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Representations of Functions in the Nevanlinna Class

Takahiko Nakazi[1]

Hokusei Gakuen University
2-3-1, Ohyachi-Nishi, Atsubetu-ku
Sapporo 004-8631, Japan
z00547@hokusei.ac.jp

Abstract

It is known that a function f in the Nevanlinna class can be represented as a quotient of two bounded analytic functions. In this paper we give a more explicit representation of f.

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§1. Introduction

Let D be the open unit disc in the complex plane $\mathbb{C}$ and let ∂D be the boundary of D. An analytic function in D is said to be of class N if the integrals $\int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta$ are bounded for $r < 1$. If f is in N, then $f(e^{i\theta})$, which we define to be $\lim_{r\to 1} f(re^{i\theta})$, exists almost everywhere on ∂D. If $\lim_{r\to 1} \int_{-\pi}^{\pi} \log^+ |f(re^{i\theta})| d\theta = \int_{-\pi}^{\pi} \log^+ |f(e^{i\theta})| d\theta$ then f is said to be of class N_+. The set of all boundary functions in N and N_+ is denoted by N and N_+ respectively. For $0 < p \leq \infty$, H^p is defined by $N_+ \cap L^p$. N is called the Nevanlinna class, N_+ is called the Smirnov class and H^p is called the Hardy space.

It is known that any function in N can be represented as a quotient of two functions in H^∞. In this paper, we show that any function F in N can be represented as a quotient of two functions ℓ and $q - k$ in H^∞ such that $|\ell|^2 + |k|^2 = 1$ and q is inner. This representation can be proved using the

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Herglotz integral of $|F|^2$ when F is in H^2. Of course if F is not in H^2 we can not use the Herglotz integral of $|F|^2$. We use a theorem of Douglas and Rudin [1, Theorem 2.1 of Chapter V]. Moreover we show that if f is a function in N and Ref is nonnegative on ∂D then there exist an inner function q and a contractive function k in H^∞ such that $f = (q+k)/(q-k)$ on ∂D. If f is in H^1 then we can show it by the Poisson integral of f. Of course, then we can choose a constant q. However for f outside H^1 we must take a different way.

If f in N has a nonnegative real part on D then f belongs to $\bigcap_{p<1} H^p$ and $f = (1+k)/(1-k)$ for some contractive function k in H^∞ and f is an outer function. We are familiar to such a function and it is very important in many areas. On the other hand, we are not familiar to a function in N whose real part is nonnegative on ∂D. If f is in H^1 and Ref is nonnegative on ∂D then Ref is nonnegative on D and f is outer. However it is not necessarily true for a function outside H^1. In fact, we have a lot of functions in H^p that are nonnegatie on ∂D when $0 < p < 1/2$. Such a function appears in extremal problems of H^1 (see [2],[4],[5],[7]). The author [4],[5],[6] described and studied functions in N_+ whose real parts are nonnegative on ∂D. Moreover we gave a necessary condition for that if Re$f \geq 0$ on ∂D then Re$f > 0$ on D for f in N_+. In this paper, we generalize results of previous papers for N_+ to N. We need several new ideas to do it.

Throughout this paper, an inner function is a unimodular function in N_+. A nonzero function f in N_+ is called outer or strongly outer if sf belongs to N_+ then s is constant when s is a bounded measurable real valued function or a nonnegative measurable function, respectively.

§2. Without the Poisson integral

For a nonzero function f in H^1 with Re$f \geq 0$ on ∂D the Poisson integral of f is defined by

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Re}\left(\frac{e^{it}+z}{e^{it}-z}\right) f(e^{it}) dt \quad (z \in D).$$

Then Re$f > 0$ on D and so f is an outer function. Hence $f = (1+k)/(1-k)$ on ∂D and k is a contraction in H^∞. The author [4] studies a nonzero function in N_+ with Re$f \geq 0$ on ∂D. He shows $f = (q+k)/(q-k)$ on ∂D where q in inner, k is a contraction and $q-k$ is outer. Moreover he [4] proves that if $|F| \leq$Ref on ∂D then Re$f > 0$ on D when Ref is outer and F is strongly outer. We try to generalize the above results.

Theorem 1. *A function f in N has a nonnegative real part almost everywhere on ∂D if and only if there exist an inner function q and a contractive function k in H^∞ such that*

$$f = \frac{q+k}{q-k} \quad \text{and} \quad Z(q-k) = \emptyset$$

where $Z(q-k) = \{z \in D : (q-k)(z) = 0\}$.

Then $\mathrm{Re} f = (1-|k|^2)/|q-k|^2$ *almost everywhere on ∂D.*

Proof. If $f \in N$, $\mathrm{Re} f \geq 0$ *a.e.* on ∂D and $\phi = (f-1)/(f+1)$, then $|\phi| \leq 1$ *a.e.* on ∂D. Here we will use a theorem of F. and R. Nevanlinna. We can write f as the following :

$$f + 1 = \frac{q_1 h_1}{s_1} \quad \text{and} \quad f - 1 = \frac{q_2 h_2}{s_2}$$

where q_i is inner, s_i is singular inner, $q_i \wedge s_i = 1$ and h_i is outer for $i = 1, 2$ (see [1, Theorem 5.5 of Chapter II]). Hence $f = (q_1 h_1 - s_1)/s_1 = (q_2 h_2 + s_2)/s_2$. Suppose $Q_1 H_1$ and $Q_2 H_2$ are inner outer factorizations of $q_1 h_1 - s_1$ and $q_2 h_2 - s_2$, respectively. Then $H_1 = H_2$ and so $s_1 Q_2 = s_2 Q_1$. Since $s_1 \wedge q_1 = s_2 \wedge q_2 = 1$, we may assume that $s_1 = s_2$. Hence

$$\phi = \frac{f-1}{f+1} = \frac{q_2 h_2 / s_2}{q_1 h_1 / s_1} = \frac{q_2 h_2}{q_1 h_1}.$$

Therefore

$$f = \frac{1+\phi}{1-\phi} = \frac{q+k}{q-k}$$

where $q = q_1$ and $k = q_2 h_2 / h_1$. Then k belongs to H^∞ because $|k| = |\phi| \leq 1$ and $Z(q-k) = \emptyset$ because $2 = (f+1) - (f-1) = q_1 h_1 / s_1 - q_2 h_2 / s_2 = (q-k) h_1 / s_1$.

The proof of Theorem 1 shows that if f is in N_+ then $q - k$ is outer. In the description of a function in N whose real part is nonnegative on ∂D, it is important to know whether $Z(q-k)$ is empty. In the previous paper [7], we described a function k such that $q - k$ is outer. We used a theorem due to Adamyan, Arov and Krein (see Lemma 6 in [1]). In §4, we describe a function k with $Z(q-k) = \emptyset$.

Suppose f is a function in N. Then $\mathrm{Re} f$ is nonnegative on D if and only if $|f-1| \leq |f+1|$ on D. The following is a generalization of this result.

Corollary 1. *When f is a function in N, $\mathrm{Re} f$ is nonnegative on ∂D if and only if there exists an inner function q such that $|f-1| \cdot |q| \leq |f+1|$ on D.*

Proof. If Re$f \geq 0$ on ∂D then by Theorem 1 $f = (q+k)/(q-k)$ for some inner q and some contractive function k in H^∞. Hence $k = (f-1)q/(f+1)$ and so $|f-1| \cdot |q| \leq |f+1|$ on D. Conversely if $|f-1| \cdot |q| \leq |f+1|$ on D for some inner q, put $k = (f-1)q/(f+1)$. Then k is analytic on D and $|k| \leq 1$ on D.

If f is a nonzero function in N then f has the following factorization : $f = qh/s$ where q is an inner function and s is a singular inner function and h is an outer function. Then we will write $I[f] = q$. For two inner functions q_1, q_2 we will write $q_1 \succ q_2$ when there exists an outer function h in H^1 such that $\bar{q}_1 q_2 = |h|/h$. If $q_1 \succ q_2$ and $q_1 \prec q_2$ then we will write that $q_1 \sim q_2$.

In the previous paper [5], we showed that if f is a nonzero function in N_+ and Re$f \geq 0$ on ∂D then $f = (q+k)/(q-k)$ and $I[f+\lambda] \sim q$ for any $\lambda > 0$ and $I[f] \prec q$. We would like to generalize this to a nonzero function in N.

Theorem 2. *Suppose that f is a nonzero function in N with Re$f \geq 0$ a.e. on ∂D and q is an inner function in Theorem 1. If $\lambda > 0$ then $I[f+\lambda] \sim q$ and $I[f] \prec q$.*

Proof. By definition, $f = (q+k)/(q-k)$ where $k \in H^\infty$, $\|k\|_\infty \leq 1$ and $Z(q-k) = \emptyset$. We suppose that f is not in N_+, and so $q-k$ is not outer. Hence for any $\lambda > 0$

$$f + \lambda = (1+\lambda)\frac{q + \frac{1-\lambda}{1+\lambda}k}{q-k}.$$

Since $s = I[q-k]$ is a nonconstant singular inner function, $I[q + \frac{1-\lambda}{1+\lambda}k] = I[f+\lambda]$. Suppose that

$$q + \frac{1-\lambda}{1+\lambda}k = s_\lambda b_\lambda h_\lambda \text{ and } q - k = sh$$

where s_λ and s are singular inner functions, b_λ is a Blaschke product, and h_λ and h are outer functions. Then

$$2q = (1+\lambda)s_\lambda b_\lambda h_\lambda + (1-\lambda)sh \text{ and } 2k = (1+\lambda)s_\lambda b_\lambda h_\lambda - (1+\lambda)sh.$$

Since we may assume that $q \wedge I[k] = 1, s_\lambda \wedge s = 1$ and so $s_\lambda = 1$, because $s \neq 1$. Thus $I[f+\lambda] = I[q + \frac{1-\lambda}{1+\lambda}k]$ for $\lambda \geq 0$. By Theorem 4 in the previous paper [7], $I[q + \frac{1-\lambda}{1+\lambda}k] \sim q$ when $\lambda > 0$ and $I[q \pm k] \prec q$.

Corollary 2. *In Theorem 2, if $I[f+\lambda]$ is a finite Blaschke product for some $\lambda > 0$, then q is also a finite Blaschke product with* $\deg q = \deg I[f+\lambda]$. *Hence f belongs to N_+.*

Proof. Theorem 2 implies that q is a finite Blaschke product. By [8, Theorem 8] $I[q-k] \prec q$ and so $q-k$ is outer. Thus $f \in N_+$.

If f is a nonzero function in N with $\mathrm{Re}f \geq 0$ on ∂D, and $f+\lambda$ is outer for some $\lambda > 0$ then by Corollary 2 $\mathrm{Re}f > 0$ on D and $f = \dfrac{1+k}{1-k}$ for some nonconstant contractive function k in H^∞. Suppose $f = \dfrac{z+k}{z-k}$ for some contractive function k in H^∞. If f is a function in N then by Corollary 2 f belongs to N_+.

If f is an outer function in N_+ and $\mathrm{Re}f \geq c$ on ∂D for some positive constant c then it clear that $\mathrm{Re}f > 0$ on D. The following theorem is known [4, Theorem 6]. We give an another proof.

Theorem 3. *Suppose F is a strongly outer function in H^1. If $|F| \leq \mathrm{Re}f$ on ∂D and f is an outer function in N_+ then* $\mathrm{Re}f > 0$ *on D.*

Proof. Put $k = 2F/f$, then k belongs to H^∞ and

$$\begin{aligned} \left|\frac{F}{|F|} - k\right|^2 &= 1 - 2\mathrm{Re}\frac{F}{|F|}\bar{k} + |k|^2 \\ &= 1 - 4\frac{|F|\mathrm{Re}f}{|f|^2} + \frac{4|F|^2}{|f|^2} \leq 1 \end{aligned}$$

because $|F| \leq \mathrm{Re}f$. By [3, Lemma 6]

$$k = \frac{h(1-Q_h)(1-w)}{1-Q_h w}$$

where h is an outer function in H^1, $F/h \geq 0$ on ∂D, w is a contractive function in H^∞ and

$$\frac{1+Q_h(z)}{1-Q_h(z)} = \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}|h(e^{it})|dt \quad (z \in D).$$

Since F is strongly outer, $h = \alpha F$ for some positive constant α. Hence

$$f = \frac{2F}{k} = \frac{2}{\alpha}\frac{1-Q_h w}{(1-Q_h)(1-w)} = \frac{1}{\alpha}\left(\frac{1+Q_h}{1-Q_h} + \frac{1+w}{1-w}\right).$$

This implies $\mathrm{Re}f > 0$ on D.

The converse of Theorem 3 is also valid [4, Theorem 6]. The proof of Theorem 3 shows the following : If F is not strongly outer in H^1 and f is outer then there exist outer function h and g in H^1 such that $f = \frac{F}{h} \cdot g$ and $\frac{F}{h} \geq 0$ on ∂D and Re$g > 0$ on D.

Theorem 4. *If $|F| \leq$Ref on ∂D then Re$f > 0$ on D when f is an outer function in N_+ and F^{-1} is in H^1.*

Proof. Since $|f| \geq$Re$f \geq |F|$ on ∂D,$|f^{-1}| \leq |F^{-1}|$ on ∂D. Since $F^{-1} \in H^1$ and f is outer, f^{-1} belongs to H^1. By the Poisson integral of f^{-1}, Re$(f^{-1})(z) > 0$ on D and so Re$f(z) > 0$ on D.

It is known that if F and F^{-1} are in H^1 then F is strongly outer. It is also knew that the converse is not true.

§3. Without the Herglotz integral

For a nonzero function F in H^1 the Herglotz integral of $|F|$ is defined by

$$f(z) = \frac{1}{2\pi} \int_0^{2\pi} \frac{e^{it} + z}{e^{it} - z} |F(e^{it})| dt \quad (z \in D).$$

Then f belongs to $\bigcap_{p<1} H^p$, Re$f \geq 0$ on $\bar{D}$ and Re$f = |F|$ on ∂D.

For a function F in H^p for $0 < p < 1$, if F is not in H^1 then we can not define the Herglotz integral of $|F|$. However, Theorem 5 shows that there exists a function f in N_+ such that Re$f = |F|$ on ∂D. In general, f is not necessarily outer. Corollary 3 gives a necessary and sufficient condition for the existence of an outer function f.

Theorem 5. *If F is a nonzero function in N then there exists a function f in N_+ such that* Re$f = |F|$ *on ∂D.*

Proof. We may assume that F is an outer function. Since $F/|F|$ is a unimodular function, by a theorem of R. G. Douglas and W. Rudin [1, Theorem 2.1 in Chapter V] there exist two inner functions q and q' such that

$$\left| \frac{(1+q)^2 F}{|(1+q)^2 F|} - q' \right| = \left| \frac{F}{|F|} - \bar{q} q' \right| \leq 1$$

because $(1+q)^2/|1+q|^2 = q$. By the proof of [1, Lemma 5.4 in Chapter IV], there exists an outer function G in H^1 such that $(1+q)^2 F/|(1+q)^2 F| = qF/|qF| = G/|G|$. Put

$$f = \frac{qF}{G} \, \frac{1+Q}{1-Q}$$

where $(1+Q)/(1-Q)$ is the Herglotz integral of $|G|$. Then $f \in N_+$ and

$$\mathrm{Re} f = \frac{qF}{G}\mathrm{Re}\frac{1+Q}{1-Q} = \frac{|qF|}{|G|}\,\frac{1-|Q|^2}{|1-Q|^2} = |F|$$

because $qF/G \geq 0$ and $|G| = (1-|Q|^2)/|1-Q|^2$.

Corollary 3. *Suppose F is a nonzero function in N and $F = q_0F_0/s_0$ where F_0 is outer, q_0 is inner and s_0 is a singular inner function.*

(1) There exist an inner function q and an outer function G in H^1 such that $qF_0/G \geq 0$ on ∂D.

(2) There exists an outer function f in N_+ such that $\mathrm{Re} f = |F|$ *on ∂D if and only if there exists an outer function G in H^1 such that $F_0/G \geq 0$ on ∂D.*

Proof. By the proof of Theorem 5, $\mathrm{Re} f = |F|$, and

$$f = \frac{qF_0}{G}\,\frac{1+Q}{1-Q} \quad \text{and} \quad \frac{qF_0}{G} \geq 0 \quad \text{on} \quad \partial D.$$

This implies (1). Since f is outer if and only if $q = 1$, (2) is valid.

Corollary 4. *If u is a nonnegative function on ∂D and $\log^+ u$ is summable then there exists a function f in N_+ such that* $\mathrm{Re} f = u$ *on ∂D.*

Proof. Put $v = u+1$ then $\log v \in L^1$. Hence there exists an outer function F in N such that $v = |F|$. By Theorem 5, there exists a function g in N_+ such that $\mathrm{Re} g = |F|$. Suppose $f = g - 1$.

Let μ be a real measure on ∂D, μ_a the absolutely continuous part and μ_s the singular part. Suppose f, f_a and f_s are the Herglotz integrals of μ, μ_a and μ_s, respectively. If μ_a is a possitive measure then $\mathrm{Re} f_a > 0$ on D and $\mathrm{Re} f \geq 0$ on ∂D. It is known that $\mathrm{Re} f > 0$ on D if and only if μ is a possitive measure.

If h is a function in N with $\mathrm{Re} h > 0$ on D and g is a function in N with $\mathrm{Re} g = 0$ on ∂D then $f = h + g$ has a nonnegative real part on ∂D. The converse is not true clearly. The following theorem is a partial converse. The statement (1) of Theorem 6 gives another proof of [5, Theorem 11].

Theorem 6. *Let f be a function in N with* $\mathrm{Re} f \geq 0$ *on ∂D.*

(1) If $\log \mathrm{Re} f$ *is in L^1 then $f = sh + g$ where h is a function in H^1 with* $\mathrm{Re} h > 0$ *on D, g is a function in N with* $\mathrm{Re} g = 0$ *on ∂D and s is a nonnegative function in N.*

(2) If both $\log \mathrm{Re} f$ *and* $\mathrm{Re} f$ *are in L^1 then $f = h + g$ where h is a function in H^1 with* $\mathrm{Re} h > 0$ *on D and g is a function N with* $\mathrm{Re} g = 0$.

Proof. (1) Since $f \in N$ and $\log \mathrm{Re} f \in L^1$, there exists an outer function F in N_+ such that $\mathrm{Re} f = |F|$ on ∂D. By Theorem 5 and the proof, there exist a nonnegative function s in N and a contractive function α in H^∞ such that $\mathrm{Re} f = |F| = s\ \mathrm{Re}\dfrac{1+\alpha}{1-\alpha}$. Put $g = f - sh$ and $h = \dfrac{1+\alpha}{1-\alpha}$. Then $\mathrm{Re} g = 0$ and $\mathrm{Re} h > 0$ on D and $h \in H^1$.

(2) If $\mathrm{Re} f$ is in L^1 then there exists an outer function F in H^1 such that $\mathrm{Re} f = |F|$. Let h be the Herglotz integral of $|F|$ then $\mathrm{Re} h = |F|$.

If F is a nonzero function in H^2 and f is the Herglotz integral of $|F|^2$, then there exists a nonconstant contractive function k in H^∞ such that $f = (1+k)/(1-k)$ and so $|F|^2 = (1-|k|^2)/|1-k|^2$. If h is an outer function in H^∞ such that $|h|^2 = 1 - |k|^2$ then

$$F = \frac{Qh}{1-k} \quad \text{and} \quad |Qh|^2 + |k|^2 = 1$$

where Q is the inner part of F. We generalize this known result for a function F in N using Theorems 1 and 5.

Theorem 7. *If F is a nonzero function in N then there exist an inner function q and a contractive functions k, ℓ in H^∞ such that*

$$F = \frac{\ell}{q-k} \quad \text{and} \quad |\ell|^2 + |k|^2 = 1$$

where $Z(q-k) = \emptyset$

Proof. By Theorem 5 there exists a function f in N_+ such that $\mathrm{Re} f = |F|^2$. By Theorem 1 there exist an inner function q and a contractive function k_1 in H^∞ such that $f = (q_1 + k_1)/(q_1 - k_1)$ and $q_1 - k_1$ is outer, because $f \in N_+$. Hence $|F|^2 = (1-|k_1|^2)/|q_1 - k_1|^2$. Put $\ell_1 = F(q_1 - k_1)$ then $|\ell_1|^2 + |k_1|^2 = 1$ and ℓ_1 belongs to N. Since $\ell_1 = \ell/s$ for some ℓ in N_+ and some singular inner function s, $\ell = F(sq_1 - sk_1)$. Put $q = sq_1$ and $k = sk_1$.

§4. Empty zero set of $q - k$

When q is inner and k is a contraction, a zero set $Z(q-k)$ of $q-k$ was important in the previous sections. We will descrive k when q is given. This is similar to the description of k such that $q-k$ is outer in [5, §8]. Put

$$\mathcal{E}(q) = \{k \in H^\infty\ ;\ q-k \text{ is outer and } \|k\|_\infty \le 1\}$$

and

$$\mathcal{E}_0(q) = \{k \in H^\infty \ ; \ Z(q-k) = \emptyset \text{ and } \|k\|_\infty \leq 1\}.$$

Then $\mathcal{E}(q) \subset \mathcal{E}_0(q)$. For any q, $\mathcal{E}(q) \neq \emptyset$ and so $\mathcal{E}_0(q) \neq \emptyset$. When q is a finite Blaschke product, it is known [8, Theorem 8] that $\mathcal{E}(q) = \mathcal{E}_0(q)$. When q is not a finite Blaschke product, we can prove $\mathcal{E}(q) \neq \mathcal{E}_0(q)$.

Theorem 8. Let q be an inner function. The function k belongs to $\mathcal{E}_0(q)$ if and only if

$$k = q\frac{1-Q_F}{1-\overline{Q}_F}\frac{1-\overline{Q}_F}{1-Q_F w}$$

where F is a function in H^1 with $Z(F) = \emptyset$ and $q\bar{F} \geq 0$ a.e. on ∂D and $\|F\|_1 = 1$, and w is a contractive function in H^∞ and $(1+Q_F)/(1-Q_F)$ is the Heglotz integral of $|F|$.

Proof. If $k \in \mathcal{E}_0(q)$ and $h = q - k$ then by Lemma 6 in [3], there exists a function F in H^1 with $\|F\|_1 = 1$ such that $q\overline{F} \geq 0$, $Z(F) = \emptyset$ and $h = F(1-Q_F)(1-w)/(1-Q_F w)$. Since $\bar{q}F = |f| = (1-|Q_F|^2)/|1-Q_F|^2$, the 'only if' part follows. The proof is reversible and so the 'if' part follows.

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