Int. Journal of Math. Analysis, Vol. 6, 2012, no. 31, 1545 - 1551

Compact Composition Operators on the Hardy-Smirnov Spaces

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Abstract

For a simply connected domain G properly contained in $\mathbb{C}$, we apply the results of [6] and [7] to give estimates for the essential norm of composition operators on the Hardy-Smirnov space $E^p(G)$. As a corollary to these results, we present characterizations of compactness of bounded composition operators on $E^p(G)$ and give an example illustrating the main results.

Mathematics Subject Classification: 47B38, 30D55, 46E15

Keywords: Hardy spaces, Hardy-Smirnov spaces, Weighted composition operators, essential norm

1 Introduction

For a simply connected domain G properly contained in $\mathbb{C}$ and a Riemann map η that takes the open unit disk D univalently onto G, for $0 < p < \infty$, we define the **Hardy-Smirnov** spaces $E^p(G)$ as the collections of functions f holomorphic on G such that

$$\sup_{0<r<1} \int_{\eta(z:|z|=r)} |f(w)|^p \, |dw| < \infty \tag{1.1}$$

When G is a Jordan domain with rectifiable boundary, $E^p(G)$ coincides with $H^p(G)$ up to an isometry [4]. In particular, $E^p(D) = H^p$. However, if the region G is an interior of a Jordan curve which is analytic except at one point, where it has a corner with interior angle α, then $E^p(G)$ is properly contained in G if $0 < \alpha < \pi$, while $H^p(G)$ is properly contained in $E^p(G)$ if $\pi < \alpha < 2\pi$ [4].

For ϕ, ψ holomorphic maps ϕ, ψ on D with $\phi(D) \subset D$, $0 < p < \infty$, a **weighted composition operator** $W_{\phi,\psi} : H^p \to H^p$ is defined as

$$W_{\phi,\psi}(f) = \psi(f \circ \phi), \qquad f \in H^p \tag{1.2}$$

In particular, if ψ is the constant function 1, then $W_{\phi,\psi}$ becomes the usual composition operator, typically denoted by C_ϕ. Multiplication operators M_ψ are also weighted composition operators with ϕ being the identity map on D. These definitions of weighted composition operators, and hence composition operators, extend naturally to other function spaces.
Let X and Y be Banach spaces. For a bounded linear operator $T : X \to Y$, the **essential norm** $||T||_{e,X\to Y}$ of T is defined to be the distance from T to the set of the compact operators

$$||T||_{e,X\to Y} = \inf \{||T - \kappa||;\ \kappa : X \to Y \quad \text{is compact}\} \tag{1.3}$$

where $||.||$ denotes the usual norm operator. We denote the essential norm of T by $||T||_{e,X}$ when $T : X \to X$. Clearly, T is compact if and only if $||T||_{e,X\to Y} = 0$. Consequently, the essential norm is often used to characterize compactness of concrete operators. The essential norm of various concrete weighted composition operators on the Hardy spaces of the unit disk has been studied by several authors (for more recent results see [1], [2], [3], [7]). In this paper, we use the results of [6] to identify composition operators on the Hardy-Smirnov space $E^p(G)$ as weighted composition operators on the the Hardy spaces H^p of the unit disk, and, subsequently, apply the recent results of [7] to give estimate of the essential norm of composition operators on $E^p(G)$. Our main result, thus, reads as follows.

Theorem 1.1.
For $\zeta \in \partial(D)$ and $r > 0$, let $S(\zeta, r)$ be the Carleson set of $\overline{D}$ defined as

$$S(\zeta, r) = \{z \in \overline{D} : |1 - z\zeta| < 1\} \tag{1.4}$$

If the disk map $\varphi : D \to D$ is given by $\varphi = \eta^{-1} \circ \phi \circ \eta$ and C_ϕ is a bounded composition operator on $E^p(G)$, then for $1 \le p < \infty$ and $Q_\varphi(z) = \dfrac{\eta'(z)}{\eta'(\varphi(z))}$ for all $z \in D$

$$\begin{aligned} ||C_\phi||^p_{e,E^p(G)} &\sim \lim_{|w|\to 1^-} \sup \int_{\partial(D)} \left|Q^*_\varphi(\zeta)\right| \left\{ \frac{1 - |w|^2}{|1 - \overline{w}\varphi^*(\zeta)|^2} \right\} dm(\zeta) \\ &\sim \limsup_{r \to 0^+} \sup_{\zeta \in \partial D} \frac{\mu_{\varphi, Q_\varphi^{1/p}}(S(\zeta, r))}{r} \end{aligned} \tag{1.5}$$

where $Q^*_\varphi, \varphi^* : \partial D \to \overline{D}$ are radial limit function of Q_φ and φ respectively, $\mu_{\varphi,\psi}$ is the pull–back measure induced by φ and ψ, and the symbol $\sim$ means that the ratios of two terms are bounded below and above by constants.

2 The essential norm of Weighted Composition Operators

In [1], Contreras and Hernández-Díaz have characterized the compactness of weighted composition operators $W_{\phi,\psi} : H^p \to H^p \, (1 < p < \infty)$ in terms of the pull-back measure. However, they have not given the estimate for the essential norm of $W_{\phi,\psi}$. The essential norm of $W_{\phi,\psi} : H^p \to H^p$ has been studied by Čučković and Zhao [2], [3]. Recently, Ueki and Luo [7] have found estimates of the essential norm of $W_{\phi,\psi} : H^p \to H^q \, (1 < p \le q < \infty)$. The one dimensional version of their results states that if $W_{\phi,\psi}$ is a bounded weighted composition operator on H^p, then

$$\begin{aligned} ||W_{\phi,\psi}||^p_{e,H^p} &\sim \lim_{|w|\to 1^-} \sup \int_{\partial(D)} |\psi^*(\zeta)|^p \left\{ \frac{1 - |w|^2}{|1 - \overline{w}\phi^*(\zeta)|^2} \right\} dm(\zeta) \\ &\sim \limsup_{r \to 0^+} \sup_{\zeta \in \partial D} \frac{\mu_{\phi,\psi}(S(\zeta, r))}{r} \end{aligned} \tag{2.1}$$

3 Main Results

Let us consider composition operators $C_\phi : E^p(G) \to E^p(G)$ for $0 < p < \infty$. Using change of variable formula (see e.g [4],Corollary, page 169), it can be verified that

$$f \in E^p(G) \leftrightarrow f(\eta(\omega))(\eta'(\omega))^{1/p} \in H^p \tag{3.1}$$

We define a weighted composition operator $W_{\varphi,p} : Hol(D) \to Hol(D)$ associated with C_ϕ as

$$W_{\varphi,p} = V_p \circ C_\phi \circ V_p^{-1} \tag{3.2}$$

where $V_p f = (\eta')^{1/p}(f \circ \eta)$, $f \in Hol(G)$.
It is easy to see that

$$(W_{\varphi,p})(f)(z) = (Q_\varphi(z))^{1/p}(f(\varphi(z))), z \in G \tag{3.3}$$

Note that, since $\eta^{'}$ never vanishes, Q_φ is holomorphic on D, hence $W_{\varphi,p}$ is a weighted composition operator on $Hol(D)$ and hence in H^p.

The following facts are extracted from the recent paper of Shapiro and Smith [6]

Remark 3.1.

(a) *V_p defines isometric similarity between $C_\phi : E^p(G) \to E^p(G)$ and $W_{\varphi,p} : H^p \to H^p$. Thus, the two are unitarily equivalent.*

(b) *C_ϕ bounded (compact) if and only if $W_{\varphi,p}$ is bounded (compact) (a direct consequence of* (a))

(c) *Boundedness and compactness of C_ϕ is independent of p, (i.e., if these properties hold for some p, $0 < p < \infty$, they hold for all p)*

(d) *Both η' and $\dfrac{1}{\eta'}$ are bounded on D if and only if every composition operator on $E^p(G)$ is bounded.*

(e) *$E^p(G)$ supports compact composition operators if and only if $\eta' \in H^1$, which can be rephrased as $\partial(G)$ having finite one-dimensional Hausdorff measure* (see [5], Theorem 10.11, pp. 320–321). *In the case when G is a Jordan domain, this condition is in turn equivalent to that $\partial(G)$ is rectifiable* (see also [5], Lemma 10.7, page 319).

Thus, given $C_\phi : E^p(G) \to E^p(G)$, where G is simply connected, it can be viewed as a weighted composition operator on H^p with weight $\psi(z) = \left(\dfrac{\eta'(z)}{\eta'(\varphi(z))}\right)^{1/p}$, where η is the Riemann map from D onto G. Making use of (1.6), we get the corresponding estimates for the essential norm of C_ϕ, which proves our main theorem. As a corollary, we can state the following characterization of compactness of bounded composition operators on $E^p(G)$ (see Corollary 3.7 [7]).

Corollary 3.1.
For a simply connected domain G and the Riemann map $\eta : D \to G$ with $\eta' \in H^1$, let $C_\phi : E^p(G) \to E^p(G)$ and $\varphi = \eta^{-1} \circ \phi \circ \eta : D \to D$ with $Q_\varphi(z) = \dfrac{\eta'(z)}{\eta'(\varphi(z))}$ for all $z \in D$.
If C_ϕ is bounded on $E^p(G)$, then the following statements are equivalent:

(a) *$C_\phi : E^p(G) \to E^p(G)$ is compact;*

(b) *Q_φ and φ satisfy*

$$\lim_{|w|\to 1^-} \int_{\partial(D)} |Q_\varphi^*(\zeta)| \left\{ \frac{1-|w|^2}{|1-\overline{w}\varphi^*(\zeta)|^2} \right\} dm(\zeta) = 0 \tag{3.4}$$

(c) *Q_φ and φ satisfy*

$$\limsup_{r\to 0^+} \sup_{\zeta\in\partial D} \frac{\mu_{\varphi, Q_\varphi^{1/p}}(S(\zeta,r))}{r} = 0 \tag{3.5}$$

Remark 3.2. *The assumption $\eta' \in H^1$ in* **Corollary 3.1** *is incorporated to ensure that $E^p(G)$ supports compact composition operators* [6].

4 Applications

We give an example verifying our main theorem for a simple geometry, where an explicit and simplified expression for the Riemann map is known [6].

Example 4.1. *Let D represent the unit disk. Let $\eta(z) = (z+1)^2$, so that $\eta(D) = G$ is a "heart-shaped" domain symmetric about the real axis, whose inward-pointing cusp has vertex at the origin. Let $\varphi(z) = \dfrac{z}{2}$, so that C_φ is compact on any Hardy space H^p. Then $Q_\varphi(z) = \dfrac{\eta'(z)}{\eta'(\varphi(z))} = \dfrac{2(z+1)}{2(z/2+1)} = \dfrac{2(z+1)}{z+2}$. Applying* **Theorem 1.1**, *we estimate the essential norm of $||C_\phi||_{e,E^p(G)}$,* *where* $\phi = \eta \circ \varphi \circ \eta^{-1}$.

$$\begin{aligned} ||C_\phi||^p_{e,E^p(G)} &\sim \lim_{|w|\to 1^-} \sup \int_{\partial D} \left|\frac{2(\zeta+1)}{\zeta+2}\right| \left\{\frac{1-|w|^2}{|1-\overline{w}\zeta/2|^2}\right\} dm(\zeta) \\ &\leq M \int_0^{2\pi} \left\{\frac{1-|w|^2}{\left|1-|\overline{w}\zeta/2|\right|^2}\right\} d\theta(\zeta) \\ &\leq M \int_0^{2\pi} \left\{\frac{1-|w|^2}{\left|1-|\overline{w}/2|\right|^2}\right\} d\theta \\ &\leq M \frac{(1-|w|^2)}{\left|1-|\overline{w}/2|\right|^2} \to 0 \quad \text{as} \quad |w| \to 1^- \end{aligned} \tag{4.1}$$

where M *is a upper bound for* $\left|\frac{2(\zeta+1)}{\zeta+2}\right|$ *on* ∂D. *Thus, it shows that* $C_\phi : E^p(G) \to E^p(G)$ *is compact by* **Corollary 3.1**.

Remark 4.1. *The* **Example 4.1** *can be easily extended for any simply connected domain* G *properly contained in* $\mathbb{C}$ *and any Riemann map* η *with* $Q_\varphi(z) = \frac{\eta'(z)}{\eta'(\varphi(z))}$ *bounded on* D *and inducing map* φ *satisfying* $||\varphi||_\infty < 1$.

Acknowledgements.

The first author would like to express his deep gratitude to his advisor the late Prof. T. A. Metzger for introducing me to the subject. He also owe a lot to his insight, enthusiasm and understanding. He would like also to express his deep gratitude to my PhD. thesis advisors Prof. Juan J. Manfredi, Prof. Christopher J. Lennard, Prof. Frank H. Beatrous, and the late Prof. Jacob Burbea for their indispensable support and guidance during the early stage of his research in the area. The publication of this manuscript was supported by ADVANCE Partnerships for Adaptation, Implementation, and Dissemination (PAID) Award: Gender Equity in STEM at Oklahoma State University, HRS-0820240.

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Received: December 11, 2011