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A Conjecture on Upper Bound of the Numerical Radius of a Bounded Linear Operator[1]

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Abstract. We prove an operator inequality which improves on either upper bound or lower bound of the numerical radius of a bounded linear operator acting on a complex Hilbert space H. We also conjecture an inequality involving upper bound of the numerical radius. If the conjecture is true then it gives better estimation of the upper bound for the numerical radius of a bounded linear operator.

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1. Introduction

Suppose T be a bounded linear operator on a complex Hilbert space H with inner product (,) and norm $\| \|$. Let W(T), $\sigma(T)$ denote respectively the numerical range, spectrum of T and $w(T)$, $r_\sigma(T)$ denote respectively the numerical radius, spectral radius of T, i.e.,

$$W(T) \ = \ \{(Tx, x) : \|x\| = 1\} \ and \ w(T) \ = \ \sup\{| \ \lambda \ | : \ \lambda \in W(T)\}.$$

It is easy to see that $w(T)$ is a norm on B(H), the Banach algebra of all bounded linear operators on H. Also $w(T)$ is equivalent to the usual operator norm $\|T\|$ on B(H) as

$$\frac{\|T\|}{2} \leq w(T) \leq \|T\| \ldots \ (1)$$

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Kittaneh [3] substantially improved on the second inequality to prove that if T is a bounded linear operator on a complex Hilbert space H then

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}} \ldots \; (2)$$

Clearly $\frac{1}{2}\|T\|+\frac{1}{2}\|T^2\|^{\frac{1}{2}} \leq \|T\|$ so that inequality (2) is sharper than the second inequality of (1). The significant part in inequality (2) is the contribution made by the second factor involving $\|T^2\|$. Some easy examples mentioned below illustrate the fact that one can not compare $w(T)$ with $\|T^2\|^{\frac{1}{2}}$. If T is a 2×2 nilpotent matrix with index 2 then one may get $w(T) = \frac{1}{2}$ and $\|T^2\| = 0$ whereas if T is a 3×3 nilpotent matrix with index 3 then one may get $w(T) = \frac{1}{\sqrt{2}}$ and $\|T^2\| = 1$. U.Haagerup and P.De La Harpe [2] estimated the numerical radius of a nilpotent operator on a Hilbert space and proved that

$$w(T) \leq \|T\| \cos\frac{\pi}{n+1}, \; where\ T^n = 0\ for\ some\ n \geq 2,$$

the equality holds when T is the n-dimensional shift on the space C^n.

Let $T = U \mid T \mid$ be the polar decomposition of T, then the Aluthge [1] transform $\tilde{T}$ of T is defined as $\tilde{T} =\mid T \mid^{\frac{1}{2}} U \mid T \mid^{\frac{1}{2}}$. Using the inequality (2) of Kittaneh, T.Yamazaki [4] obtained an inequality concerning operator norm $\|T\|$, numerical radius $w(T)$ and Aluthge transform $\tilde{T}$ of T as follows

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2}w(\tilde{T}).$$

Letting $\tilde{T}_0 = T$ and $\tilde{T}_n = \tilde{\tilde{T}}_{n-1}$ for natural number n, Yamazaki also proved that

$$w(T) \leq \sum_{n=1}^{\infty} \frac{1}{2^n}\|\tilde{T}_{n-1}\|.$$

2. Main Results

We first prove the following theorem

Theorem 2.1. *Let T be a bounded linear operator on a complex Hilbert space H. Then either there exists some $n_0 \in N$ such that*

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}}$$

or for all $n \in N$

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < w(T).$$

Proof. We know that $\frac{\|T\|}{2} \leq w(T) \leq \|T\|$. We first note that if $w(T) = \|T\|/2$ then it satisfies the first inequality (for $n_0 = 1$) and if $w(T) = \|T\|$ then it satisfies the second inequality stated in the theorem.
If $w(T) > \frac{\|T\|}{2}$ then by Archimedean property there exists $n \in N$ such that

$$n(w(T) - \frac{\|T\|}{2}) > \frac{1}{2^2}\|T^2\|^{\frac{1}{2}}.$$

So

$$w(T) > \frac{1}{2}\|T\| + \frac{1}{2^2 n}\|T^2\|^{\frac{1}{2}}.$$

Let $S = \{n \in N : w(T) > \frac{1}{2}\|T\| + \frac{1}{2^2 n}\|T^2\|^{\frac{1}{2}}\}$. Then $S \neq \phi$ and so S has a least element $k_1 \in N$. If $k_1 \neq 1$ then

$$\frac{1}{2}\|T\| + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} < w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2^2(k_1 - 1)}\|T^2\|^{\frac{1}{2}}$$

or if $k_1 = 1$ then we get

$$w(T) > \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}}.$$

In both cases i.e., for $k_1 > 1$ and $k_1 = 1$ we have $w(T) > \frac{1}{2}\|T\| + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}}$. Again by Archimedean property there exists $n \in N$ such that

$$n(w(T) - \frac{1}{2}\|T\| - \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}}) > \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}}.$$

As before we can find a least element $k_2 \in N$ such that if $k_2 \neq 1$ then

$$\frac{1}{2}\|T\| + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3 k_2}\|T^{2^2}\|^{\frac{1}{2^2}} < w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3(k_2 - 1)}\|T^{2^2}\|^{\frac{1}{2^2}}$$

or if $k_2 = 1$ then

$$w(T) > \frac{1}{2}\|T\| + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}}.$$

Proceeding in this way we get a sequence of natural numbers $\{k_n\}$ such that either of the following two cases arise
Case 1. $k_n \neq 1$ *for some* n. In this case

$$\frac{1}{2}\|T\| + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n k_{n-1}}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} + \frac{1}{2^{n+1} k_n}\|T^{2^n}\|^{\frac{1}{2^n}} < w(T)$$

and

$$w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2^2 k_1}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n k_{n-1}}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} + \frac{1}{2^{n+1}(k_n - 1)}\|T^{2^n}\|^{\frac{1}{2^n}}.$$

This is an improvement on Kittenah's inequality as well as any other known inequality involving lower and upper bounds of numerical radius.

Case 2. $k_n = 1\ \forall n \in N$. In this case for all $n \in N$

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < w(T).$$

This is also an improvement of any known inequality involving lower bound of the numerical radius.

If Case 1 holds we get the existence of $n_0 \in N$ such that

$$w(T) \le \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^{n_0}}\|T^{2^{n_0-1}}\|^{\frac{1}{2^{n_0-1}}}$$

and if Case 2 holds then we get $\forall n \in N$

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} < w(T).$$

This completes the proof.

Remark 2.2. If T is a normaloid operator or if T is an n-dimensional shift on the space C^n ($n > 2$), then $k_1 = 1$. Also if $w(T) > \frac{3}{4}\|T\|$ i.e., $w(T)$ is nearer to $\|T\|$ than $\frac{1}{2}\|T\|$ then $k_1 = 1$.

Remark 2.3. In the proof of theorem if $k_1 = 2$ then

$$\frac{1}{2}\|T\| + \frac{1}{2^2.2}\|T^2\|^{\frac{1}{2}} < w(T) \le \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} < \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}$$

and if $k_1 = 1, k_2 = 2$ then

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3.2}\|T^{2^2}\|^{\frac{1}{2^2}} < w(T) \le \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}}$$

so that

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3.2}\|T^{2^2}\|^{\frac{1}{2^2}} < w(T) \le \frac{1}{2}\|T\| + \frac{3}{8}\|T^2\|^{\frac{1}{2}} < \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}.$$

Thus in both cases $k_1 = 2$ and $k_1 = 1, k_2 = 2$ we see that the inequality obtained is better than Kittaneh's inequality $w(T) \le \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}$.

Thus if $k_1 = 2$ then

$$\frac{1}{2}\|T\| - \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} \le \|T\| - w(T) \le \frac{1}{2}\|T\| - \frac{1}{2^3}\|T^2\|^{\frac{1}{2}}.$$

and if $k_1 = 1, k_2 = 2$ then

$$\frac{1}{2}\|T\| - \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} - \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}} \le \|T\| - w(T) \le \frac{1}{2}\|T\| - \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} - \frac{1}{2^4}\|T^{2^2}\|^{\frac{1}{2^2}}$$

so that we can measure the difference $\|T\| - w(T)$ in some sense.

Remark 2.4. (i) When Case 2 holds then we have

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} + \ldots \leq w(T).$$

(ii) There is nothing special in considering $\|T\|, \|T^2\|^{\frac{1}{2}}, \|T^{2^2}\|^{\frac{1}{2^2}}, \ldots \|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}}, \ldots$. One can also consider $\|T\|, \|T^2\|^{\frac{1}{2}}, \|T^3\|^{\frac{1}{3}}, \ldots \|T^n\|^{\frac{1}{n}}, \ldots$.

Example 2.5. Let us consider a nilpotent operator

$$T = S_5 = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

Then $\|T\| = \|T^2\| = \|T^3\| = \|T^4\| = 1,\ T^5 = 0,\ w(T) = \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}$ so that

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3.2}\|T^{2^2}\|^{\frac{1}{2^2}} < w(T) \leq \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \frac{1}{2^3}\|T^{2^2}\|^{\frac{1}{2^2}}.$$

Thus T satisfies the inequalities under case 1.
If we take

$$T = S_4 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

then $\|T\| = \|T^2\| = \|T^3\| = 1,\ T^4 = 0,\ w(T) = \cos\frac{\pi}{5} = 0.80901699$ so that

$$\frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} < w(T).$$

Thus T satisfies the inequalities under case 2.

Example 2.6. Consider a bounded linear operator T for which $\|T\|^n = \|T^n\|\ \forall\, n$, such an operator is called normaloid operator. It is known that $w(T) = \|T\|$ iff $\|T\|^n = \|T^n\|\ \forall\ n$ so that T will satisfy the inequality under case 2. In fact for such an operator T

$$w(T) = \frac{1}{2}\|T\| + \frac{1}{2^2}\|T^2\|^{\frac{1}{2}} + \ldots + \frac{1}{2^n}\|T^{2^{n-1}}\|^{\frac{1}{2^{n-1}}} + \ldots\ldots = \|T\|.$$

Conjecture 2.7. Suppose T be a bounded linear operator on a Hilbert space H. If T is not nilpotent then we conjecture that

$$w(T) \leq \sum_{n=0}^{\infty} \frac{1}{2^{n+1}}\|T^{2^n}\|^{\frac{1}{2^n}}.$$

If the conjecture is true then the equality $w(T) = \frac{1}{2}\|T\| + \frac{1}{2}\|T^2\|^{\frac{1}{2}}$ in the Kittaneh's inequality (2) holds iff T^2 is normaloid.

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