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Compatible Maps in G-Metric Spaces

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Abstract

The aim of this paper is to introduce the notion of compatible maps in G-metric spaces and then prove a common fixed point theorem for these mappings. The main result is also illustrated by an example.

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1. Introduction

In 1992, Dhage [1] introduced the concept of D – metric spaces. Recently, Mustafa and Sims [4] had shown that most of the results concerning Dhage's D – metric spaces are invalid and thereafter , they introduced a new generalized metric space structure, and called it as G – metric spaces. For more details on G – metric spaces one can refer to the papers [5]- [7].

Now we give preliminaries and basic definitions which are used throughout the paper.

In 2006, Mustafa and Sims [5] introduced the concept of G-metric spaces as follows: Let X be a nonempty set. Let G: $X \times X \times X \rightarrow R^+$ be a function satisfying the following properties:

(G1) G(x, y, z) = 0 if x = y = z,

(G2) $0<G(x, x, y)$ for all x, $y \in X$ with $x \neq y$,

(G3) $G(x, x, y) \leq G(x, y, z)$ for all x, y, $z \in X$ with $z \neq y$,

(G4) G(x, y, z) = G(x, z, y) = G(y, z, x) = … (symmetry in all three variables),

(G5) $G(x, y, z) \leq G(x, a, a) + G(a, y, z)$ for all x, y, z, $a \in X$, (rectangle inequality).

The function G is called a generalized metric or, more specifically, a G – metric on X, and the pair (X, G) is called a G – metric space.

The following definitions and results are given by Mustafa and Sims [5].

1.1 Definition. Let (X,G) be a G-metric space then for $x_0 \in X$, r > 0,the G-ball with center x_0 and radius r is, $B_G(x_0,r) = \{y \in X : G(x_0,y,y) < r\}$.

1.1 Proposition. Let (X,G) be a G-metric space, then for any $x_0 \in X$ and r > 0,we have,

(i) if $G(x_0,x,y) < r$, then x, $y \in B_G(x_0,r)$.

(ii) if $y \in B_G(x_0,r)$,then there exists a $\delta > 0$ such that $B_G(y, \delta) \subseteq B(x_0,r)$.

It follows from (ii) of the above proposition that the family of G-balls $\{ B_G(x, r) ; x \in X, r>0 \}$ is the base of a topology τ (G) on X, the G-metric topology.

1.2 Definition. Let (X,G) and (X′,G′) be G-metric spaces .A function f: $X \rightarrow X'$ is G-continuous at a point $x_0 \in X$ if $f^{-1}(B_{G'}(f(x_0),r)) \in \tau(G)$ for all r > 0.We say f is G-continuous if it is G-continuous at all points of X.

1.2 Proposition. Let (X,G) and (X′,G′) be G-metric spaces. A function f: $X \rightarrow X'$ is G-continuous at a point $x \in X$ if and only if it is G-sequentially continuous at x ;that is, whenever (x_n) is G-convergent to x we have $(f(x_n))$ is G-convergent to f(x).

1.3 Proposition. Let (X,G) be a G-metric space, then the function G(x,y,z) is jointly continuous in all three of its variables.

1.3 Definition. Let (X, G) be a G – metric space. A sequence $\{x_n\}$ in X is G–convergent to x if $\lim_{m,n\to\infty} G(x, x_n, x_m) = 0$; i.e., for each $\in > 0$ there exists an N such that $G(x, x_n, x_m) < \in$ for all $m, n \geq N$.

1.4 Proposition. Let (X, G) be a G – metric space, then for a sequence $\{x_n\}$ in X and point $x \in X$ the following are equivalent:

(i) $\{x_n\}$ is G convergent to x,

(ii) $G(x_n, x_n, x) \to 0$ as $n \to \infty$,

(iii) $G(x_n, x, x) \to 0$ as $n \to \infty$,

(iv) $G(x_m, x_n, x) \to 0$ as $m, n \to \infty$.

1.4 Definition. Let (X, G) be a G – metric space. A sequence $\{x_n\}$ in X is called G – Cauchy if, for each $\in > 0$ there exists a positive integer N such that $G(x_n, x_m, x_l) < \in$ for all $n, m, l \geq N$.

1.5 Proposition. In a G – metric space (X, G) the following are equivalent:

(i) the sequence $\{x_n\}$ is G – Cauchy,

(ii) for each $\in > 0$ there exists a positive integer N such that $G(x_n, x_m, x_m) < \in$ for all $n, m \geq N$.

1.6 Proposition. A G – metric space (X, G) is G – complete if and only if (X, d_G) is a complete metric space, where d_G is a metric on X, associated with G.

2. Compatible maps in G- metric spaces

There has been a considerable interest to study common fixed point for a pair (or family) of mappings satisfying contractive conditions in various spaces. Several interesting and elegant results were obtained in this direction by various authors. It was the turning point in the "fixed point arena" when the notion of commutativity was used by Jungck [2] to obtain common fixed point theorems in metric spaces. This result was further generalized and extended in various ways by many authors.

In particular, now we look in the context of common fixed point theorem in G- metric spaces. Start with the following contraction conditions:

Let T be a mapping from a complete metric space (X, G) into itself and consider the following condition:

(*) $G(Tx, Ty, Tz) \leq \alpha.G(x, y, z)$ for all $x, y, z \in X$, where $0 \leq \alpha < 1$.

It is clear that every self mapping T of X satisfying condition (*) is continuous. Now we focus to generalize the condition (*) for a pair of self maps . Let S and T be self maps of X satisfying the followings:

(**) $G(Sx, Sy, Sz) \leq \alpha\, G(Tx, Ty, Tz)$ for all $x, y, z \in X$, where $0 \leq \alpha < 1$.

To prove the existence of common fixed points for (**), it is necessary to add some additional assumptions such as :

(i) construction of the sequence $\{x_n\}$

(ii) some mechanism to obtain common fixed point.

Most of the common fixed point theorems have the following steps:

(i) contraction

(ii) continuity of functions (either one or both) and

(iii) commuting/ minimal commuting .

In some cases condition (ii) can be relaxed but condition (i) and (iii) are unavoidable.

In 1986**,** Jungck[4] introduced the concept of compatible maps in metric spaces as follows:

Two self-mappings f and g of a metric space (X, d) are said to be compatible if $\lim_{n\to\infty} d(fgx_n, gfx_n) = 0$, whenever $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some t in X .

Now we introduce the concept of compatible maps in G – metric space as follows:

2.1 Definition. Let f and g be maps from a G – metric space (X, G) into itself. The maps f and g are said to be compatible map if there exists a sequence $\{x_n\}$ such that

$\lim_{n\to\infty} G(fgx_n, gfx_n, gfx_n) = 0$ or $\lim_{n\to\infty} G(gfx_n, fgx_n, fgx_n) = 0$ whenever $\{x_n\}$ is sequence in X such that $\lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Now, we come to our main result for a pair of compatible maps.

2.1 Theorem. Let (X, G) be a complete G – metric space. Let f and g be self mappings of X satisfying the following conditions:

(2.1) $f(X) \subseteq g(X)$,

(2.2) f or g is continuous,

(2.3) $G(fx, fy, fz) \le qG(gx, gy, gz)$, for every x, y, z in X and $0 \le q < 1$.

Then f and g have a unique common fixed point in X provided f and g are compatible maps.

Proof. Let x_0 be an arbitrary point in X. By (2.1), one can choose a point x_1 in X such that $fx_0 = gx_1$, In general choose x_{n+1} such that $y_n = fx_n = gx_{n+1}$, n=0,1,2,….

From (2.3), we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le q\, G(gx_n, gx_{n+1}, gx_{n+1})$$
$$= qG(fx_{n-1}, fx_n, fx_n)$$

Continuing in the same way, we have

$$G(fx_n, fx_{n+1}, fx_{n+1}) \le q^n G(fx_0, fx_1, fx_1).$$

Therefore, for all n, m $\in$ N, n< m, we have by rectangle inequality that

$$G(y_n, y_m, y_m) \le G(y_n, y_{n+1}, y_{n+1}) + G(y_{n+1}, y_{n+2}, y_{n+2}) + G(y_{n+2}, y_{n+3}, y_{n+3}) + \text{---} + G(y_{m-1}, y_m, y_m) \le (q^n + q^{n+1} + \text{---------} + q^{m-1})\, G(y_0, y_1, y_1) \le \frac{q^n}{(1-q)} G(y_0, y_1, y_1).$$

Letting as n, m $\to \infty$ we have $\lim_{n,m\to\infty} G(y_n, y_m, y_m) = 0$. Thus $\{y_n\}$ is a G – Cauchy sequence in X. Since (X, G) is complete G – metric space, therefore, there exists a point z $\in$ X such that $\lim_{n\to\infty} y_n = z$ and $\lim_{n\to\infty} y_n = \lim_{n\to\infty} fx_n = \lim_{n\to\infty} gx_{n+1} = z$. Since the mapping f or g is continuous, for definiteness one can assume that g is continuous ,therefore $\lim_{n\to\infty} g fx_n = gz$. Further, f and g are compatible, therefore, $\lim_{n\to\infty}$

$G(gfx_n, fgx_n, fgx_n) = 0$, implies $\lim_{n\to\infty} fgx_n = gz$. From (2.3), we have $G(fgx_n, fx_n, fx_n) \leq qG(ggx_n, gx_n, gx_n)$.

Proceeding limit as $n \to \infty$, we have $gz = z$.

Again from (2.3) , we have $G(fx_n, fz, fz) \leq qG(gx_n, gz, gz)$.

Taking limit as $n \to \infty$, we have $z = fz$. Therefore, we have $gz = fz = z$. Thus z is a common fixed point of f and g.

Uniqueness . We assume that z_1 ($\neq z$) be another common fixed point of f and g .

Then $G(z, z_1, z_1) > 0$ and $G(z, z_1, z_1) = G(fz, f z_1, f z_1)) \leq qG(gz, g z_1, g z_1) = qG(z, z_1, z_1) < G(z, z_1, z_1)$, a contradiction ,therefore $z = z_1$. Hence uniqueness follows.

2.1 Example. Let $X = [-1, 1]$ and let G be the G – metric on $X \times X \times X$ defined as follows:

$G(x, y, z) = (|x-y| + |y-z| + |z-x|)$, for all x, y, z in X. Then (X, G) is a G – metric space. Define $f(x) = \frac{x}{6}$ and $g(x) = \frac{x}{2}$ .Here we note that, f is Continuous and $f(X) \subseteq g(X)$ .Also, $G(fx, fy, fz)) \leq q\, G(gx, gy, gz)$, holds for all $x, y, z \in X$, $\frac{1}{3} \leq q < 1$ and 0 is the unique common fixed point of f and g.

References

[1] B.C. Dhage, Generalized metric spaces and mappings with fixed point, Bull. Calcutta Math. Soc. 84 (1992), 329-336.

[2] G. Jungck, Commuting mappings and fixed point, Amer. Math. Monthly, 83(1976), 261-263.

[3] G.Jungck, Compatible mappings and common fixed points, Int. J. Math. Math. Sci 9(1986), 771-779.

[4] Mustafa and B.Sims, Some remarks concerning D-metric spaces, Proceedings of International Conference on Fixed Point Theory and applications, Yokohama Publishers, Valencia Spain, July 13-19(2004), 189-198.

[5] Z.Mustafa and B.Sims, A new approach to a generalized metric spaces, J. Nonlinear Convex Anal., 7(2006), 289-297.

[6] Z.Mustafa and H.Obiedat and F.Awawdeh, Some fixed point theorems for mappings on complete G-metric spaces, Fixed point theorey and applications Volume2008, Article ID18970,12 pages.

[7] Z. Mustafa,W. Shatanawi and M.Bataineh, Existence of fixed points results in G-metric spaces, International Journal of Mathematics and Mathematical Sciences,Volume 2009, Article ID. 283028, 10 pages.

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