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Some Properties of Operator Classes

(M,k), $(M,k)^*$, (A,k) and $(A,k)^*$

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Abstract

In this paper we will define a new classes of operators, which we denote by (A,k) and $(A,k)^*$. We give some relations between classes (M,k), $(M,k)^*$, (A,k) and $(A,k)^*$. Also, some spectral characterizations of (A,k), $(A,k)^*$ classes of operators are given.

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1. Introduction

Let us denote by H the complex Hilbert space and $B(H)$ the space of all bounded linear operators defined in Hilbert space H. In the following we will mention some known classes of operators defined in Hilbert space H. Let T be an operator in $B(H)$. The operator T is called normal if it satisfies the following condition $T^*T = TT^*$. The operator T is called quasi-normal if $T(T^*T) = (T^*T)T$, it is hyponormal if $T^*T \geq TT^*$, which is equivalent to the condition $\|Tx\| \geq \|T^*x\|$, for all x in H. We say that an operator T is quasi-hyponormal if the following condition $T^{*2}T^2 \geq (T^*T)^2$ holds and the last one is equivalent with $\|T^2x\| \geq \|T^*Tx\|$, for all x in H. We say that an operator T is of (M,k) class if $T^{*k}T^k \geq (T^*T)^k$, for $k \geq 2$, which is equivalent to the condition $\|T^kx\| \geq \left\|(T^*T)^{\frac{k}{2}}x\right\|$, for all x in H and $k \geq 2$ (see[4]) . It is known that the $(M,2)$ class coincides with the class of quasi-hyponormal operators. But, the class of hyponormal operators does not coincide with (M,k), for any k (see [3]). We say that an operator T is of $(M,k)^*$ class if $T^{*k}T^k \geq (TT^*)^k$, for $k \geq 1$, which is equivalent to the condition $\|T^kx\| \geq \left\|(TT^*)^{\frac{k}{2}}x\right\|$, for all x in H and $k \geq 1$ (see[4]). It is known that the $(M,1)^*$ class coincides with the class of hyponormal operators. We will define a new classes of operators which we denote by (A,k), respectively $(A,k)^*$. The operator T is of (A,k) class if $\|T^kx\|^2 \geq \|T^*Tx\|^k$, for all unit vectors x in H and $k \geq 2$. We observe that the $(A,2)$ class coincides with the class of quasi-hyponormal operators. The operator T is of $(A,k)^*$ class if $\|T^kx\|^2 \geq \|TT^*x\|^k$, for all unit vectors x in H and $k \geq 1$. We see that the $(A,2)^*$ class coincides with the $(M,2)^*$ class. The spectrum, the point spectrum, the approximate point spectrum, the joint point spectrum and joint approximate point spectrum of an operator T are denoted by $\sigma(T)$, $\sigma_p(T)$, $\sigma_{ap}(T)$, $\sigma_{jp}(T)$, $\sigma_{ja}(T)$, respectively. We give some relations between classes (M,k), $(M,k)^*$, (A,k) and $(A,k)^*$. For example, we show that every operator T from the (M,k), $(k \geq 3)$ class, belongs to the (A,k) class and every operator T from the $(M,k)^*$ class belongs to the $(A,k)^*$ class. Also, some spectral characterizations of (A,k) $\left((A,k)^*\right)$ class of operators are given.

Theorem A. (Hölder-McCarthy inequality [2]). Let A be a positive operator. Then the following inequalities hold for all x in H

$i)$ $\langle A^r x, x\rangle \le \langle Ax, x\rangle^r \|x\|^{2(1-r)}$, for $0 < r \le 1$

$ii)$ $\langle A^r x, x\rangle \ge \langle Ax, x\rangle^r \|x\|^{2(1-r)}$, for $r \ge 1$.

Theorem B.**([1])**. Let $\lambda \ne 0$, and $\{x_n\}$ be a sequence of vectors. Then the following assertions are equivalent

$i)$. $(T-\lambda)x_n \to 0$ and $(T^* - \overline{\lambda})x_n \to 0$.

$ii)$ $(|T| - |\lambda|)x_n \to 0$ and $(U - e^{i\theta})x_n \to 0$.

$iii)$ $(|T^*| - |\lambda|)x_n \to 0$ and $(U \ - e^{-i\theta})x_n \to 0$.

2. Operator classes (M,k) and (A,k) in Hilbert space

In this section we will show some properties of (A,k) and (M,k) classes.

Proposition 2.1. For each positive integer $k \ge 2$ an operator T belongs to the (M,k) class if and only if

$$T^{*k}T^k + 2\lambda(T^*T)^k + \lambda^2 T^{*k}T^k \ge 0,$$

holds for all $\lambda \in R$.

Proof. Let $\lambda \in R$ and $x \in H$ be given. Then $T \in (M,k)$, if and only if

$$\left\|(T^*T)^{\frac{k}{2}}x\right\| \le \|T^k(x)\| \Leftrightarrow 4\left\|(T^*T)^{\frac{k}{2}}x\right\|^4 - 4\cdot\|T^kx\|^2\cdot\|T^kx\|^2 \le 0$$

$$\Leftrightarrow \|T^kx\|^2 + 2\lambda\left\|(T^*T)^{\frac{k}{2}}x\right\|^2 + \lambda^2\|T^kx\|^2 \ge 0$$

$$\Leftrightarrow \langle(T^kx, T^kx\rangle + 2\lambda\langle(T^*T)^{\frac{k}{2}}x, (T^*T)^{\frac{k}{2}}x\rangle + \lambda^2\langle T^kx, T^kx\rangle \ge 0$$

$$\Leftrightarrow \langle(T^{*k}T^k + 2\lambda(T^*T)^k + \lambda^2T^{*k}T^k)x, x\rangle \ge 0$$

$$\Leftrightarrow T^{*k}T^k + 2\lambda(T^*T)^k + \lambda^2T^{*k}T^k \ge 0.$$

The proof is completed.■

Corollary 2.1. If $k=2$, we get the following relation $T^{*2}T^2 \geq (T^*T)^2$ if and only if $T^{*2}T^2 + 2\lambda(T^*T)^2 + \lambda^2 T^{*2}T^2 \geq 0$, for all $\lambda \in R$, which is the definition of the quasi-hyponormal operator.

Proposition 2.2. If the operator T belongs to the (M,k) class, where $k \geq 2$, then it belongs to the (A,k) class.

Proof. Let $T \in (M,k)$, then for every unit vectors $x \in H$ we have

$$
\begin{aligned}
\|T^k x\|^4 &= \langle T^k x, T^k x\rangle^2 = \langle T^{*k}T^k x, x\rangle^2 \\
&\geq \langle (T^*T)^k x, x\rangle^2, \quad \text{because } T \in (M,k) \\
&\geq \langle (T^*T)^2 x, x\rangle^k \quad \text{(Hölder-McCarthy inequality)} \\
&= \langle T^*Tx, T^*Tx\rangle^k = \|T^*Tx\|^{2k}.
\end{aligned}
$$

Thus $\|T^k x\|^2 \geq \|T^*Tx\|^k$, respectively $T \in (A,k)$ .Therefore the proof is completed.■

Corollary 2.2. If the operator T belongs to the $(M,k)^*$ class, where $k \geq 1$, then T belongs to the $(A,k+1)$ class.

Proof. This proof follows from theorem 3.8 in [3] and proposition 2.2. ■

Proposition 2.3. Let T be an operator in (A,k). Then the following relation

$$0 \neq \lambda \in \sigma_p(T) \Rightarrow \bar{\lambda} \in \sigma_p(T^*),$$

holds .

Proof. Let $T \in (A,k)$, $\lambda \in \sigma_p(T)$, $\lambda \neq 0$ and let $x \in H$ a unit vector such that $Tx = \lambda x$. Then we have

$$\|T^*Tx\|^k \leq \|T^k x\|^2 \tag{1}$$

$$\|T^*Tx\|^k = |\lambda|^k \|T^*x\|^k \tag{2}$$

$$\|T^k x\|^2 = |\lambda|^{2k} \quad . \tag{3}$$

Now from relation (1), (2) and (3) it follows that

$$\|T^*x\| \leq |\lambda| \tag{4}$$

Hence

$$
\begin{aligned}
\|T^*x - \overline{\lambda x}\|^2 &= \langle T^*x - \bar{\lambda}x, T^*x - \overline{\lambda x}\rangle = \langle T^*x, T^*x\rangle - \langle T^*x, \bar{\lambda}x\rangle - \langle \bar{\lambda}x, T^*x\rangle + \langle \bar{\lambda}x, \bar{\lambda}x\rangle \\
&= \|T^*x\|^2 - \lambda\langle x, Tx\rangle - \bar{\lambda}\langle Tx, x\rangle + |\lambda|^2\langle x, x\rangle \\
&\leq |\lambda|^2 - |\lambda|^2\|x\|^2 - |\lambda|^2\|x\|^2 + |\lambda|^2\|x\|^2 = |\lambda|^2 - |\lambda|^2 - |\lambda|^2 + |\lambda|^2 = 0.
\end{aligned}
$$

Therefore we have $T^*x = \bar{\lambda}x$, consequently $\bar{\lambda} \in \sigma_p(T^*)$.■

Lemma 2.1. If T is a bilateral weighted shift operator, with weighted sequence ω_n, $(Te_n = \omega_n e_{n+1})$, then T is in (A,k) class if and only if

$$|\omega_n| \cdot |\omega_{n+1}| \cdot \ldots \cdot |\omega_{n+k-1}| \geq |\omega_n|^k, \quad n \in Z \text{ and } k \geq 2.$$

Proof. The proof follows immediately from the definition of (A,k) class. ■

Lemma 2.2. If T is a regular bilateral weighted shift operator, with weighted sequence $\omega_n \neq 0$, $(Te_n = \omega_n e_{n+1})$, then $T^{-1} \in (A,k)$ if and only if

$$|\omega_{n-1}|^k \geq |\omega_{n-1}| \cdot |\omega_{n-2}| \cdot \ldots \cdot |\omega_{n-k}|, \quad n \in Z, |\omega_n| \neq 0 \text{ and } k \geq 2.$$

Proof. The proof follows immediately from the definition of (A,k) class. ■
By the following example we show that there exists an operator T which belongs to the (A,k) class but its inverse T^{-1} is not element of (A,k).

Example 2.1. Let T is a regular bilateral weighted shift operator, with weighted sequence (ω_n) given as follows

$$\omega_n = \begin{cases} \frac{1}{2}, & for\ n \leq -1 \\ 1, & for\ n = 0 \\ \frac{1}{2}, & for\ n = 1 \\ 2, & for\ n = 2 \\ \frac{1}{4}, & for\ n = 3 \\ 16, & for\ n \geq 4 \end{cases}.$$

From lemma 2.1 and lemma 2.2, after some calculations it follows that $T \in (A,3)$, but $T^{-1} \notin (A,3)$.

Theorem 2.1. Let $T \in (A,k)$ and let $T^n, (n > k)$ be a compact operator, then it follows that T is compact too.

Proof. Let $T \in (A,k)$, $k \geq 2$ and let $\dfrac{T^{n-k}x}{\|T^{n-k}x\|}$ be a unit vector in Hilbert space H. Then we have

$$\|T^*T(T^{n-k}x)\|^k \leq \|T^n x\|^2 \cdot \|T^{n-k}x\|^{k-2} \quad . \tag{5}$$

Let $(x_m)\in H$ be weakly convergent sequence with limit 0 in H. From compactness of T^n and inequality (5) we get the following relation

$$\left\|T^*TT^{n-k}x_m\right\|^k \le \left\|T^n x_m\right\|^2\cdot\left\|T^{n-k}x_m\right\|^{k-2}\to 0,\ m\to\infty .$$

So, the operator T^*TT^{n-k} is a compact operator from which follows that $T^{*n-1}T^{n-1}$ is also compact, respectively T^{n-1} is a compact operator. Now, if we repeat this procedure, we conclude the T is a compact operator.■

Proposition 2.4. Let $T\in(A,k)$ be a regular operator, for $k\ge 2$. Then the approximate point spectrum lies in the disc

$$\sigma_{ap}\subseteq\left\{\lambda\in C:\frac{1}{\|T\|^{k-1}\sqrt{\left\|(T^*T)^{-1}\right\|^k}}\le|\lambda|\le\|T\|\right\}.$$

Proof. Let $T\in(A,k)$ be a regular operator, for $k\ge 2$. Then for every unit vector x in H, we have

$$\|x\|^k=\left\|(T^*T)^{-1}(T^*T)x\right\|^k\le\left\|(T^*T)^{-1}\right\|^k\cdot\left\|T^*Tx\right\|^k\le\left\|(T^*T)^{-1}\right\|^k\cdot\left\|T^kx\right\|^2\le$$

$$\le\left\|(T^*T)^{-1}\right\|^k\cdot\left\|T^{k-1}\right\|^2\cdot\|Tx\|^2 .$$

$$\|Tx\|\ge\frac{1}{\left\|T^{k-1}\right\|\sqrt{\left\|(T^*T)^{-1}\right\|^k}}\ge\frac{1}{\|T\|^{k-1}\sqrt{\left\|(T^*T)^{-1}\right\|^k}} . \qquad (6)$$

Now, assume that $\lambda\in\sigma_{ap}$. Then there exists a sequence (x_n), $\|x_n\|=1$, such that $\|(T-\lambda)x_n\|\to 0,\ when\ n\to\infty$. Therefore by (6) we have

$$\|Tx_n-\lambda x_n\|\ge\|Tx_n\|-|\lambda|\|x_n\|\ge\frac{1}{\|T\|^{k-1}\sqrt{\left\|(T^*T)^{-1}\right\|^k}}-|\lambda| . \qquad (7)$$

Now, when $n\to\infty$, from relation (7) we have $|\lambda|\ge\dfrac{1}{\|T\|^{k-1}\sqrt{\left\|(T^*T)^{-1}\right\|^k}}$,

respectively $\sigma_{ap}\subseteq\left\{\lambda\in C:\dfrac{1}{\|T\|^{k-1}\sqrt{\left\|(T^*T)^{-1}\right\|^k}}\le|\lambda|\le\|T\|\right\}$.■

Corollary 2.3. Let T be a regular quasi-hyponormal operator. Then the following relation

$$\sigma_{ap} \subseteq \left\{ \lambda \in C : \frac{1}{\|T\| \cdot \|(T^*T)^{-1}\|} \leq |\lambda| \leq \|T\| \right\}$$

holds.

Proposition 2.5. If the operator $T \in B(H)$ satisfies condition $|T^k|^{\frac{4}{k}} \geq |T|^4$, for $k \geq 2$, then operator T belongs to the (A,k) class.

Proof. Assume that the operator T satisfies the condition

$$|T^k|^{\frac{4}{k}} \geq |T|^4, \text{ for } k \geq 2.$$

Then, for every unit vector $x \in H$, we have

$$\|T^k x\|^4 = \langle T^k x, T^k x\rangle^2 = \langle T^{*k} T^k x, x\rangle^2 = \langle |T^k|^2 x, x\rangle^2$$

$$\geq \langle |T^k|^{\frac{4}{k}} x, x\rangle^k \quad \text{(Hölder-McCarthy inequality)}$$

$$\geq \langle |T|^4 x, x\rangle^k = \langle (T^*T)^2 x, x\rangle^k = \langle T^*Tx, T^*Tx\rangle^k = \|T^*T\|^{2k}.$$

Thus $\|T^k x\|^2 \geq \|T^*Tx\|^k$, for $k \geq 2$, respectively $T \in (A,k)$. Therefore the proof is completed. ■

Corollary 2.4. If the operator $T \in B(H)$ satisfies condition $|T^2|^2 \geq |T|^4$, then it is quasi-hyponormal.

Theorema 2.2. If $T \in B(H)$ belongs to the class (A,k), $k \geq 2$ and $\lambda \neq 0$, then for a sequence of unit vectors $\{x_n\}$, $(T-\lambda)x_n \to 0$ implies $(T^* - \overline{\lambda})x_n \to 0$, $\sigma_p(T) - \{0\} = \sigma_{jp}(T) - \{0\}$ and $\sigma_{ap}(T) - \{0\} = \sigma_{ja}(T) - \{0\}$.

Proof. We only need to prove that $(T^* - \overline{\lambda})x_n \to 0$ by Theorem B. By the assumption $(T-\lambda)x_n \to 0$ and $(T^k - \lambda^k)x_n \to 0$, we have

$$\|Tx_n\| \to |\lambda| \text{ and } \|T^k x_n\| \to |\lambda|^k. \tag{8}$$

Since $T \in B(H)$ belongs to the (A,k) class, then we have

$$\left\| |T|^2 x_n \right\|^2 = \left\| T^* T x_n \right\|^2 \le \left\| T^k x_n \right\|^{\frac{4}{k}} \to \left(|\lambda|^k \right)^{\frac{4}{k}} = |\lambda|^4 .$$

Thus

$$\left\| |T|^2 x_n \right\|^2 \to |\alpha| \le |\lambda|^4 . \tag{9}$$

Now, from relation (9) we have:

$$\left\| \left(|T|^2 - |\lambda|^2 \right) x_n \right\|^2 = \langle \left(|T|^2 - |\lambda|^2 \right) x_n , \left(|T|^2 - |\lambda|^2 \right) x_n \rangle$$

$$= \langle |T|^2 x_n , |T|^2 x_n \rangle - \langle |T|^2 x_n , |\lambda|^2 x_n \rangle - \langle |\lambda|^2 x_n , |T|^2 x_n \rangle + \langle |\lambda|^2 x_n , |\lambda|^2 x_n \rangle$$

$$= \left\| |T|^2 x_n \right\|^2 - |\lambda|^2 \langle T x_n , T x_n \rangle - |\lambda|^2 \langle T x_n , T x_n \rangle + |\lambda|^4 \langle x_n , x_n \rangle ,$$

$$= \left\| |T|^2 x_n \right\|^2 - 2|\lambda|^2 \| T x_n \|^2 + |\lambda|^4$$

$$\to |\alpha| - 2|\lambda|^2 \cdot |\lambda|^2 + |\lambda|^4 \le |\lambda|^4 - 2|\lambda|^4 + |\lambda|^4 = 0 .$$

respectively

$$\left\| \left(|T|^2 - |\lambda|^2 \right) x_n \right\|^2 \to 0 . \tag{10}$$

From the relation (10) we have

$$\left(|T| - |\lambda| \right) x_n = \left(|T| + |\lambda| \right)^{-1} \cdot \left(|T|^2 - |\lambda|^2 \right) x_n \to 0 .$$

Thus

$$\left(U - e^{i\theta} \right) |\lambda| x_n = U \left(|\lambda| - |T| \right) x_n + \left(U|T| - e^{i\theta} |\lambda| \right) x_n \to 0 .$$

So, that $\left(T^* - \bar{\lambda} \right) x_n \to 0$, by Theorem B. Therefore the proof is completed. ■

3. Operator classes $(M,k)^*$ and $(A,k)^*$ in Hilbert space

In this section we will show some properties of $(M,k)^*$ and $(A,k)^*$ classes.

Proposition 3.1. If the operator T belongs to the class $(A,1)^*$, then it is hyponormal operator.

Proof. Let $T \in (A,1)^*$, then for every unit vectors $x \in H$ we have

$$\|Tx\|^2 \geq \|TT^*x\|$$

$$\langle Tx,Tx\rangle \geq \langle TT^*x,TT^*x\rangle^{\frac{1}{2}}$$

$$\langle Tx,Tx\rangle - \langle TT^*x,TT^*x\rangle^{\frac{1}{2}} \geq 0$$

$$\langle T^*Tx,x\rangle - \langle (TT^*)^2x,x\rangle^{\frac{1}{2}} \geq 0 \text{ ,(Hölder-McCarthy ineq.)}$$

$$\langle T^*Tx,x\rangle - \langle TT^*x,x\rangle \geq 0$$

$$\langle (T^*T - TT^*)x,x\rangle \geq 0 .$$

Thus $T^*T \geq TT^*$, respectively T is hyponormal operator. ■

Corollary 3.1. If the operator T belongs to the $(A,1)^*$ class, then it belongs to the $(M,1)^*$ class.

Proposition 3.2. If the operator T belongs to the $(M,k)^*$ class, for $k \geq 3$, then it belongs to the $(A,k)^*$ class.

Proof. Let $T \in (M,k)^*$, then for every unit vectors $x \in H$ we have :

$$\begin{aligned}\|T^kx\|^4 &= \langle T^kx,T^kx\rangle^2 = \langle T^{*k}T^kx,x\rangle^2 \\ &\geq \langle (TT^*)^kx,x\rangle^2, \quad \text{because } T \in (M,k)^* \\ &\geq \langle (TT^*)^2x,x\rangle^k \quad \textbf{(}\text{Hölder-McCarthy inequality}\textbf{)} \\ &= \langle TT^*x,TT^*x\rangle^k = \|TT^*x\|^{2k} .\end{aligned}$$

Thus $T \in (A,k)^*$. Therefore the proof is completed. ■

Corollary 3.2. If the operator T have dense range in Hilbert space and belongs to the $(M,k+1)$ class, for $k \geq 2$, then it belongs to the $(A,k)^*$ class.

Proof. This proof follows from theorem 3.8 in [3] and proposition 3.2. ■

Lemma 3.1. If T is bilateral weighted shift operator, with weighted sequence ω_n, $(Te_n = \omega_n e_{n+1})$, then it is $(A,k)^*$ class if and only if

$$|\omega_n| \cdot |\omega_{n+1}| \cdot \ldots \cdot |\omega_{n+k-1}| \geq |\omega_{n-1}|^k , \quad n \in Z \text{ and } k \geq 1 .$$

Proof. This follows immediately from the definition of $(A,k)^*$ class. ■

Example 3.1. Let T is a bilateral weighted shift operator, with weighted sequence (ω_n) given as follows

$$\omega_n = \begin{cases} \frac{1}{5} \; p\ddot{e}r \; n \le -1 \\ 5 \; p\ddot{e}r \; n = 0 \\ \frac{1}{5} \; p\ddot{e}r \; n = 1 \\ 25 \; p\ddot{e}r \; n \ge 2 \end{cases}.$$

From lemma 2.1 and lemma 3.1, after some calculation it follows that $T \in (A,3)^*$, but $T \notin (A,3)$.

Example 3.2. Let T is a bilateral weighted shift operator, with weighted sequence (ω_n) given as follows

$$\omega_n = \begin{cases} \frac{1}{2}, \; for \; n \le -1 \\ 1, \; for \; n = 0 \\ \frac{1}{2}, \; for \; n = 1 \\ 2, \; for \; n = 2 \\ \frac{1}{4}, \; for \; n = 3 \\ 16, \; for \; n \ge 4 \end{cases}.$$

From lemma 2.1 and lemma 3.1, after some calculation it follows that $T \in (A,3)$, but $T \notin (A,3)^*$.

Lemma 3.2. If T is a regular bilateral weighted shift operator, with weighted sequence $\omega_n \ne 0$, ($Te_n = \omega_n e_{n+1}$), then $T^{-1} \in (A,k)^*$ if and only if

$$|\omega_n|^k \ge |\omega_{n-1}| \cdot |\omega_{n-2}| \cdot \ldots \cdot |\omega_{n-k}|, \quad n \in Z, |\omega_n| \ne 0 \; and \; k \ge 1.$$

Proof. This follows immediately from the definition of $(A,k)^*$ class. ■

Proposition 3.3. Let $T \in (A,k)^*$ be a regular operator, for $k \ge 1$. Then the approximate point spectrum lies in the disc

$$\sigma_{ap} \subseteq \left\{ \lambda \in C : \frac{1}{\|T\|^{k-1}\sqrt[k]{\|(TT^*)^{-1}\|}} \le |\lambda| \le \|T\| \right\}.$$

Proof. The proof of the Proposition is similar to that of Proposition 2.4. ■

Proposition 3.4. If the operator $T \in B(H)$ satisfies condition $|T^k|^{\frac{4}{k}} \ge |T^*|^4$, for $k \ge 2$, then operator T belongs to the $(A,k)^*$ class.

Proof. The proof of the Proposition is similar to that of Proposition 2.5. ■

Theorem 3.1. If $T \in (A,k)^*$, then T is normaloid for every positive integer $k \ge 1$.

Proof. Suppose T belongs to the $(A,k)^*$ class, for $k \ge 1$. Then we have

$$\|TT^*x\|^k \le \|T^k x\|^2,\ \|x\| = 1$$

$$\sup_{\|x\|=1}\|TT^*x\|^k \le \sup_{\|x\|=1}\|T^k x\|^2$$

$$\|TT^*\|^k \le \|T^k\|^2.$$

Now, since $\|T\|^2 = \|T^*\|^2 = \|T^*T\| = \|TT^*\|$, we have $\|T\|^{2k} \le \|T^k\|^2$, respectively $\|T\|^k \le \|T^k\| \le \|T\|^k$.

Therefore

$$\|T^k\| = \|T\|^k. \tag{11}$$

Now, we prove the equality (11) for $k+1$. Since $\dfrac{TT^*x}{\|TT^*x\|}$ is a unit vector, then we have

$$\|TT^*TT^*x\|^k \le \|T^kTT^*x\|^2 \cdot \|TT^*x\|^{k-2} \Rightarrow \|(TT^*)^2 x\|^k \le \|T^{k+1}\|^2 \cdot \|T^*x\|^2 \cdot \|TT^*x\|^{k-2} \Rightarrow$$

$$\Rightarrow \|TT^*\|^{2k} \le \|T^{k+1}\|^2 \cdot \|T\|^2 \cdot \|T\|^{2(k-2)} \Rightarrow \|T\|^{4k} \le \|T^{k+1}\|^2 \cdot \|T\|^{2k-2} \Rightarrow$$

$$\Rightarrow \|T\|^{2k+2} \le \|T^{k+1}\|^2 \Rightarrow \|T\|^{k+1} \le \|T^{k+1}\| \le \|T\|^{k+1}.$$

Therefore

$$\left\|T^{k+1}\right\| = \left\|T\right\|^{k+1}. \tag{12}$$

Now from the relation (12) it follows that $\left\|T^k\right\| = \left\|T\right\|^k$, for every positive integer $k \geq 1$. Therefore

$$r(T) = \lim_{k\to\infty}\left(\left\|T^k\right\|\right)^{\frac{1}{k}} = \lim_{k\to\infty}\left\|T\right\|^{k\cdot\frac{1}{k}} = \left\|T\right\|,$$

and the proof is completed. ■

In the following we denote by $(QD)(P_n)$ the class of quasi-diagonal operator with respect to the sequence $(P_n)_{n\in N}$ of orthogonal projections such that $P_n \to 1$, strongly (see [7]).

Theorem 3.2. If $T^*T = \lambda I + K'$, were $\lambda \in C$, K' is a compact operator, and T^* is a quasi-normal, then $TT^* = \beta I - K''$ for some complex number β and some compact operator K''.

Proof. For $\alpha > \left\|T^*T\right\| = \left\|T\right\|^2$ we have

$$\left(\alpha I - T^*T\right)^{-1} = \frac{I}{\alpha} + \frac{1}{\alpha}T^*\left(\alpha I - TT^*\right)^{-1}T$$

$$\left(\alpha I - T^*T\right)^{-1} = \frac{I}{\alpha} + \frac{1}{\alpha}\left(\alpha I - TT^*\right)^{-1}T^*T\ . \tag{13}$$

Since T^*T is a Fredholm operator and $indT^*T = 0$, it follows that there exist a compact operator K, such that $T^*T + K$ is an invertible operator. Now by (13), we have

$$\left(\alpha I - T^*T\right)^{-1} = \frac{I}{\alpha} + \frac{1}{\alpha}\left(\alpha I - TT^*\right)^{-1}\left(T^*T + K\right) - \frac{1}{\alpha}(\alpha I - TT^*)^{-1}K$$

were $K_1 = -\frac{1}{\alpha}(\alpha I - TT^*)^{-1}K$ is a compact operator,

$$\left(\alpha I - T^*T\right)^{-1} = \frac{I}{\alpha} + \frac{1}{\alpha}\left(\alpha I - TT^*\right)^{-1}\left(T^*T + K\right) + K_1$$

$$\frac{1}{\alpha}\left(\alpha I - TT^*\right)^{-1} = \left[\left(\alpha I - T^*T\right)^{-1} - K_1 - \frac{I}{\alpha}\right]\left(T^*T + K\right)^{-1}$$

$$\left(\alpha I - TT^*\right)^{-1} = \alpha\left[\left(\alpha I - T^*T\right)^{-1} - K_1 - \frac{I}{\alpha}\right]\left(T^*T + K\right)^{-1}.$$

So, $\left(\alpha I - TT^*\right)^{-1} \in (QD)(P_n)$, for every sequence $(P_n)_{n\in N}$ of orthogonal projections, for which $P_n \to 1$ strongly. Therefore $\alpha I - TT^* = \gamma I + K''$ (see [7]) , respectively $TT^* = (\alpha - \gamma)I - K'' = \beta I - K''$. ■

Proposition 3.5. Let $T \in (A,k)^*$ and T^k be a compact operator for some $k \geq 1$, then T is compact too.

Proof. From the fact that $T \in (A,k)^*$ for $k \geq 1$, we have

$$\|TT^* x\|^k \leq \|T^k x\|^2, \text{ for every } x \in H, \|x\| = 1 \text{ and } k \geq 1. \tag{14}$$

Let $(x_n) \in H$ be weakly convergent sequence with limit 0 in H. From compactness of T^k and relation (14) we get the following relation

$$\|TT^* x_n\|^k \leq \|T^k x_n\|^2 \to 0,\ n \to \infty$$

$$\|TT^* x_n\| \to 0,\ n \to \infty .$$

From the last relation it follows that TT^* is a compact operator , respectively T is a compact operator . ■

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