\lambda_1}{n}}(\mu)\| \sum_{k=0}^{k=\infty} 2^{k(\alpha-n+\frac{s\lambda_1}{np}+\frac{s(p-1)}{p})}$$

$$\leq Cr^{\alpha-n+\frac{s\lambda_1}{np}+\frac{s(p-1)}{p}} \|f : L^{p,\frac{s\lambda_1}{n}}(\mu)\|, \alpha - n + \frac{s\lambda_1}{np} + \frac{s(p-1)}{p} < 0.$$

Base on the fact that $s = \frac{pq(n-\alpha)}{pq+p-q}$ and $\frac{s\lambda_1}{np} = \frac{\lambda_2}{q}$, we have

$$-\frac{\lambda_2}{q} + \alpha - n + \frac{s\lambda_1}{np} + \frac{s(p-1)}{p} + \frac{s}{q} = 0.$$

Therefore

$$\left(\frac{1}{\lambda_2}\int_{B(a,r)} |I_\alpha f_2(x)|^q d\mu(y)\right)^{\frac{1}{q}} \leq Cr^{-\frac{\lambda_2}{q}+\alpha-n+\frac{s\lambda_1}{np}+\frac{s(p-1)}{p}+\frac{s}{q}} \|f : L^{p,\frac{s\lambda_1}{n}}(\mu)\|$$

$$= C\|f : L^{p,\frac{s\lambda_1}{n}}(\mu)\|.$$

By Minkowski's inequality, Theorem 3.2. is completely proved.

If we choose $s = n$, then we will get the following result which can be viewed as Spanne type for non-homogeneous space.

Corollary 3.3 *Let* $1 < p < \frac{n}{\alpha}$, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, $0 < \lambda_1 < n$, *and* $\mu \in GC(n)$. *The operator* I_α *is bounded from* $L^{p,\lambda_1}(\mu)$ *to* $L^{q,\lambda_2}(\mu)$ *if and only if* $\frac{\lambda_1}{p} = \frac{\lambda_2}{q}$.

We use the following lemma when we prove the necessary condition for the boundedness I_α from $L^{p,\lambda}(\mu)$ to $L^{q,\lambda}(\mu)$. The proof of this lemma is similar with Lemma 3.1.

Lemma 3.4 *Let* $0 < \lambda < s$. *If* $B_o := B(a_o, r_o)$, *then* $\chi_{B_o} \in L^{p,\lambda}(\mu)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|\chi_{B_o} : L^{p,\lambda}(\mu)\| \leq Cr_o^{\frac{s-\lambda}{p}}.$$

In the next theorem, we will give the necessary and sufficient condition for the boundedness I_α from $L^{p,\lambda}(\mu)$ to $L^{q,\lambda}(\mu)$.

Theorem 3.5 *Let* $1 < p < q < \infty$ *and* $0 < \lambda < s$. *The Operator* I_α *is bounded from* $L^{p,\lambda}(\mu)$ *to* $L^{q,\lambda}(\mu)$ *if and only if* $\frac{1}{q} = \frac{1}{p} - \frac{\alpha-n+s}{s-\lambda}$.

Proof. *Necessity.* Assume that I_α is bounded from $L^{p,\lambda}(\mu)$ to $L^{q,\lambda}(\mu)$ and $B_o := B(a_o, r_o)$ is an arbitrary ball in R^d. By using the same process as in theorem 3.2. we get

$$\mu(B_o)^{1+\frac{1}{q}} \leq C(r_o^s)^{\frac{s-\lambda}{sp}+\frac{\lambda}{qs}+\frac{n-\alpha}{s}}.$$

Since $\mu \in GC(s)$, we have $1 + \frac{1}{q} = \frac{s-\lambda}{sp} + \frac{\lambda}{qs} + \frac{n-\alpha}{s}$. Thus, $\frac{1}{q} = \frac{1}{p} - \frac{\alpha-n+s}{s-\lambda}$.

Sufficiency. Let $B := B(x,r)$ is ball in R^d and $f \in L^{p,\lambda}(\mu)$. Suppose that $I_\alpha f(x) = I_\alpha f_1(x) + I_\alpha f_2(x)$ where $f_1(x) = f\chi_B$ and $f_2(x) = f\chi_{B^c}$. For f_1, we have the following estimate:

$$|I_\alpha f_1(x)| \leq \sum_{k=-\infty}^{k=-1} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y)$$

$$\leq C \sum_{k=-\infty}^{k=-1} (2^k r)^{\alpha+s-n} \frac{1}{(2^{k+1}r)^s} \int_{B(a,2^{k+1}r)} |f(y)| d\mu(y)$$

$$\leq C r^{\alpha+s-n} M^s f(x) \sum_{k=-\infty}^{k=-1} (2^k)^{\alpha+s-n} = C r^{\alpha+s-n} M^s f(x), \alpha+s-n > 0.$$

By Hölder's inequality and the fact that $\mu \in GC(s)$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha-n+s}{s-\lambda}$, we have the following estimate:

$$|I_\alpha f_2(x)| \leq \sum_{k=0}^{k=\infty} \int_{2^k r \leq |x-y| < 2^{k+1} r} \frac{|f(y)|}{|x-y|^{n-\alpha}} d\mu(y)$$

$$\leq C \sum_{k=0}^{k=\infty} (2^k r)^{\alpha-n} \left(\int_{B(a,2^{k+1}r)} |f(y)|^p d\mu(y) \right)^{1/p} \left(\int_{B(a,2^{k+1}r)} d\mu(y) \right)^{\frac{p-1}{p}}$$

$$\leq C \sum_{k=0}^{k=\infty} (2^k r)^{\alpha-n+\frac{s(p-1)}{p}+\frac{\lambda}{p}} \left(\frac{1}{(2^{k+1}r)^\lambda} \int_{B(a,2^{k+1}r)} |f(y)|^p d\mu(y) \right)^{1/p}$$

$$\leq C r^{\alpha-n+\frac{s(p-1)}{p}+\frac{\lambda}{p}} \|f : L^{p,\lambda}(\mu)\| \sum_{k=0}^{k=\infty} (2^k)^{\alpha-n+\frac{s(p-1)}{p}+\frac{\lambda}{p}}$$

$$= C r^{\alpha-n+\frac{s(p-1)}{p}+\frac{\lambda}{p}} \|f : L^{p,\lambda}(\mu)\|, \alpha-n+\frac{s(p-1)}{p}+\frac{\lambda}{p} = \frac{\lambda-s}{q} < 0.$$

Combining the two estimates, we get

$$|I_\alpha f(x)| \leq C r^{\alpha-n} (r^s M^s f(x) + r^{\frac{s(p-1)+\lambda}{p}} \|f : L^{p,\lambda}(\mu)).$$

Assuming that $f \neq 0$ a.e. , we choose $r = \left(\frac{M^s f(x)}{\|f : L^{p,\lambda}(\mu)\|} \right)^{\frac{p}{\lambda-s}}$. Then, we have

$$|I_\alpha f(x)| \leq C M^s f(x)^{1-\frac{p(\alpha-n+s)}{s-\lambda}} \|f : L^p(\mu)\|^{\frac{p(\alpha-n+s)}{s-lambda}}$$

$$\leq CM^s f(x)^{\frac{p}{q}} \|f : L^p(\mu)\|^{1-\frac{p}{q}}$$

By using the boundedness of M^s on $L^p(\mu)$, Theorem 3.5. is completely proved.

If we choose $s = n$, then we will get the following result which can be viewed as Adams type for non-homogeneous space.

Corollary 3.6 *Let $1 < p < \frac{n}{\alpha}$, $0 < \lambda < p - n\alpha$, and $\mu \in GC(n)$. The operator I_α is bounded from $L^{p,\lambda}(\mu)$ to $L^{q,\lambda}(\mu)$ if and only if $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n-\lambda}$.*

4 Olsen Type Inequalities

In studying a Schrödinger equation with perturbed potentials W on R^n particularly, for $n = 3$, Olsen proved the following theorem [5].

Theorem 4.1 *(Olsen) Let $1 < p < \frac{n}{\alpha}$ and $0 < \lambda < n - p\alpha$. If $W \in L^{\frac{n-\lambda}{\alpha},\lambda}(R^n)$, then the operator W is bounded on $L^{p,\lambda}(R^n)$. Moreover, there exists a constant $C > 0$ such that*

$$\|W.I_\alpha f : L^{p,\lambda}(R^n)\| \leq C\|W : L^{\frac{n-\lambda}{\alpha},\lambda}(R^n)\|\|f : L^{p,\lambda}(R^n)\|$$

The results of studies about the boundedness of $W.I_\alpha$ on R^n can be seen in [7, 10]. In this paper, we will present here boundedness of $W.I_\alpha$ in non-homogeneous Lebesgue space and non-homogeneous classic Morrey space $\mu \in GC(s), s > 0$.

Theorem 4.2 *Let $s > 0$ and $1 < p < \infty$. If $W \in L^{\frac{s}{\alpha-n+s}}(\mu)$, then the operator W is bounded in $L^p(\mu)$. Moreover, there exists a constant $C > 0$ such that*

$$\|W.I_\alpha f : L^p(\mu)\| \leq C\|W : L^{\frac{s}{\alpha-n+s}}(\mu)\|\|f : L^p(\mu)\|.$$

Proof. Let q satisfy $q > p$ and $s = \frac{pq(n-\alpha)}{pq+p-q}$. By Hölder's inequality, we have

$$\left(\int_{R^d} |W.I_\alpha f(x)|^p d\mu(x)\right)^{\frac{1}{p}} \leq C\left(\int_{R^d} |W(x)|^{\frac{pq}{q-p}} d\mu(x)\right)^{\frac{q-p}{pq}} \left(\int_{R^d} |f(x)|^q d\mu(x)\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_{R^d} |W(x)|^{\frac{s}{\alpha-n+s}} d\mu(x)\right)^{\frac{\alpha-n+s}{s}} \left(\int_{R^d} |f(x)|^q d\mu(x)\right)^{\frac{1}{q}}.$$

By the boundedness of I_α from $L^p(\mu)$ to $L^q(\mu)$ (Theorem 2.2), we get

$$\|W.I_\alpha f : L^p(\mu)\| \leq \|W : L^{\frac{s}{\alpha-n+s}}(\mu)\|\|f : L^p(\mu)\|.$$

This completes the proof of the theorem.

Theorem 4.3 *Let* $1 < p < q < \infty$, $s = \frac{pq(n-\alpha)}{pq+p-q}$ *and* $0 < \lambda_1 < \frac{np}{q}$. *If* $W \in L^{\frac{s}{\alpha-n+s}}(\mu)$, *then the operator* W *is bounded in* $L^{p,\frac{s\lambda_1}{n}}(\mu)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|W.I_\alpha f : L^{p,\frac{s\lambda_1}{n}}(\mu)\| \leq C\|W : L^{\frac{s}{\alpha-n+s}}(\mu)\|\|f : L^{p,\frac{s\lambda_1}{n}}(\mu)\|.$$

Proof. Let λ_2 satisfy $\frac{s\lambda_1}{np} = \frac{\lambda_2}{q}$. By Hölder's inequality, we have

$$\left(\frac{1}{r^{\frac{s\lambda_1}{n}}}\int_{R^d}|W.I_\alpha f(x)|^p d\mu(x)\right)^{\frac{1}{p}}$$

$$\leq Cr^{-\frac{s\lambda_1}{np}+\frac{\lambda_2}{q}}\left(\int_{R^d}|W(x)|^{\frac{pq}{q-p}}d\mu(x)\right)^{\frac{q-p}{pq}}\left(\frac{1}{r^{\lambda_2}}\int_{R^d}|f(x)|^q d\mu(x)\right)^{\frac{1}{q}}$$

$$\leq C\left(\int_{R^d}|W(x)|^{\frac{s}{\alpha-n+s}}d\mu(x)\right)^{\frac{\alpha-n+s}{s}}\left(\frac{1}{r^{\lambda_2}}\int_{R^d}|f(x)|^q d\mu(x)\right)^{\frac{1}{q}}.$$

By the boundedness of I_α from $L^{p,\frac{s\lambda_1}{n}}(\mu)$ to $L^{q,\lambda_2}(\mu)$ (Theorem 3.2), we get

$$\|W.I_\alpha f : L^{p,\frac{s\lambda_1}{n}}(\mu)\| \leq C\|W : L^{\frac{s}{\alpha-n+s}}(\mu)\|\|f : L^{p,\frac{s\lambda_1}{n}}(\mu)\|.$$

This completes the proof of the theorem.

Theorem 4.4 *Let* $s > 0$, $0 < \lambda < s$ *and* $1 < p < \infty$. *If* $W \in L^{\frac{s-\lambda}{\alpha-n+s}}(\mu)$, *then the operator* W *is bounded in* $L^{p,\lambda}(\mu)$. *Moreover, there exists a constant* $C > 0$ *such that*

$$\|W.I_\alpha f : L^{p,\lambda}(\mu)\| \leq C\|W : L^{\frac{s-\lambda}{\alpha-n+s}}(\mu)\|\|f : L^{p,\lambda}(\mu)\|.$$

Proof. Let q satisfy $q > p$ and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha-n+s}{s-\lambda}$. By Hölder's inequality, we have

$$\left(frac1r^{\lambda}\int_{R^d}|W.I_\alpha f(x)|^p d\mu(x)\right)^{\frac{1}{p}}$$

$$leqC\left(frac1r^{\lambda}\int_{R^d}|W(x)|^{\frac{pq}{q-p}}d\mu(x)\right)^{\frac{q-p}{pq}}\left(\frac{1}{r^{\lambda}}\int_{R^d}|f(x)|^q d\mu(x)\right)^{\frac{1}{q}}$$

$$\leq C\left(\frac{1}{r^{\lambda}}\int_{R^d}|W(x)|^{\frac{s}{\alpha-n+s}}d\mu(x)\right)^{\frac{\alpha-n+s}{s}}\left(\frac{1}{r^{\lambda}}\int_{R^d}|f(x)|^q d\mu(x)\right)^{\frac{1}{q}}.$$

By the boundedness of I_α from $L^{p,\lambda}(\mu)$ to $L^{q,\lambda}(\mu)$ (Theorem 3.5), we get

$$\|W.I_\alpha f : L^{p,\lambda}(\mu)\| \leq C\|W : L^{\frac{s-\lambda}{\alpha-n+s}}(\mu)\|\|f : L^{p,\lambda}(\mu)\|.$$

This completes the proof of the theorem.

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