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Vector Sequence Space S_2^{π}

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Abstract. In this paper we study the various proerties of the space S_2^{π}. The sequences are taken with terms in a commutative Banach algebra X with identity e and we have obtained the following results. S_2^{π} non separable Banach space which is solid and has monotone norm.

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Introduction

The paper [1] deals with the various proerties of the vector sequence space S_2. The paper [8] is identified some vectors sequence spaces with monotone norms. Matrix mappings of some rate spaces are given in [4] , [5] . The rate sequence space is also studied in [3] . In this paper we study the various proerties of the space S_2^{π}.

Notation

A Banach algebra is a complete normed algebra, *i.e.* an algebra which is a Banach space. Let X be a commutative Banach algebra with identity e. Then $ae = ea$, for each a, $a \in X$ and $\|e\| = 1$. Any sequence , whose k^{th} term is (x_k) , will be by (x) or $(x_k) \in X$. Then $x = (x_k)$ is a vector sequence. We recall that, in [8] , for any real valued sequence $\pi = (\pi_k)$ with

$$0 < \pi_k < \pi_{k+1} \text{ and } \sup_k \pi_k = \infty,$$

we have

$$(\ell^{\infty})_{\pi} = \left\{ x = (x_k) : |||x|||_{\pi} = \sup_k \left\| \frac{x_k}{\pi_k} \right\| < \infty \right\}.$$

Let X be a Banach algebra. Then X is said to have monotone norm if

$$\left|\left\|x^{(m)}\right\|\right| \geq \left|\left\|x^{(n)}\right\|\right| \text{ for } m > n \text{ and } \|x\| = \sup \left|\left\|x^{(m)}\right\|\right|.$$

$I.\ S_2^{\pi}\quad SPACE$

Definition 1.1: Let X be a commutative Banach algebra with identity e. S_2^{π} is the set of all sequences (x_k) with $x_k \in X$ and $\frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k} \leq N$ for each k, where $N > 0$ is a constant. Let $\|.\|$ be norm in X. The norm of the sequence is $|\|.\||$.

Definition 1.2: $(c_{02})^{\pi} = \left\{ (x_k) \in S_2^{\pi} : \frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k} \to 0 \text{ as } k \to \infty \right\}$.

$II.\ PROPERTIES\ OF\ S_2^{\pi}\ SPACE$

Proposition 2.1: $(\ell^{\infty})_{\pi} \subset S_2^{\pi}$.

Proof: Let $x \in (\ell^{\infty})_{\pi}$ .Then we have the following implications

$$\left\| \frac{x_k}{\pi_k} \right\| \leq N \text{ for each } k.$$

But

$$\frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k} \leq \left\| \frac{x_k}{\pi_k} \right\| \leq N \text{ for each } k.$$

This implies that

$$\frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k} < \infty$$

$$\Rightarrow \quad x \in S_2^{\pi}$$
$$\Rightarrow \quad (\ell^{\infty})_{\pi} \subset S_2^{\pi}.$$

This completes the proof.

Theorem 2.2: *S_2^{π} is a normed space.*

Proof: Put $|\|x\|| = \sup\limits_k \frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k}$. We have

$$\frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k} \geq 0 \Rightarrow |\|x\|| \geq 0 \tag{1}$$

$$\frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k} = 0 \Leftrightarrow |\|x\|| = 0$$

Hence

$$|\|x\|| = 0 \Leftrightarrow x = 0 \tag{2}$$

$$|\|\alpha x\|| = \sup_k \frac{\left\| \alpha \frac{x_k}{\pi_k} \right\|}{2^k} = |\alpha| \ \sup_k \frac{\left\| \frac{x_k}{\pi_k} \right\|}{2^k} = |\alpha| \ \ |\|x\|| \tag{3}$$

where α is a scalar.

(4) $$|||x+y||| = \sup_k \left\{ \frac{\left\| \frac{x_k}{\pi_k} + \frac{y_k}{\pi_k} \right\|}{2^k} \right\} \le \sup_k \frac{\left\| \frac{x_k}{\pi_k} \right\| + \left\| \frac{y_k}{\pi_k} \right\|}{2^k} = |||x||| + |||y|||$$

Thus $|||x+y||| \le |||x||| + |||y|||$.

From (1), (2), (3) and (4), $|||x|||$ is the norm of x.This completes the proof.

Theorem 2.3: *S_2^π is a Banach space.*

Proof: Let $\left\{x^{(n)}\right\}_{n=1}^{\infty}$ be a Cauchy sequence in S_2^π where $x^{(n)} = \left(x_1^{(n)}, x_2^{(n)}, ...\right)$ for each n. In other words,

$$x^{(1)} = \left(x_1^{(1)}, x_2^{(1)}, ...\right)$$

$$x^{(2)} = \left(x_1^{(2)}, x_2^{(2)}, ...\right)$$

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$$\left|\left|\left|x^{(n)} - x^{(m)}\right|\right|\right| \le \varepsilon \quad \forall\, n, m \ge n_0.$$

$$\frac{\left\| \frac{x_k^{(n)} - x_k^{(m)}}{\pi_k} \right\|}{2^k} < \varepsilon \qquad \forall\, n, m \ge n_0.$$

$$\left\| \frac{x_k^{(n)} - x_k^{(m)}}{\pi_k} \right\| \le \varepsilon\, 2^k \quad \forall\, n, m \ge n_0.$$

$\left\{x^{(n)}\right\}_{n=1}^{\infty}$ be a Cauchy sequence in X. But X is complete. Hence $x_k^{(n)} \to x_k$ as $n \to \infty$. Take $x = \left\{\frac{\frac{x_k}{\pi_k}}{2^k}\right\}$. Then $x \in S_2^\pi$ and $x^{(n)} \to x$ in S_2^π.

Theorem 2.4: *S_2^π has monotone norm.*

Proof:

Let $m > n$. Consider the sequence

$$x^{(n)} = (x_1, x_2, ..., x_n, 0, 0, ...)$$

But then

$$x^{(m)} = (x_1, x_2, ..., x_n, ..., x_m, 0, 0, ...)\,.$$

Hence

$$\left|\left|\left|x^{(n)}\right|\right|\right| = \sup\left\{ \frac{\left\|\frac{x_1}{\pi_1}\right\|}{2}, \frac{\left\|\frac{x_2}{\pi_2}\right\|}{2^2}, ..., \frac{\left\|\frac{x_n}{\pi_n}\right\|}{2^n}, 0, 0, ... \right\}$$

$$\left|\left\|x^{(m)}\right\|\right| = \sup\left\{\frac{\left\|\frac{x_1}{\pi_1}\right\|}{2}, \frac{\left\|\frac{x_2}{\pi_2}\right\|}{2^2}, ..., \frac{\left\|\frac{x_n}{\pi_n}\right\|}{2^n}, ..., \frac{\left\|\frac{x_m}{\pi_m}\right\|}{2^m}, 0, 0, ...\right\}$$

For $m > n$,we have $\left|\left\|x^{(m)}\right\|\right| \geq \left|\left\|x^{(n)}\right\|\right|$. Also,

$$\lim_{n\to\infty}\left|\left\|x^{(n)}\right\|\right| = \sup\left|\left\|x^{(n)}\right\|\right| = |\|x\||.$$

Theorem 2.5: S_2^π *is solid.*
Proof:
Let

$$\left\|\frac{u_k}{\pi_k}\right\| \leq \left\|\frac{x_k}{\pi_k}\right\| \text{ for each } k.$$

and let $x = (x_k) \in S_2^\pi$. Hence $\frac{\left\|\frac{u_k}{\pi_k}\right\|}{2^k} \leq \frac{\left\|\frac{x_k}{\pi_k}\right\|}{2^k}$ for each k. But $\left(\frac{x_k}{\pi_k}\right) \in S_2$, because $x \in S_2^\pi$. That is $\frac{\left\|\frac{u_k}{\pi_k}\right\|}{2^k} \leq N$. Therefore $u = (u_k) \in S_2^\pi$. This completes the proof.
Theorem 2.6: $(c_{02})^\pi$ *has AK property.*
Proof: Let $x = (x_k) \in (c_{02})^\pi$, but then $\left(\frac{x_k}{\pi_k}\right) \in c_{02}$, and hence

$$\sup_{k>n}\frac{\left\|\frac{x_k}{\pi_k}\right\|}{2^k} \to 0 \text{ as } k \to \infty. \tag{5}$$

$$\begin{aligned}\left|\left\|x - x^{(n)}\right\|\right| &= \sup_{k>n}\frac{\left\|\frac{x_k}{\pi_k}\right\|}{2^k} \to 0 \text{ as } k \to \infty, \text{ by using } (5)\\ &\Rightarrow x^{(n)} \to x \text{ as } n \to \infty,\end{aligned}$$

implying that $(c_{02})^\pi$ has AK. This completes the proof.
Theorem 2.7: S_2^π *is not separable.*
Proof: Let K be any dense subset of $(\ell^\infty)_\pi$. Let B be the set of all these sequences whose terms are 0 or 1. Then B is an uncountable subset of $(\ell^\infty)_\pi$. Define a surjection $f : B \to K$ by $f(x) = t_x$, $\forall x = (x_1, x_2, ...x_n, ...) \in B$ with

$$|\|x - t_x\|| < \frac{1}{2}.$$

Let $x,\ y \in B$ with $x \neq y$. But then $|\|x - y\|| = \sup_k\frac{\left\|\frac{x_k}{\pi_k} - \frac{y_k}{\pi_k}\right\|}{2^k} = 1$. Now

$$|\|x - y\|| \leq |\|x - t_x\|| + |\|t_x - y\||$$

and so

$$|\|y - t_x\|| \geq |\|x - y\|| - |\|x - t_x\|| \geq 1 - \frac{1}{2} = \frac{1}{2}.$$

But $f(y) = t_y$ with $|||y - t_y||| < \frac{1}{2}$. Hence $t_x \neq t_y$ or equivalently, $f(x) \neq f(y)$. Thus, f is a bijection. Since B is uncountable it follows that

$$f(B) = K$$

is uncountable. Consequently, S_2^π cannot be separable.

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