Int. Journal of Math. Analysis, Vol. 6, 2012, no. 29, 1423 - 1429

Fuzzy Stability of Functional Equations

Won-Gil Park

Department of Mathematics, College of Education
Mokwon University, Daejeon 302-729, Korea
wgpark@mokwon.ac.kr

Afshin Erami

Department of Mathematics, Marvdasht Branch
Islamic Azad University, Marvdasht, Iran
eramiafshin@yahoo.com

Habib Hoseini

Department of Mathematics, Firoozabad Branch
Islamic Azad University, Firoozabad, Iran
mmmomid@gmail.com

Anoshiravan Ghaffaripour and Hassan Azadi Kenary

Department of Mathematics, College of Sciences
Yasouj University, Yasouj 75914-353, Iran
an-ghaffaripour@mail.yu.ac.ir, azadi@mail.yu.ac.ir

Abstract. In [2] Azadi Kenary et al. proved the Hyers-Ulam-Rassias stability of the following quadratic functional equation:

$$f(3x+y)+f(3x-y)=f(x+y)+f(x-y)+16f(x)$$

in non-Archimedean normed spaces.
In this paper, using the fixed point method, we prove the Hyers-Ulam-Rassias stability of the above functional equation in fuzzy Banach spaces.

Mathematics Subject Classification: 39B22, 39B52, 39B82, 46S10, 47S10, 46S40

Keywords: Hyers-Ulam-Rassias stability, Fuzzy normed space

1. **Introduction and preliminaries**

A classical question in the theory of functional equations is the following: *When is it true that a function which approximately satisfies a functional equation must be close to an exact solution of the equation?*. If the problem accepts a solution, we say that the equation is *stable*. The first stability problem concerning group homomorphisms was raised by Ulam [8] in 1940. In the next year, Hyers [1] gave a positive answer to the above question for additive groups under the assumption that the groups are Banach spaces. In 1978, Rassias [6] proved a generalization of Hyers' theorem for additive mappings.

Refer to ([2]-[8]) for detailed information on stability of functional equations.

Definition 1.1. *Let X be a real vector space. A function $N : X \times \mathbb{R} \to [0,1]$ is called a fuzzy norm on X if for all $x, y \in X$ and all $s, t \in \mathbb{R}$,*
$(N1)$ $N(x,t) = 0$ *for* $t \leq 0$*;*
$(N2)$ $x = 0$ *if and only if* $N(x,t) = 1$ *for all* $t > 0$*;*
$(N3)$ $N(cx,t) = N\left(x, \frac{t}{|c|}\right)$ *if* $c \neq 0$*;*
$(N4)$ $N(x+y, c+t) \geq min\{N(x,s), N(y,t)\}$*;*
$(N5)$ $N(x,.)$ *is a non-decreasing function of* $\mathbb{R}$ *and* $\lim_{t\to\infty} N(x,t) = 1$*;*
$(N6)$ *for* $x \neq 0$, $N(x,.)$ *is continuous on* $\mathbb{R}$*.*

The pair (X,N) is called a fuzzy normed vector space (briefly, FNS).

Definition 1.2. *Let (X,N) be an FNS. A sequence $\{x_n\}$ in X is said to be convergent if there exists an $x \in X$ such that $\lim_{t\to\infty} N(x_n - x, t) = 1$ for all $t > 0$. In this case, x is called the limit of the sequence $\{x_n\}$ in X and we denote it by $N\text{-}\lim_{t\to\infty} x_n = x$.*

Definition 1.3. *Let (X,N) be an FNS. A sequence $\{x_n\}$ in X is called Cauchy if for each $\epsilon > 0$ and each $t > 0$ there exists an $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ and all $p > 0$, we have*

$$N(x_{n+p} - x_n, t) > 1 - \epsilon.$$

It is well known that every convergent sequence in an FNS is Cauchy. If each Cauchy sequence is convergent, then the fuzzy norm is said to be complete and the fuzzy normed vector space is called a fuzzy Banach space.

We say that a mapping $f : X \to Y$ between FNS X and Y is continuous at a point $x \in X$ if for each sequence $\{x_n\}$ converging to $x_0 \in X$, then the sequence $\{f(x_n)\}$ converges to $f(x_0)$. If $f : X \to Y$ is continuous at each $x \in X$, then $f : X \to Y$ is said to be continuous on X.

Theorem 1.4. *Let (X,d) be a complete generalized metric space and $J : X \to X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then, for all $x \in X$, either $d(J^n x, J^{n+1} x) = \infty$ for all nonnegative integers n or there exists a positive integer n_0 such that*

(a) *$d(J^nx, J^{n+1}x) < \infty$ for all $n \geq n_0$;*
(b) *the sequence $\{J^nx\}$ converges to a fixed point y^* of J;*
(c) *y^* is the unique fixed point of J in the set*

$$Y = \{y \in X : d(J^{n_0}x, y) < \infty\};$$

(d) *$d(y, y^*) \leq \frac{1}{1-L} d(y, Jy)$ for all $y \in Y$.*

In this paper, using the fixed point and direct methods, we prove the Hyers-Ulam-Rassias stability of the following functional equation:

$$f(3x + y) + f(3x - y) = f(x + y) + f(x - y) + 16f(x) \tag{1.1}$$

in fuzzy normed spaces.

2. FNS-stability of functional equation (1.1)

Throughout this subsection, using the fixed point alternative approach we prove the Hyers-Ulam-Rassias stability of functional equation (1.1) in fuzzy Banach spaces. In this subsection, assume that X is a vector space and that (Y, N) is a fuzzy Banach space.

Theorem 2.1. *Let $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{3}, \frac{y}{3}\right) \leq \frac{L}{9}\varphi(x, y)$$

for all $x, y \in X$. Let $f : X \to Y$ is a mapping satisfying

$$N\left(f(3x \pm y) - f(x \pm y) - 16f(x), t\right) \geq \frac{t}{t + \varphi(x, y)} \tag{2.1}$$

for all $x, y \in X$ and all $t > 0$. Then the limit

$$Q(x) := N\text{-}\lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right)$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(18 - 18L)t}{(18 - 18L)t + L\varphi(x, 0)}. \tag{2.2}$$

Proof. Putting $y = 0$ and then replacing x by $\frac{x}{3}$ in (2.1), we have

$$N\left(f(x) - 9f\left(\frac{x}{3}\right), \frac{t}{2}\right) \geq \frac{t}{t + \varphi\left(\frac{x}{3}, 0\right)} \tag{2.3}$$

for all $x \in X$ and $t > 0$. Consider the set $S := \{g : X \to Y\}$ and the generalized metric d in S defined by

$$d(f, g) := \inf\left\{\mu \in \mathbb{R}^+ : N(g(x) - h(x), \mu t) \geq \frac{t}{t + \varphi(x, 0)} \text{ for all } x \in X \text{ and all } t > 0\right\},$$

where $\inf \emptyset = +\infty$. It is easy to show that (S, d) is complete (see [5, Lemma 2.1]). Now, we consider a linear mapping $J : S \to S$ such that

$$Jg(x) := 9g\left(\frac{x}{3}\right)$$

for all $x \in X$. Let $g, h \in S$ be such that $d(g, h) = \epsilon$. Then $N(g(x)-h(x), \epsilon t) \ge \frac{t}{t+\varphi(x,0)}$ for all $x \in X$ and $t > 0$. Hence

$$\begin{aligned} N(Jg(x) - Jh(x), L\epsilon t) &= N\left(9g\left(\frac{x}{3}\right) - 9h\left(\frac{x}{3}\right), L\epsilon t\right) \\ &= N\left(g\left(\frac{x}{3}\right) - h\left(\frac{x}{3}\right), \frac{L\epsilon t}{9}\right) \ge \frac{\frac{Lt}{9}}{\frac{Lt}{9} + \varphi\left(\frac{x}{3}, 0\right)} \ge \frac{t}{t+\varphi(x,0)} \end{aligned}$$

for all $x \in X$ and all $t > 0$. Thus $d(g, h) = \epsilon$ implies that $d(Jg, Jh) \le L\epsilon$. This means that $d(Jg, Jh) \le Ld(g, h)$ for all $g, h \in S$. It follows from (2.3) that

$$N\left(f(x) - 9f\left(\frac{x}{3}\right), \frac{Lt}{18}\right) \ge \frac{t}{t+\varphi(x,0)}.$$

This means that $d(f, Jf) \le \frac{L}{18}$. By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$9Q\left(\frac{x}{3}\right) = Q(x) \tag{2.4}$$

for all $x \in X$. The mapping A is a unique fixed point of J in the set $\Omega = \{h \in S : d(g, h) < \infty\}$. This implies that Q is a unique mapping satisfying (2.4) such that there exists $\mu \in (0, \infty)$ satisfying

$$N(f(x) - Q(x), \mu t) \ge \frac{t}{t+\varphi(x,0)}$$

for all $x \in X$ and all $t > 0$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right) = Q(x)$$

for all $x \in X$.

(3) $d(f,Q) \leq \frac{1}{1-L} d(f,Jf)$ with $f \in \Omega$, which implies the inequality $d(f,Q) \leq \frac{L}{18-18L}$. This implies that the inequality (2.2) holds. Furthermore, since

$$\begin{aligned} & N\left(Q(3x \pm y) - Q(x \pm y) - 16fQx), t\right) \\ & \geq N\text{-}\lim_{n\to\infty} N\left(9^n f\left(\frac{3x \pm y}{3^n}\right) - 9^n f\left(\frac{x \pm y}{3^n}\right) - 16.9^n f\left(\frac{x}{3^n}\right), t\right) \\ & \geq \lim_{n\to\infty} \frac{\frac{t}{9^n}}{\frac{t}{9^n} + \frac{L^n}{9^n}\varphi(x,y)} \to 1 \end{aligned}$$

for all $x, y \in X$ and all $t > 0$. Thus the mapping $Q : X \to Y$ is a mapping satisfying (1.1), as desired. □

Corollary 2.2. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying*

$$N\left(f(3x \pm y) - f(x \pm y) - 16f(x), t\right) \geq \frac{t}{t + \theta\left(\|x\|^p + \|y\|^p\right)} \tag{2.5}$$

for all $x, y \in X$ and all $t > 0$. Then the limit

$$Q(x) := N\text{-}\lim_{n\to\infty} 9^n f\left(\frac{x}{3^n}\right)$$

exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(2 \cdot 3^p - 18)t}{(2 \cdot 3^p - 18)t + \theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.1 by taking $\varphi(x,y) := \theta(\|x\|^p + \|y\|^p)$ for all $x, y \in X$. Then we can choose $L = 3^{2-p}$ and we get the desired result. □

Theorem 2.3. *Let $\varphi : X^2 \to [0,\infty)$ be a function such that there exists an $L < 1$ with*

$$\varphi(x,y) \leq 9L\varphi\left(\frac{x}{3}, \frac{y}{3}\right)$$

for all $x, y \in X$. Let $f : X \to Y$ be a mapping satisfying (2.1). Then $Q(x) := N\text{-}\lim_{n\to\infty} \frac{f(3^n x)}{9^n}$ exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that

$$N(f(x) - Q(x), t) \geq \frac{(18 - 18L)t}{(18 - 18L)t + \varphi(x,0)} \tag{2.6}$$

for all $x \in X$ and all $t > 0$.

Proof. Let (S,d) be the generalized metric space defined as in the proof of Theorem 4.1. Consider the linear mapping $J : S \to S$ such that $Jg(x) := \frac{g(3x)}{9}$

for all $x \in X$. Let $g, h \in S$ be such that $d(g,h) = \epsilon$. Then $N(g(x)-h(x), \epsilon t) \geq \frac{t}{t+\varphi(x,0)}$ for all $x \in X$ and all $t > 0$. Hence

$$N(Jg(x) - Jh(x), L\epsilon t) = N\left(\frac{g(3x)}{9} - \frac{h(3x)}{9}, L\epsilon t\right)$$
$$= N\left(g(3x) - h(3x), 9L\epsilon t\right) \geq \frac{9Lt}{9Lt + \varphi(3x,0)} \geq \frac{t}{t + \varphi(x,0)}$$

for all $x \in X$ and $t > 0$. Thus $d(g,h) = \epsilon$ implies that $d(Jg, Jh) \leq L\epsilon$. This means that $d(Jg, Jh) \leq Ld(g,h)$ for all $g, h \in S$. It follows from (2.3) that

$$N\left(\frac{f(3x)}{9} - f(x), \frac{t}{18}\right) \geq \frac{t}{t + \varphi(x,0)}.$$

Therefore $d(f, Jf) \leq \frac{1}{18}$. By Theorem 1.4, there exists a mapping $Q : X \to Y$ satisfying the following:

(1) Q is a fixed point of J, that is,

$$9Q(x) = Q(3x) \tag{2.7}$$

for all $x \in X$. The mapping Q is a unique fixed point of J in the set $\Omega = \{h \in S : d(g,h) < \infty\}$. This implies that Q is a unique mapping satisfying (2.7) such that there exists $\mu \in (0, \infty)$ satisfying $N(f(x) - Q(x), \mu t) \geq \frac{t}{t+\varphi(x,0)}$ for all $x \in X$ and $t > 0$.

(2) $d(J^n f, Q) \to 0$ as $n \to \infty$. This implies the equality

$$N\text{-}\lim_{n\to\infty} \frac{f(3^n x)}{9^n} = Q(x)$$

for all $x \in X$.

(3) $d(f, Q) \leq \frac{d(f,Jf)}{1-L}$ with $f \in \Omega$, which implies the inequality $d(f, A) \leq \frac{1}{18-18L}$. This implies that the inequality (2.6) holds. The rest of the proof is similar to that of the proof of Theorem 2.1. □

Corollary 2.4. *Let $\theta \geq 0$ and let p be a real number with $p > 1$. Let X be a normed vector space with norm $\|\cdot\|$. Let $f : X \to Y$ be a mapping satisfying (2.5). Then the limit $Q(x) := N\text{-}\lim_{n\to\infty} \frac{f(3^n x)}{9^n}$ exists for each $x \in X$ and defines a unique quadratic mapping $Q : X \to Y$ such that*

$$N(f(x) - Q(x), t) \geq \frac{(18 - 2\cdot 3^p)t}{(18 - 2\cdot 3^p)t + \theta\|x\|^p}$$

for all $x \in X$.

Proof. The proof follows from Theorem 2.3 by taking

$$\varphi(x,y) := \theta\left(\|x\|^p + \|y\|^p\right)$$

for all $x, y \in X$. Then we can choose $L = 3^{p-2}$ and we get the desired result. □

References

[1] D. H. Hyers, *On the stability of the linear functional equation*, Proc. Nat. Acad. Sci. U.S.A. **27** (1941) 222–224.

[2] H. Azadi Kenary, D. Y. Shin, J. R. Lee and H. Hoseini, *Fixed point and Hyers-Ulam stability of functional equations*, Int. J. Math. Anal. **5** (2011) no. 37, 1827–1833.

[3] H. Azadi Kenary, *On the stability of a cubic functional equation in random normed spaces*, J. Math. Extension **4** (2009) 1–11.

[4] H. Azadi Kenary, *Stability of a Pexiderial functional equation in random normed spaces*, Rend. Circ. Mat. Palermo 60 (2011) 59–68, DOI:10.1007/s12215-011-0027-5.

[5] D. Mihet and V. Radu, *On the stability of the additive Cauchy functional equation in random normed spaces*, J. Math. Anal. Appl. **343** (2008) 567–572.

[6] Th. M. Rassias, *On the stability of the linear mapping in Banach spaces*, Proc. Amer. Math. Soc. **72** (1978) 297–300.

[7] F. Skof, *Local properties and approximation of operators*, Rend. Sem. Mat. Fis. Milano **53** (1983) 113–129.

[8] S. M. Ulam, *Problems in Modern Mathematics*, Science Editions, John Wiley and Sons, 1964.

Received: December, 2011