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On *n Power Class (Q)* Operators

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Abstract

In this paper we introduce the new class $n\, power\, class(Q)$ operators acting on a Hilbert space H. An operator $T \in L(H)$ is $n\, power class(Q)$ if $T^{*2}T^{2n} = (T^*T^n)^2$. We investigate some basic properties of such operator. In general a $n\, power\, class(Q)$ operator need not be a normal operator .

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1 Introduction

Throughout this paper H is a Hilbert space and $L(H)$ is the algebra of all bounded linear operators acting on H . An operator $T \in L(H)$ is called $class(Q)$ if $T^{*2}T^2 = (T^*T)^2$, T is called normal if $T^*T = TT^*$, T is n-normal if $T^*T^n = T^nT^*$,

T is n power quasi normal if $T^n(T^*T)=(T^*T)T^n$ and T is quasi n normal if $T(T^*T^n)=(T^*T^n)T$.

2 Main Results

In this section we investigate some properties of operators in $n\ power class(Q)$.

Theorem 2.1 If $T \in n\ power class(Q)$ then so are

(i) kT for any real number k.

(ii) any $S \in L(H)$ that is unitarily equivalent to T.

(iii) the restriction T/M of T to any closed subspace M of H that reduces T.

Proof . (i) The proof is straightforward.

(ii) Let $S \in L(H)$ be unitarily equivalent to T then there is a unitary operator $U \in L(H)$ such that $S^{2n} = U^*T^{2n}U$ which implies that $S^* = U^*T^*U$.

Thus, $S^{*^2}S^{2n} = U^*T^*UU^*T^*US^{2n} = U^*T^*UU^*T^*UU^*T^{2n}U = U^*(T^*)^2T^{2n}U$

and $(S^*S^n)^2 = (U^*T^*UU^*T^nU)^2 = (U^*T^*T^nU)^2$

$= (U^*T^*T^nU)\ (U^*T^*T^nU) = U^*(T^*T^n)^2U$

Since $T^{*^2}T^{2n} = (T^*T^n)^2$ we have $S^{*^2}S^{2n} = (S^*S^n)^2$.

Thus $S \in n\ power class(Q)$.

(iii) By [2] we have

$$\left(T/M\right)^{*^2}\left(T/M\right)^{2n} = \left(T^{*^2}/M\right)\left(T^{2n}/M\right) = \left(T^{*^2}T^{2n}/M\right)$$

$$= (T^*T^n)^2/M = \left[\left(T/M\right)^*\left(T/M\right)^n\right]^2$$

Thus $T/M \in n\ power class(Q)$.

The following example shows that if unitarily equivalence in theorem 2.1 (ii) is replaced by similarity then the result is need not be true.

Example 2.2 Consider the two operators $T = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix}$ and $X = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acting on the two dimensional Hilbert space then $T \in 2\ power class(Q)$. Now

$X^{-1} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$ and by direct decomposition we show that $XTX^{-1} = \begin{pmatrix} 2 & -1 \\ 0 & 1 \end{pmatrix} = S$ (say). Now again by direct decomposition we show that $S^{*^2} S^4 = \begin{pmatrix} 64 & -60 \\ -48 & 46 \end{pmatrix}$ while $(S^* S^2)^2 = \begin{pmatrix} 88 & -72 \\ -48 & 40 \end{pmatrix}$. Thus S is similar to T but $S \notin 2\, power\, class(Q)$.

The following example shows that the sum and product of $2\, power class(Q)$ operators are not $2\, power class(Q)$.

Example 2.3 Consider the operators $S = \begin{pmatrix} i & 1 \\ 0 & -i \end{pmatrix}$ and $T = \begin{pmatrix} i & 0 \\ 1 & -i \end{pmatrix}$ are $2\, power class(Q)$ operators on the complex Hilbert space . But $S + T$ and ST are not $2\, power class(Q)$.

Remark 2.4 If $T \in n\, power\, class(Q)$ such that $T^2 = 0$ then it is not necessarily that $T = 0$. Consider $T = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}$ acting on R^2 which is not normal.

Theorem 2.5 If $T \in L(H)$ is n-normal then $T \in n\, power\, class(Q)$.

Proof. Since T is n-normal then $T^* T^n = T^n T^*$

Pre multiply by T^* and post multiply by T^n on both sides we get,

$$T^* T^* T^n T^n = T^* T^n T^* T^n$$

$$T^{*^2} T^{2n} = (T^* T^n)^2$$

Hence $T \in n\, power\, class(Q)$.

The following example shows that an operator of $2\, power class(Q)$ need not be 2 normal.

Example 2.6 If $T = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$ be an operator acting on three dimensional complex Hilbert space. Then T is $2\, power class(Q)$ but it is not 2 normal.

The following examples show that a $2\, powerclass(Q)$ need not be $3\, powerclass(Q)$ and vice versa.

Example 2.7 Consider the operator $T=\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on 2 dimensional complex Hilbert space which is $2\, powerclass(Q)$ but not $3\, powerclass(Q)$.

Example 2.8 Consider the operator $T=\begin{pmatrix} 1 & 1 \\ -1 & 0 \end{pmatrix}$ acting on 2 dimensional Hilbert space which is $3\, powerclass(Q)$ but not $2\, powerclass(Q)$.

Theorem 2.9 If T is $n\, power\, class(Q)$ and T is quasi n normal then T is $n+1\ power\, class(Q)$.

Proof. If T is $n\, power\, class(Q)$ then $T^{*2}T^{2n}=(T^*T^n)^2$

Post multiply by T^2 on both sides

$$T^{*2}T^{2n}T^2=(T^*T^n)^2T^2$$

$$T^{*2}T^{2n+2}=(T^*T^n)(T^*T^n)TT$$

Since T is quasi n normal we have

$$T^{*2}T^{2(n+1)}=(T^*T^n)T(T^*T^n)T=(T^*T^{n+1})^2$$

Hence $T\in n+1\ power\, class(Q)$.

The following example shows that the condition that T is quasi n normal is necessary.

Example 2.10 Consider the operator $T=\begin{pmatrix} i & 2 \\ 0 & -i \end{pmatrix}$ acting on R^2 which is $2\, powerclass(Q)$, not quasi n normal and not $3\, powerclass(Q)$.

Theorem 2.11 Let $T_1, \dots\dots, T_m$ be n normal operators in $L(H)$. Then $(T_1 \oplus \dots\dots \oplus T_m)$ and $(T_1 \otimes \dots\dots \otimes T_m)$ are $n\, power\, class(Q)$ operators.

Proof .

$$(T_1 \oplus T_2 \oplus \dots\dots \oplus T_m)^{*2}(T_1 \oplus T_2 \oplus \dots\dots \oplus T_m)^{2n}$$

$$=(T_1 \oplus T_2 \oplus \dots\dots \oplus T_m)^*(T_1 \oplus T_2 \oplus \dots\dots \oplus T_m)^*(T_1 \oplus T_2 \oplus \dots\dots \oplus T_m)^n(T_1 \oplus T_2 \oplus \dots \oplus T_m)^n$$

$$=(T_1 \oplus T_2 \oplus \dots\dots \oplus T_m)^*(T_1^* \oplus T_2^* \oplus \dots\dots \oplus T_m^*)(T_1^n \oplus T_2^n \oplus \dots\dots \oplus T_m^n)(T_1 \oplus T_2 \oplus \dots \oplus T_m)^n$$

$$=(T_1\oplus T_2\oplus........\oplus T_m)^*\left(T_1^*T_1^n\oplus T_2^*T_2^n\oplus.......\oplus T_m^*T_m^n\right)(T_1\oplus T_2\oplus....\oplus T_m)^n$$

$$=(T_1\oplus T_2\oplus........\oplus T_m)^*\left(T_1^nT_1^*\oplus T_2^nT_2^*\oplus.......\oplus T_m^nT_m^*\right)(T_1\oplus T_2\oplus....\oplus T_m)^n$$

$$=(T_1\oplus T_2\oplus........\oplus T_m)^*\left(T_1^n\oplus T_2^n\oplus.......\oplus T_m^n\right)\left(T_1^*\oplus T_2^*\oplus......\oplus T_m^*\right)(T_1\oplus T_2\oplus....\oplus T_m)^n$$

$$=(T_1\oplus T_2\oplus........\oplus T_m)^*(T_1\oplus T_2\oplus.......\oplus T_m)^n(T_1\oplus T_2\oplus......\oplus T_m)^*(T_1\oplus T_2\oplus....\oplus T_m)^n$$

$$=\left((T_1\oplus T_2\oplus........\oplus T_m)^*(T_1\oplus T_2\oplus......\oplus T_m)^n\right)^2$$

Hence $(T_1\oplus\ldots\ldots.\oplus T_m)$ is $n\ power\, class(Q)$.

Now $x_1,x_2,\ldots\ldots..x_m\in H$,

$$(T_1\otimes T_2\otimes.......\otimes T_m)^{*^2}(T_1\otimes T_2\otimes........\otimes T_m)^{2n}(x_1\otimes x_2\otimes......\otimes x_m)$$

$$=(T_1\otimes T_2\otimes..\otimes T_m)^*(T_1\otimes T_2\otimes..\otimes T_m)^*(T_1\otimes T_2\otimes..\otimes T_m)^n(T_1\otimes T_2\otimes..\otimes T_m)^n(x_1\otimes x_2\otimes..\otimes x_m)$$

$$=(T_1\otimes T_2\otimes..\otimes T_m)^*\left(T_1^*\otimes T_2^*\otimes..\otimes T_m^*\right)\left(T_1^n\otimes T_2^n\otimes..\otimes T_m^n\right)\left(T_1^n\otimes T_2^n\otimes..\otimes T_m^n\right)(x_1\otimes x_2\otimes..\otimes x_m)$$

$$=(T_1\otimes T_2\otimes...\otimes T_m)^*\left(T_1^*T_1^nT_1^nx_1\otimes T_2^*T_2^nT_2^nx_2\otimes......\otimes T_m^*T_m^nT_m^nx_m\right)$$

$$=(T_1\otimes T_2\otimes...\otimes T_m)^*\left(T_1^nT_1^*T_1^nx_1\otimes T_2^nT_2^*T_2^nx_2\otimes......\otimes T_m^nT_m^*T_m^nx_m\right)$$

$$=(T_1\otimes T_2\otimes..\otimes T_m)^*\left(T_1^n\otimes T_2^n\otimes..T_m^n\right)\left(T_1^*\otimes T_2^*\otimes..\otimes T_m^*\right)\left(T_1^n\otimes T_2^n\otimes..\otimes T_m^n\right)(x_1\otimes x_2\otimes..\otimes x_m)$$

$$=\left((T_1\otimes T_2\otimes........\otimes T_m)^*(T_1\otimes T_2\otimes......\otimes T_m)^n\right)^2(x_1\otimes x_2\otimes......\otimes x_m)$$

$$=\left((T_1\otimes T_2\otimes........\otimes T_m)^*(T_1\otimes T_2\otimes......\otimes T_m)^n\right)^2$$

Hence $(T_1\otimes\ldots\ldots..\otimes T_m)$ is $n\ power\, class(Q)$.

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