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Hybrid Descent-Like Halpern Iteration Methods for Two Nonexpansive Mappings and Semigroups on Two Sets

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Abstract

In this paper, we introduce some new iteration methods based on the hybrid method in mathematical programming, the Mann's iterative method and the Halpern's method for finding a fixed point of a nonexpansive mapping and a common fixed point of a nonexpansive semigroup Hilbert spaces.

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1. Introduction

Let H be a real Hilbert space with the scalar product and the norm denoted by the symbols $\langle ., . \rangle$ and $\|.\|$, respectively, and let C be a nonempty closed and convex subset of H. Denote by $P_C(x)$ the metric projection from $x \in H$ onto C. Let T be a nonexpansive mapping on C, i.e., $T : C \to C$ and

$\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$. We use $F(T)$ to denote the set of fixed points of T, i.e., $F(T) = \{x \in C : x = Tx\}$. We know that $F(T)$ is nonempty, if C is bounded, for more details see [1].

Let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C, that is,
(1) for each $t > 0, T(t)$ is a nonexpansive mapping on C;
(2) $T(0)x = x$ for all $x \in C$;
(3) $T(t_1 + t_2) = T(t_1) \circ T(t_2)$ for all $t_1, t_2 > 0$; and
(4) for each $x \in C$, the mapping $T(.)x$ from $(0, \infty)$ into C is continuous.

Denote by $\mathcal{F} = \cap_{t>0} F(T(t))$ the set of common fixed points for the semigroup $\{T(t) : t > 0\}$. We know that $\mathcal{F}$ is a closed convex subset in H and $\mathcal{F} \neq \emptyset$ if C is compact (see, [2]).

For finding a fixed point of a nonexpansive mapping T on C, in 1953, Mann [3] proposed the following method:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ x_{n+1} &= \alpha_n x_n + (1 - \alpha_n) T x_n, \end{aligned} \tag{1.1}$$

that converges only weakly, in general (see [4] for an example). In 1967, Halpern [5] firstly proposed the following iteration process:

$$x_{n+1} = \beta_n u + (1 - \beta_n) T x_n, \quad n \geq 0, \tag{1.2}$$

where u, x_0 are two fixed elements in C and $\{\beta_n\} \subset (0, 1)$. He pointed out that the conditions $\lim_{n\to\infty} \beta_n = 0$ and $\sum_{n=0}^{\infty} \beta_n = \infty$ are necessary in the sense that, if the iteration (1.2) converges to a fixed point of T, then these conditions must be satisfied. Further, the iteration method was investigated by Lions [6], Reich [7], Wittmann [8] and Song [9]. Recently, Alber [10] proposed the following descent-like method

$$x_{n+1} = P_C(x_n - \mu_n [x_n - T x_n]), n \geq 0, \tag{1.3}$$

and proved that if $\{\mu_n\} : \mu_n > 0, \mu_n \to 0$, as $n \to \infty$ and $\{x_n\}$ is bounded, then:

(i) there exists a weak accumulation point $\tilde{x} \in C$ of $\{x_n\}$;
(ii) all weak accumulation points of $\{x_n\}$ belong to $F(T)$; and
(iii) if $F(T)$ is a singleton, i.e., $F(T) = \{\tilde{x}\}$, then $\{x_n\}$ converges weakly to $\tilde{x}$.

To obtain strong convergence for (1.1), Nakajo and Takahashi [11] introduced the hybrid Mann's iteration method:

$$\begin{aligned} x_0 &\in C, \\ y_n &= \alpha_n x_n + (1-\alpha_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), \end{aligned} \tag{1.4}$$

where $\{\alpha_n\} \subset [0,a]$ for some $a \in [0,1)$. They showed that $\{x_n\}$ defined by (1.4) converges strongly to $P_{F(T)}(x_0)$ as $n \to \infty$. Recently, Yanes and Xu [12] adapted the iteration process (1.2) as follows:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_n &= \beta_n x_0 + (1-\beta_n)Tx_n, \\ C_n &= \{z \in C : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0). \end{aligned} \tag{1.5}$$

They proved that if T is a nonexpansive mapping on a closed convex subset C with $F(T) \neq \emptyset$ and the sequence $\{\beta_n\} \subset (0,1)$ is chosen such that $\lim_{n\to\infty} \beta_n = 0$, then the sequence $\{x_n\}$ defined by (1.5) converges strongly to $P_{F(T)}(x_0)$ as $n \to \infty$.

For finding an element $p \in \mathcal{F}$, Nakajo and Takahashi [11] also introduced an iteration procedure as follows:

$$\begin{aligned} x_0 &\in C \quad \text{any element}, \\ y_n &= \alpha_n x_n + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)x_n ds, \\ C_n &= \{z \in C : \|y_n - z\| \le \|x_n - z\|\}, \\ Q_n &= \{z \in C : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{C_n \cap Q_n}(x_0), n \ge 0, \end{aligned} \tag{1.6}$$

where $\alpha_n \in$ [0,a] for some $a \in$ [0,1) and $\{t_n\}$ is a positive real number divergent sequence. Under the conditions on $\{\alpha_n\}$ and $\{t_n\}$, the sequence $\{x_n\}$ defined by (1.6) converges strongly to $P_{\mathcal{F}}(x_0)$.

If $C \equiv H$, then C_n and Q_n in (1.4)-(1.6) are two halfspaces. So, the projection x_{n+1} onto $C_n \cap Q_n$ in these methods can be described by an explicit formula [13]. Clearly, if C is a proper subset of H, then C_n and Q_n in (1.4)-(1.6) are not two halfspaces. Then, the following big problem is posed: how to construct the closed convex subsets C_n and Q_n and if we can express x_{n+1} of (1.4)-(1.6) in a similar form as in [13]? This problem is solved very recently in [14] and [15]. In this works, C_n and Q_n are replaced by two halfspaces and y_n is the right hand side of (1.3) with a modification.

In this paper, using the idea, we introduce the following new iteration processes:

$$\begin{aligned} x_0 &\in H \quad \text{any element}, \\ z_n &= \alpha_n P_C(x_n) + (1-\alpha_n) P_C T P_C(x_n), \\ y_n &= \beta_n x_0 + (1-\beta_n) P_C T z_n, \\ H_n &= \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \ge 0; \end{aligned} \tag{1.7}$$

and

$$\begin{aligned} x_0 &\in H \quad \text{any element}, \\ z_n &= \alpha_n P_C(x_n) + (1-\alpha_n) \frac{1}{t_n}\int_0^{t_n} T(s) P_C(x_n) ds, \\ y_n &= \beta_n x_0 + (1-\beta_n) \frac{1}{t_n}\int_0^{t_n} T(s) z_n ds, \\ H_n &= \{z \in H : \|y_n - z\|^2 \le \|x_n - z\|^2 \\ &\quad + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \ge 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \ge 0, \end{aligned} \tag{1.8}$$

for a nonexpansive mapping $T : C \to H$ and a nonexpansive semigroup $\{T(t) : t > 0\}$ on C, respectively. We shall prove the strong convergence of the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.7) and (1.8) to a fixed point of T and a common fixed point of the nonexpansive semigroup $\{T(t) : t > 0\}$, respectively.

Later, the symbols $\rightharpoonup$ and $\to$ denote weak and strong convergences, respectively.

2. Strong convergence to a fixed point of nonexpansive mappings

We formulate the following facts needed in the proof of our results.

Lemma 2.1 [16]. *Let H be a real Hilbert space H. There holds the following identity*: $\|x-y\|^2 = \|x\|^2 - \|y\|^2 - 2\langle x-y, y\rangle$.

Lemma 2.2 [12]. *Let C be a nonempty closed convex subset of a real Hilbert space H. For any $x \in H$, there exists a unique $z \in C$ such that $\|z-x\| \leq \|y-x\|$ for all $y \in C$, and $z = P_C(x)$ if and only if $\langle z-x, y-z\rangle \geq 0$ for all $y \in C$, where P_C is the metric projection of H on C.*

Lemma 2.3. (Demiclosedness principle) [17]. *If C is a nonempty closed convex subset of a real Hilbert space H, T is a nonexpansive mapping on C, $\{x_n\}$ is a sequence in C such that $x_n \rightharpoonup x$ and $x_n - Tx_n \to 0$, then $x - Tx = 0$.*

Lemma 2.4 [17]. *Every Hilbert space H has Randon-Riesz property or Kadec-Klee property, that is, for a sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$ and $\|x_n\| \to \|x\|$, then there hodls $x_n \to x$.*

Now, we are in a position to prove the following result.

Theorem 2.5. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\alpha_n \to 1$ and $\beta_n \to 0$. Then, the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$ defined by (1.7) converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.*

Proof. First, note that

$$\|y_n - z\|^2 \leq \|x_n - z\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, z\rangle)$$

is equivalent to

$$\langle (1-\beta_n)x_n + \beta_n x_0 - y_n, z\rangle \leq \langle x_n - y_n, x_n\rangle - \frac{1}{2}\|y_n - x_n\|^2 + \frac{\beta_n}{2}\|x_0\|^2.$$

Thus, H_n is a halfspace. It is clear that $F(T_1) = F(T_1P_C) := \{p \in H : T_1P_C(p) = p\}$ for any mapping T_1 from C into C. Taking $T_1 = P_CT$ and using Lemma 2.6 in [15] with $S = P_CT$, we have that $F(T) = F(P_CTP_C)$. Hence, by the convexity of $\|.\|^2$ and the nonexpansive property of P_C, we obtain for any $p \in F(T)$ that $p = P_CTP_C(p)$, and hence

$$\begin{aligned}
\|z_n - p\|^2 &= \|\alpha_n P_C(x_n) - p + (1-\alpha_n)P_CTP_C(x_n)\|^2 \\
&= \|\alpha_n(P_C(x_n) - P_C(p)) + (1-\alpha_n)[P_CTP_C(x_n) - P_CTP_C(p)]\|^2 \\
&\leq \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|P_C(x_n) - P_C(p)\|^2 \\
&\leq \|x_n - p\|^2.
\end{aligned}$$

By the similar argument and Lemma 2.1 with $x = x_0 - p$ and $y = x_n - p$, we also obtain

$$\begin{aligned}\|y_n - p\|^2 &= \|\beta_n x_0 + (1-\beta_n)P_C T z_n - p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|P_C T z_n - P_C T p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|z_n - p\|^2\\ &\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|x_n - p\|^2\\ &= \|x_n - p\|^2 + \beta_n(\|x_0 - p\|^2 - \|x_n - p\|^2)\\ &= \|x_n - p\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle).\end{aligned}$$

Therefore, $p \in H_n$ for all $n \ge 0$. It means that $F(T) \subset H_n$ for all $n \ge 0$.

Next, we show by mathematical induction that $F(T) \subset H_n \cap W_n$ for each $n \ge 0$. For $n = 0$, we have $W_0 = H$, and hence $F(T) \subset H_0 \cap W_0$. Suppose that x_i is given and $F(T) \subset H_i \cap W_i$ for some $i > 0$. There exists a unique element $x_{i+1} \in H_i \cap W_i$ such that $x_{i+1} = P_{H_i \cap W_i}(x_0)$. Therefore, by Lemma 2.2,

$$\langle x_{i+1} - x_0, p - x_{i+1}\rangle \ge 0$$

for each $p \in H_i \cap W_i$. Since $F(T) \subset H_i \cap W_i$, we get $F(T) \subset W_{i+1}$. So, we have $F(T) \subset H_{i+1} \cap W_{i+1}$.

Further, since $F(T)$ is a nonempty closed convex subset of H, there exists a unique element $u_0 \in F(T)$ such that $u_0 = P_{F(T)}(x_0)$. From $x_{n+1} = P_{H_n \cap W_n}(x_0)$, we obtain

$$\|x_{n+1} - x_0\| \le \|z - x_0\|$$

for every $z \in H_n \cap W_n$. As $u_0 \in F(T) \subset W_n$, we get

$$\|x_{n+1} - x_0\| \le \|u_0 - x_0\| \quad n \ge 0. \tag{2.1}$$

This implies that $\{x_n\}$ is bounded. So, $\{P_C T P_C(x_n)\}$, $\{z_n\}$ and $\{Tz_n\}$ are also bounded.

Now, we show that

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{2.2}$$

From the definition of W_n and Lemma 2.2, it follows that $x_n = P_{W_n}(x_0)$. As $x_{n+1} \in H_n \cap W_n$, we have

$$\|x_{n+1} - x_0\| \ge \|x_n - x_0\| \quad n \ge 0.$$

Therefore, $\{\|x_n - x_0\|\}$ is a nondecreasing and bounded sequence. So, there exists $\lim_{n\to\infty}\|x_n - x_0\| = c$. On the other hand, from $x_{n+1} \in W_n$, we have

$\langle x_n - x_0, x_{n+1} - x_n\rangle \geq 0$ and hence

$$\begin{aligned}\|x_n - x_{n+1}\|^2 &= \|x_n - x_0 - (x_{n+1} - x_0)\|^2 \\ &= \|x_n - x_0\|^2 - 2\langle x_n - x_0, x_{n+1} - x_0\rangle + \|x_{n+1} - x_0\|^2 \\ &\leq \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 \quad \forall n \geq 0.\end{aligned}$$

Thus, (2.2) is followed from the last inequality and $\lim_{n\to\infty} \|x_n - x_0\| = c$.

Since $\alpha_n \to 1$ and $\{x_n\}, \{P_C T P_C(x_n)\}$ are bounded, we have from (1.7) that

$$\lim_{n\to\infty} \|z_n - P_C(x_n)\| = \lim_{n\to\infty} (1 - \alpha_n)\|P_C(x_n) - P_C T P_C(x_n)\| = 0. \tag{2.3}$$

On the other hand, since $x_{n+1} \in H_n$ we have that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + \beta_n(\|x_0\| + 2\langle x_n - x_0, x_{n+1}\rangle)\}.$$

Therefore, from (2.2), the boundedness of $\{x_n\}$, $\beta_n \to 0$ and the last inequality, it follows that

$$\lim_{n\to\infty} \|y_n - x_{n+1}\| = 0. \tag{2.4}$$

This together with (2.2) implies that

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{2.5}$$

Noticing that $P_C T z_n = y_n - \beta_n(x_n - P_C T z_n) + \beta_n(x_n - x_0)$, we have

$$\|x_n - P_C T z_n\| \leq \|x_n - y_n\| + \beta_n\|x_n - P_C T z_n\| + \beta_n\|x_n - x_0\|.$$

From (2.1) and the last inequality, it follows that

$$\|x_n - P_C T z_n\| \leq \frac{1}{1 - \beta_n}\bigg(\|x_n - y_n\| + \beta_n\|u_0 - x_0\|\bigg).$$

By $\beta_n \to 0$ ($\beta_n \leq 1 - \beta$ for some $\beta \in (0, 1)$), (2.5) and the last inequality, we obtain

$$\lim_{n\to\infty} \|x_n - P_C T z_n\| = 0. \tag{2.6}$$

Further, we have that $P_C T z_n = P_C P_C T z_n$, and hence

$$\begin{aligned}\|z_n - P_C T z_n\| &\leq \|z_n - P_C(x_n)\| + \|P_C(x_n) - P_C P_C(T z_n)\| \\ &\leq \|z_n - P_C(x_n)\| + \|x_n - P_C T z_n\|.\end{aligned}$$

So, from (2.3), (2.6) and the last inequality, it follows that

$$\lim_{n\to\infty} \|z_n - P_C T z_n\| = 0. \tag{2.7}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that convegers weakly to some element $p \in H$ as $j \to \infty$. From (2.6) and (2.7), we also have that $\{z_{n_j}\}$ converges weakly to p. Since $\{z_n\} \subset C$, we obtain that $p \in C$. By Lemmas 2.3 and (2.7), $p \in F(P_C T) = F(T)$ by Lemma 2.6 in [15] with S replaced by T.

Now, from (2.1) and the weakly lower semicontinuity of the norm it implies that

$$\|x_0 - u_0\| \leq \|x_0 - p\| \leq \liminf_{j\to\infty} \|x_0 - x_{n_j}\| \leq \limsup_{j\to\infty} \|x_0 - x_{n_j}\| \leq \|x_0 - u_0\|.$$

Thus, we obtain $\lim_{j\to\infty} \|x_0 - x_{n_j}\| = \|x_0 - u_0\| = \|x_0 - p\|$. This implies $x_{k_j} \to p = u_0$ by Lemma 2.4. By the uniqueness of the projection $u_0 = P_{F(T)}(x_0)$, we have that $x_n \to u_0$. From (2.5) and (2.6)-(2.7), we also get $y_n \to u_0$ and $z_n \to u_0$, respectively. This completes the proof.

Corollary 2.6. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in [0,1] such that such that $\beta_n \to 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned} x_0 &\in H \quad \textit{any element}, \\ y_n &= \beta_n x_0 + (1-\beta_n) P_C T P_C(x_n), \\ H_n &= \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\quad + \beta_n(\|x_0\| + 2\langle x_n - x_0, z\rangle)\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \geq 0, \end{aligned}$$

converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.

Proof. By putting $\alpha_n \equiv 1$ in Theorem 2.5, we obtain the conclusion.

Corollary 2.7. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $T : C \to H$ be a nonexpansive mapping such that $F(T) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in [0,1] such that $\alpha_n \to 1$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned} x_0 &\in H \quad \textit{any element}, \\ y_n &= P_C T(\alpha_n P_C(x_n) + (1-\alpha_n) P_C T P_C(x_n)), \\ H_n &= \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \geq 0, \end{aligned}$$

converge strongly to the same point $u_0 = P_{F(T)}(x_0)$, as $n \to \infty$.

Proof. By putting $\beta_n \equiv 0$ in Theorem 2.5, we obtain the conclusion.

3. **Strong convergence to a common fixed point of nonexpansive semigroups**

We need the following Lemma in the proof of our result.

Lemma 3.1 [18]. *Let C be a nonempty bounded closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C. Then, for any $h > 0$*

$$\limsup_{t\to\infty} \sup_{y\in C} \left\| T(h)\left(\frac{1}{t}\int_0^t T(s)yds\right) - \frac{1}{t}\int_0^t T(s)yds \right\| = 0.$$

Now, we prove the following result.

Theorem 3.2. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\alpha_n\}$ and $\{\beta_n\}$ are sequences in [0,1] such that $\alpha_n \to 1$ and $\beta_n \to 0$, and $\{t_n\}$ is a positive real divergent sequence. Then, the sequences $\{x_n\}, \{z_n\}$ and $\{y_n\}$, defined by (1.8), converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, as $n \to \infty$.*

Proof. For each $p \in \mathcal{F}$, we have

$$p = P_C(p) = T(s)P_C(p)$$

for each $s > 0$ and hence from (1.8) and the convexity of $\|.\|^2$ we obtain that

$$\begin{aligned}
\|z_n - p\|^2 &= \left\| \alpha_n(P_C(x_n) - p) + (1-\alpha_n)\left(\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds - p\right) \right\|^2 \\
&= \Bigg\| \alpha_n(P_C(x_n) - P_C(p)) \\
&\quad + (1-\alpha_n)\left(\frac{1}{t_n}\int_0^{t_n} [T(s)P_C(x_n) - T(s)P_C(p)]ds\right) \Bigg\|^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\left(\frac{1}{t_n}\int_0^{t_n} \|T(s)P_C(x_n) - T(s)P_C(p)\|ds\right)^2 \\
&\le \alpha_n\|x_n - p\|^2 + (1-\alpha_n)\|P_C(x_n) - P_C(p)\|^2 \\
&\le \|x_n - p\|^2.
\end{aligned}$$

By the similar argument, we also obtain

$$\begin{aligned}
\|y_n - p\|^2 &= \left\|\beta_n(x_0 - p) + (1-\beta_n)\left(\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds - p\right)\right\|^2 \\
&\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\left\|\frac{1}{t_n}\int_0^{t_n} [T(s)z_n - T(s)p]ds\right\|^2 \\
&\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|z_n - p\|^2 \\
&\le \beta_n\|x_0 - p\|^2 + (1-\beta_n)\|x_n - p\|^2 \\
&= \|x_n - p\|^2 + \beta_n(\|x_0 - p\|^2 - \|x_n - p\|^2) \\
&\le \|x_n - p\|^2 + \beta_n(\|x_0\|^2 + 2\langle x_n - x_0, p\rangle).
\end{aligned}$$

Therefore, $p \in H_n$ for $n \ge 0$. It means that $\mathcal{F} \subset H_n$ for $n \ge 0$. As in the proof of Theorem 2.5, we can obtain the following properties:

(i) $\mathcal{F} \subset H_n \cap W_n$,

$$\|x_{n+1} - x_0\| \le \|u_0 - x_0\|, u_0 = P_{\mathcal{F}}(x_0) \tag{3.1}$$

for $n \ge 0$. This implies that $\{x_n\}$ is bounded. So,

$$\left\{\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds\right\}, \{z_n\} \quad \text{and} \quad \left\{\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\}$$

are also bounded.

(ii)

$$\lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.2}$$

$$\lim_{n\to\infty} \|z_n - P_C(x_n)\| = 0. \tag{3.3}$$

$$\lim_{n\to\infty} \|y_n - x_{k+1}\| = 0. \tag{3.4}$$

$$\lim_{n\to\infty} \|y_n - x_n\| = 0. \tag{3.5}$$

$$\lim_{n\to\infty}\left\|x_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| = \lim_{n\to\infty}\left\|z_n - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| = 0. \tag{3.6}$$

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_j}\}$ of $\{x_n\}$ that converges weakly to some element $p \in H$ as $j \to \infty$. So, by (3.6), the subsequence $\{z_{n_j}\}$ also converges weakly to p and hence $p \in C$.

On the other hand, for each $h > 0$, we have that

$$\begin{aligned}\|T(h)z_n - z_n\| \le& \left\|T(h)z_n - T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right)\right\| \\ &+ \left\|T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| \\ &+ \left\|\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds - z_n\right\| \\ \le& 2\left\|\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds - z_n\right\| \\ &+ \left\|T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\|.\end{aligned} \tag{3.7}$$

Let $C_0 = \{z \in C : \|z - u_0\| \le 2\|x_0 - u_0\|\}$. Since $u_0 = P_{\mathcal{F}}(x_0) \in C$, we have from (1.8) and (3.1) that

$$\begin{aligned}\|z_n - u_0\| =& \left\|\alpha_n(P_C(x_n) - u_0) + (1-\alpha_n)\left[\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds - u_0\right]\right\| \\ =& \Bigg\|\alpha_n[P_C(x_n) - P_C(u_0)] \\ &+ (1-\alpha_n)\left[\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds - \frac{1}{t_n}\int_0^{t_n} T(s)P_C(u_0)ds\right]\Bigg\| \\ \le& \alpha_n\|x_n - u_0\| + (1-\alpha_n)\left\|\frac{1}{t_n}\int_0^{t_n}[T(s)P_C(x_n) - T(s)P_C(u_0)]ds\right\| \\ \le& \|x_n - x_0\| + \|x_0 - u_0\| \\ \le& 2\|x_0 - u_0\|.\end{aligned}$$

So, C_0 is a nonempty bounded closed convex subset. It is easy to verify that and $\{T(t) : t > 0\}$ is a nonexpansive semigroup on C_0. By Lemma 3.1, we get

$$\lim_{n\to\infty}\left\|T(h)\left(\frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right) - \frac{1}{t_n}\int_0^{t_n} T(s)z_n ds\right\| = 0$$

for every fixed $h > 0$ and hence by (3.6)-(3.7) we obtain that

$$\lim_{n\to\infty}\|T(h)z_n - z_n\| = 0$$

for each $h > 0$. By Lemma 2.3, $p \in F(T(h))$ for all $h > 0$. It means that $p \in \mathcal{F}$. As in the proof of Theorem 2.5, by using (3.1)-(3.6), we also obtain that the sequences $\{x_n\}, \{y_n\}$ and $\{z_n\}$, defined by (1.8), converge strongly to u_0 as $n \to \infty$. This completes the proof.

Corollary 3.3. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\beta_n\}$ is a sequence in [0,1] such that $\beta_n \to 0$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned} x_0 &\in H \quad \text{any element}, \\ y_n &= \beta_n x_0 + (1-\beta_n)\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds, \\ H_n &= \{z \in H : \|y_n - z\|^2 \leq \|x_n - z\|^2 \\ &\quad + \beta_n(\|x_0\| + 2\langle x_n - x_0, z\rangle)\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \geq 0, \end{aligned}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, as $n \to \infty$.
Proof. By putting $\alpha_n \equiv 1$ in Theorem 3.2, we obtain the conclusion.

Corollary 3.4. *Let C be a nonempty closed convex subset in a real Hilbert space H and let $\{T(t) : t > 0\}$ be a nonexpansive semigroup on C such that $\mathcal{F} = \cap_{t>0} F(T(t)) \neq \emptyset$. Assume that $\{\alpha_n\}$ is a sequence in [0,1] such that $\alpha_n \to 1$. Then, the sequences $\{x_n\}$ and $\{y_n\}$, defined by*

$$\begin{aligned} x_0 &\in H \quad \text{any element}, \\ y_n &= \frac{1}{t_n}\int_0^{t_n} T(s)\Big[\alpha_n P_C(x_n) + (1-\alpha_n)\frac{1}{t_n}\int_0^{t_n} T(s)P_C(x_n)ds\Big]ds, \\ H_n &= \{z \in H : \|y_n - z\| \leq \|x_n - z\|\}, \\ W_n &= \{z \in H : \langle x_n - z, x_0 - x_n\rangle \geq 0\}, \\ x_{n+1} &= P_{H_n \cap W_n}(x_0), n \geq 0, \end{aligned}$$

converge strongly to the same point $u_0 = P_{\mathcal{F}}(x_0)$, as $n \to \infty$.
Proof. By putting $\beta_n \equiv 0$ in Theorem 3.2, we obtain the conclusion.

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References

[1] E.F. Browder, Fixed-point theorems for noncompact mappings in Hilbert spaces, Proceed. Nat. Acad. Sci. USA 53: 1272-1276 (1965)

[2] R. DeMarr, Common fixed points for commuting contraction mappings, Pacific J. Math. 13: 1139-1141 (1963)

[3] W.R. Mann, Mean value methods in iteration, Proc. Amer. Math. Soc. 4: 506-510 (1953)

[4] A. Genel, J. Lindenstrass, An example concerning fixed points, Israel J. Math. 22: 81-86 (1975)

[5] B. Halpern, Fixed points of nonexpanding mapps, Bull. Am.Math. Soc. 73: 957-961 (1967)

[6] P.L. Lions, Approximation de points fixes de contractions, C.R. Acad. Sci. Sér A-B Paris 284: 1357-1359 (1977)

[7] S. Reich, Strong convergence theorem for resolvants of accretive operators in Banach spaces, J. Math. Anal. Appl. 75: 287-292 (1980)

[8] R. Wittmann, Approximation of fixed points of nonexpansive mappings, Arch. Math. 59: 486-491 (1992)

[9] Y. Song, A new sufficient condition for strong convergence of Halpern type iterations, Appl. Math. Comput. 198(2,1): 721-728 (2007)

[10] Ya.I. Alber, On the stability of iterative approximations to fixed points of nonexpansive mappings, J. Math. Anal. Appl. 328: 958-971 (2007)

[11] K. Nakajo and W. Takahashi, Strong convergence theorem for nonexpansive mappings and nonexpansive semigroup, J. Math. Anal. Applic. 279: 372-379 (2003)

[12] C. Martinez-Yanes, H.K. Xu, Strong convergence of the CQ method for fixed iteration processes, Nonl. Anal. 64: 2400-2411 (2006)

[13] M.V. Solodov, B.F. Svaiter, Forcing strong convergence of proximal point iterations in Hilbert space, Math. Progr. 87: 189-202 (2000)

[14] Nguyen Buong, Strong convergence theorem for nonexpansive semigroup in Hilbert space, Nonl. Anal. 72(12): 4534-4540 (2010)

[15] Nguyen Buong, Strong convergence theorem of an iterative method for variational inequalities and fixed point problems in Hilbert spaces, Applied Math. and Comp., 217: 322-329 (2010)

[16] G. Marino and H.K. Xu, Weak and strong convergence theorems for stric pseudo-contractions in Hilbert spaces, J. Math. Anal. Appl. 329: 336-346 (2007)

[17] K. Goebel and W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge Studies in Advanced Math., V. 28, Cambridge Univ. Press, Cambridge 1990.

[18] T. Shimizu, W. Takahashi, Strong convergence to common fixed points of families of nonexpansive mappings, J. Math. Anal. Appl. 211: 71-83 (1997)

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