Int. Journal of Math. Analysis, Vol. 6, 2012, no. 29, 1431 - 1441

Maximal and Minimal Positive Solutions of a Nonlocal Boundary Value Problem of a Fractional Order Differential Equation

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Abstract

In this paper we study the existence of positive solution for the fractional order differential equation $D^{\beta}u(t) + f(t,\ u(t)) = 0,\ t \in\ (0,1)\ ,\ \beta\ \in\ (1,2)$, with the nonlocal conditions $I^{\gamma}\ u(t)|_{t=0}\ =\ 0,\ \gamma \in\ (0,1],\ u(1)\ =\ \sum_{j=1}^{p}\ b_j\ u(\eta_j),\ \eta_j \in (a,\ b) \subset\ (0,\ 1)$ where f is L^1-Caratheodory. The corresponding integral condition problem will be considered. The maximal and minimal solutions will be study.

Keywords: Fractional differential equation, positive solution, Green's function, maximal and minimal solutions

1 Introduction

Problems with non-local conditions have been extensively studied by several authors in the last two decades. The reader is referred to ([2]-[3]), ([6]-[13]) and references therein.
In [1], the author studied the existence of at least one positive solution for the three-point boundary-value problem

$$\begin{cases} D^{\beta}u(t)\ +\ f(t,\ u(t))\ = 0\ ,\ \beta\ \in\ (1,2),\ t\ \in\ (0,1), \\ u(0) = 0,\ u(1)\ =\ k\ u(\eta),\ 0\ <\ \eta\ < 1,\ 0\ <\ k\ \eta^{\beta-1}\ <\ 1, \end{cases}$$

(a) $f : [0,\ 1] \times [0,\ \infty)$ is nonnegative and continuous and either

(b) $0 \leq \overline{\lim}_{u \to +\infty} \max_{t \in [0,\ 1]} \dfrac{f(t,\ u)}{u} < (1 - k\eta^{\beta-1})\Gamma(\beta+1)$, and $f(t,\ 0) \not\equiv 0,\ t \in (0,\ 1)$

or

(c) $$\underline{\lim}_{u \to 0^+} \min_{t \in [0,\ 1]} \frac{f(t,\ u)}{u} > \lambda_1,\ \overline{\lim}_{u \to +\infty} \max_{t \in [0,\ 1]} \frac{f(t,\ u)}{u} < \lambda_1.$$

In this work we omit the conditions (b) and (c), relax condition (a) and study, when f is L^1-Carathèodory, the existence of at least one positive solution for the nonlocal boundary value problem of fractional-order differential equation

$$D^{\beta} u(t) + f(t,\ u(t)) = 0,\ t \in (0, 1),\ \beta \in (1, 2), \tag{1}$$

with the nonlocal conditions

$$I^{\gamma}\ u(t)|_{t=0} = 0,\ \gamma \in (0, 1],\ u(1) = \sum_{j=1}^{p} b_j\ u(\eta_j),\ \eta_j \in (a,\ b) \subset (0,\ 1). \tag{2}$$

As an application, we deduce the existence of solution for the nonlocal problem of the differential equation (1) with the nonlocal integral condition

$$I^{\gamma}\ u(t)|_{t=0} = 0,\ \gamma \in (0, 1],\ u(1) = \int_a^b u(s)\ ds. \tag{3}$$

The maximal and minimal solutions will be study.

2 Preliminary Notes

Let $C(I)$ denotes the class of continuous functions and $L^1(I)$ denotes the class of Lebesgue integrable functions on the interval $I = [0, T]$. Let $\Gamma(.)$ denotes the gamma function.

Definition 1.1 The fractional-order integral of the function $f \in L^1[0, T]$ of order $\beta > 0$ is defined by (see [17])

$$I_a^{\beta}\ f(t) = \int_a^t \frac{(t - s)^{\beta - 1}}{\Gamma(\beta)}\ f(s)\ ds,$$

Definition 1.2 The Riemann-Liouville fractional-order derivative of f of order $\beta \in (0, 1)$ is defined as (see [16] and [17])

$$D_a^{\beta}\ f(t) = \frac{d}{dt} \int_a^t \frac{(t - s)^{-\beta}}{\Gamma(1 - \beta)}\ f(s)\ ds.$$

Definition 1.3 The function $f : [0, T] \times R \to R$ is called L^1-Caratheodory if
(i) $t \to f(t, x)$ is measurable for each $x \in R$,
(ii) $x \to f(t, x)$ is continuous for almost all $t \in [0, T]$,
(iii) there exists $m \in L^1[0, T]$ such that $|f| \leq m$.

3 Existence of solution

Lemma 2.1 If $0 < \sum_{j=1}^{p} b_j \eta_j^{\beta-1} < 1$, then the solution of the problem (1)-(2) can be represent by the integral equation

$$u(t) = \frac{A\ t^{\beta-1}}{\Gamma(\beta)}\{\int_0^1 (1-s)^{\beta-1} f(s,u(s))\, ds + A \sum_{j=1}^{p} b_j\ \eta_j^{\beta-1} \int_0^1 (1-s)^{\beta-1} f(s,u(s))\, ds$$

$$-A \sum_{j=1}^{p} b_j \int_0^{\eta_j} (\eta_j - s)^{\beta-1} f(s,u(s))\, ds\} - \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds. \quad (4)$$

where $A = (1 - \sum_{j=1}^{p} b_j\ \eta_j^{\beta-1})^{-1}$.
proof. Write equation (1) in the form

$$\frac{d^2}{dt^2} I^{2-\beta} u(t) = -\ f(t,\ u(t))$$

Integrating both sides of equation (1), we obtain

$$I^{2-\beta} u(t) = C_2 + tC_1 - I^2 f(t,\ u(t)).$$

Operating by I^β, we get

$$I^2 u(t) = C_2 \frac{t^\beta}{\Gamma(\beta+1)} + C_1 \frac{t^{\beta+1}}{\Gamma(\beta+2)} - I^{2+\beta} f(t,\ u(t)).$$

Differentiating the last relation twice, we obtain

$$u(t) = C_1 \frac{t^{\beta-1}}{\Gamma(\beta)} + C_2 \frac{t^{\beta-2}}{\Gamma(\beta-1)} - I^\beta f(t,\ u(t))$$

Operating on both sides of the above equation by I^γ, we obtain

$$I^\gamma u(t) = C_1 \frac{t^{\gamma+\beta-1}}{\Gamma(\gamma+\beta)} + C_2 \frac{t^{\gamma+\beta-2}}{\Gamma(\gamma+\beta-1)} - I^{\gamma+\beta} f(t,\ u(t))$$

from which we deduce that $C_2 = 0$ and

$$u(t) = C_1 \frac{t^{\beta-1}}{\Gamma(\beta)} - I^\beta f(t,\ u(t)).$$

Also from the relation $u\ (\ 1\ ) = \sum_{j=1}^{p} b_j\ u(\eta_j)$, we have

$$C_1 \frac{1}{\Gamma(\beta)} - \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds = C_1 \sum_{j=1}^{p} b_j \frac{\eta_j^{\beta-1}}{\Gamma(\beta)} - \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds$$

$$C_1\left(\frac{1-\sum_{j=1}^{p}\ b_j\eta_j^{\beta-1}}{\Gamma(\beta)}\right)=\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(s))ds-\sum_{j=1}^{p}b_j\int_0^{\eta_j}\frac{(\eta_j-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(s))\,ds$$

then

$$C_1=(\frac{\Gamma(\beta)}{1-\sum_{j=1}^{p}\ b_j\eta_j^{\beta-1}})\{\int_0^1\frac{(1-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(s))\,ds-\sum_{j=1}^{p}\ b_j\int_0^{\eta_j}\frac{(\eta_j-s)^{\beta-1}}{\Gamma(\beta)}f(s,u(s))\,ds\}$$

and

$$u(t)=\frac{t^{\beta-1}}{\Gamma(\beta)}\{\int_0^1(1-s)^{\beta-1}\,f(s,u(s))\,ds+A\sum_{j=1}^{p}\,b_j\eta_j^{\beta-1}\int_0^1(1-s)^{\beta-1}\,f(s,u(s))\,ds$$

$$-\ A\sum_{j=1}^{p}\ b_j\ \int_0^{\eta_j}(\eta_j-s)^{\beta-1}\ f(s,u(s))\ ds\}\ -\ \int_0^t\frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f(s,u(s))\ ds.$$

Now we can write equation (4) in the formula

$$u(t)=\int_0^1 G(t,\ s)\ f(t,\ u(s))\ ds,\qquad G(t,\ s)=G_1(t,\ s)+G_2(t,\ s)\qquad(5)$$

where

$$G_1(t,\ s)=\begin{cases}\frac{t^{\beta-1}(1-s)^{\beta-1}-(t-s)^{\beta-1}}{\Gamma(\beta)},\ 0\le\ s\le\ t\le\ 1,\\ \\ \frac{t^{\beta-1}(1-s)^{\beta-1}}{\Gamma(\beta)},\ 0\le\ t\le\ s\le\ 1\end{cases}$$

$$G_2(t,\ s)=\begin{cases}\frac{A}{\Gamma(\beta)}\left[\sum_{0\le\ s\le\ \eta_j}(b_j\eta_j^{\beta-2}t^{\beta-1}(1-s)^{\beta-1}-b_j\ t^{\beta-1}(\eta_j-s)^{\beta-1})\right],\ t\in\ [0,\ 1],\\ \\ \frac{A}{\Gamma(\beta)}\left[\sum_{\eta_j\le\ s\le\ 1}b_j\eta_j^{\beta-1}t^{\beta-1}(1-s)^{\beta-1}\right],\ t\in\ [0,\ 1],\end{cases}$$

Lemma 2.2 The function $G(t,\ s)$ satisfies $G(t,\ s)>0$, for $t,\ s\in\ (0,\ 1)$.
Proof. For $0\le\ s\le\ t\le\ 1$, we have

$$t^{\beta-1}(1-s)^{\beta-1}\ =\ (t-t\ s)^{\beta-1}>(t-s)^{\beta-1}\qquad(6)$$

Thus, $G_1(t,\ s)\ >\ 0$ for $t,\ s\in\ (0,\ 1)$. Furthermore, we have

$$b_j\eta_j^{\beta-1}t^{\beta-1}(1-s)^{\beta-1}=b_j\ t^{\beta-1}(\eta_j-\eta_j\ s)^{\beta-2}>b_j\ t^{\beta-1}(\eta_j-s)^{\beta-2}\qquad(7)$$

So, $G_2(t,\ s)\ge\ 0$ for $t,\ s\in\ [0,\ 1]$.
Then we get that $G(t,\ s)>0$ for $t,\ s\in\ (0,\ 1)$.

Definition 2.1 The function u is called a solution of the fractional-order functional integral equation (4), if $u\in\ C[0,1]$ and satisfies (4).

For the existence of the solution we have the following theorem

Theorem 2.1 Assume that the the function f is L^1-Carathèodory. Then the nonlocal boundary value problem (1)-(2) has at least one positive continuous solution $u \in C[0,1]$.

Proof. Define a subset $Q_r^+ \subset C[0,1]$ by

$Q_r^+ = \{u(t) > 0, \text{ for each } t \in [0,1], \ \|u\| \leq r\}$, where $r = \frac{(1 + A + A \sum_{j=1}^p b_j)||m||_{L^1}}{\Gamma(\beta)}$.

The set Q_r^+ is nonempty, closed and convex.

Let $T : Q_r^+ \to Q_r^+$ be an operator defined by

$$Tu(t) = \frac{A\, t^{\beta-1}}{\Gamma(\beta)} \left\{ \int_0^1 (1-s)^{\beta-1} f(s,u(s))\, ds - \sum_{j=1}^p b_j \int_0^{\eta_j} (\eta_j - s)^{\beta-1} f(s,u(s))\, ds \right\}$$
$$- \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds.$$

For $u \in Q_r^+$, it is clear that T is continuous operator .

Now, let $u \in Q_r^+$, then

$$\begin{aligned}
(Tu)(t) &\leq A\, t^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\
&+ A\, t^{\beta-1} \sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds + \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\
&\leq A \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\
&+ A \sum_{j=1}^p b_j \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds + \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\
&\leq (1 + A + A \sum_{j=1}^p b_j) \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds \\
&\leq \frac{(1 + A + A \sum_{j=1}^p b_j)}{\Gamma(\beta)} \int_0^1 (1-s)^{\beta-1}\, m(s)\, ds \\
&\leq \frac{(1 + A + A \sum_{j=1}^p b_j)}{\Gamma(\beta)} \int_0^1 m(s)\, ds \ \leq \ \frac{(1 + A + A \sum_{j=1}^p b_j)||m||_{L^1}}{\Gamma(\beta)} = r
\end{aligned}$$

Then $\{Tu(t)\}$ is uniformly bounded in Q_r^+.

In what follows we show that T is a completely continuous operator.

For $t_1, t_2 \in (0,1)$, $t_1 < t_2$ such that $|t_2 - t_1| < \delta$ we have

$$|Tu(t_2) - Tu(t_1)| = |A\, t_2^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\, ds$$
$$- \ A t_2^{\beta-1} \sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)) ds - \int_0^{t_2} \frac{(t_2 - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)) ds$$

$$
\begin{aligned}
- &\ A t_1^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds + A t_1^{\beta-1} \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j - s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds \\
+ &\ \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\ ds| \\
\leq &\ | \int_0^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)))\ ds - \int_0^{t_1} \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)))\ ds\ | \\
+ &\ A\ |t_2^{\beta-1}\ -\ t_1^{\beta}| \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} |f(s,u(s))|\ ds \\
+ &\ A\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds \\
\leq &\ | \int_0^{t_1} \left(\frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)}\ -\ \frac{(t_1-s)^{\beta-1}}{\Gamma(\beta)} \right) f(s,u(s)))\ ds \\
+ &\ \int_{t_1}^{t_2} \frac{(t_2-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s)))\ ds\ | \\
+ &\ A\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} |f(s,u(s))|\ ds \\
+ &\ A\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \sum_{j=1}^{p} b_j \int_0^{\eta_j} \frac{(\eta_j-s)^{\beta-1}}{\Gamma(\beta)} |f(s,u(s))| ds \\
\leq &\ \frac{1}{\Gamma(\beta)} \int_0^{t_1} \left((t_2-s)^{\beta-1}\ -\ (t_1-s)^{\beta-1} \right) m(s)\ ds \\
+ &\ \frac{1}{\Gamma(\beta)} \int_{t_1}^{t_2} (t_2-s)^{\beta-1} m(s)\ ds\ +\ \frac{A}{\Gamma(\beta)}\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \int_0^1 (1-s)^{\beta-1} m(s)\ ds \\
+ &\ \frac{A}{\Gamma(\beta)}\ |t_2^{\beta-1}\ -\ t_1^{\beta-1}| \sum_{j=1}^{p} b_j \int_0^{\eta_j} (\eta_j-s)^{\beta-1} m(s) ds.
\end{aligned}
$$

Hence the class of functions $\{Tu(t)\}$ is equi-continuous. By Arzela-Ascolis Theorem $\{Tu(t)\}$ is relatively compact. Since all conditions of Schauder Theorem are hold, then T has a fixed point in Q_r^+. Therefor the integral equation (4) has at least one positive continuous solution $u \in C(0,1)$.

Now,

$$
\lim_{t\to 0} u(t) = A \lim_{t\to 0} t^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds
$$

$$
-A \lim_{t\to 0} t^{\beta-1} \sum_{j=1}^{p} b_j \int_0^{\eta_j} (\eta_j - s)^{\beta-1} f(s,u(s))\ ds - \lim_{t\to 0} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))\ ds = u(0) = 0,
$$

$$
\lim_{t\to 1} u(t) = A \lim_{t\to 1} t^{\beta-1} \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta)} f(s,u(s))ds
$$

$$
-A \lim_{t\to 1} t^{\beta-1} \sum_{j=1}^{p} b_j \int_0^{\eta_j} (\eta_j - s)^{\beta-1}\ f(s,u(s))\ ds - \lim_{t\to 1} \int_0^t \frac{(t-s)^{\beta-1}}{\Gamma(\beta)}\ f(s,u(s))\ ds = u(1).
$$

Then the integral equation (4) has at least one positive continuous solution $u \in C[0,1]$.
To complete the proof operating on both sides of equation (4) by $I^{2-\beta}$, we get

$$I^{2-\beta}u(t) = \frac{A\ t}{\Gamma(\beta-1)}\{\int_0^1 (1-s)^{\beta-1} f(s,u(s))\ ds - k\int_0^\eta (\eta-s)^{\beta-1} f(s,u(s))\ ds\} - I^2 f(t,u(t))$$

Differentiating the above relation twice we obtain the differential equation (1).
Also from (4) we have $I^\gamma\ u(t)|_{t=0} = 0,\ \gamma \in (0,1]$ and

$$\sum_{j=1}^p b_j u(\eta_j) = \frac{A\ \sum_{j=1}^p b_j \eta_j^{\beta-2}}{\Gamma(\beta-1)}\{\int_0^1 (1-s)^{\beta-1} f(s,u(s))\ ds - \sum_{j=1}^p b_j \int_0^{\eta_j} (\eta_j-s)^{\beta-1} f(s,u(s)) ds\}$$

$$\begin{aligned}
&- \sum_{j=1}^p b_j \int_0^{\eta_j} \frac{(\eta_j-s)^{\beta-1}}{\Gamma(\beta-1)}\ f(s,u(s))\ ds \\
&= \frac{1}{\Gamma(\beta-1)}\{A(\sum_{j=1}^p b_j \eta_j^{\beta-1} - 1 + \sum_{j=1}^p b_j \eta_j^{\beta-1}) \sum_{j=1}^p b_j \int_0^{\eta_j} (\eta_j-s)^{\beta-1}\ f(s,u(s))\ ds \\
&+ (\frac{\sum_{j=1}^p b_j \eta_j^{\beta-1}}{1 - \sum_{j=1}^p b_j \eta_j^{\beta-1}}) \int_0^1 (1-s)^{\beta-1}\ f(s,u(s))\ ds\} \\
&= \frac{1}{\Gamma(\beta-1)}\{-\ A \sum_{j=1}^p b_j \int_0^{\eta_j} (\eta_j-s)^{\beta-1}\ f(s,u(s))\ ds \\
&+ (-\ 1 + \frac{1}{1 - \sum_{j=1}^p b_j \eta_j^{\beta-1}}) \int_0^1 (1-s)^{\beta-1}\ f(s,u(s))\ ds\} \\
&= \frac{A}{\Gamma(\beta-1)} \left\{ \int_0^1 (1-s)^{\beta-1}\ f(s,u(s))\ ds - \sum_{j=1}^p b_j \int_0^{\eta_j} (\eta_j-s)^{\beta-1}\ f(s,u(s))\ ds \right\} \\
&- \int_0^1 \frac{(1-s)^{\beta-1}}{\Gamma(\beta-1)}\ f(s,u(s))\ ds = u(1)
\end{aligned}$$

The proof is complete.■

4 Nonlocal integral condition

Let $u \in C[0,1]$ be the solution of the nonlocal problem (1)-(2). Let $b_j = t_j - t_{j-1}$, $\eta_j \in (t_{j-1}, t_j)$, $a = t_0 < t_1 < t_2, ... < t_p = b$ then the nonlocal condition (2) will be

$$u(1) = \sum_{j=1}^p (t_j - t_{j-1})\ u(\eta_j).$$

From the continuity of the solution u of the nonlocal problem (1)-(2) we can obtain

$$u(1) \;=\; \lim_{p\to\infty}\sum_{j=1}^{p}\,(t_j - t_{j-1})\;u(\eta_j).$$

and the nonlocal condition (2) transformed to the integral one

$$I^{\gamma}\;u(t)|_{t=0} \;=\; 0,\;\gamma\in\;(0,1],\;u(1)\;=\;\int_a^b\;u(s)\;ds. \tag{8}$$

Now, we have the following Theorem

Theorem 3.1 Let the assumptions of Theorem 2.1 are satisfied. Then there exist at least one solution $u \in AC[0,1]$ of the nonlocal problem with integral condition,

$$D^{\beta}u(t)\;+\;f(t,\;u(t)) = 0,\;t\in\;(0,1)\;,\;\beta\;\in\;(1,2),$$

$$I^{\gamma}\;u(t)|_{t=0}\;=\;0,\;\gamma\in\;(0,1],\;u(1)\;=\;\int_a^b\;u(s)\;ds.$$

5 Maximal and minimal solutions

Here we study the existence of the maximal and minimal solutions of the fractional-order integral equation (4).

Definition 4.1 Let n be a solution of the integral equation (4) in $[0,1]$, then n is said to be a maximal solution of (4) if, for every solution u of (4) existing on $[0,1]$, the inequality $u(t) \le n(t), t\in[0,1]$, holds.

A minimal solution may be define similarly by reversing the last inequality.

From Theorem 2.1 we get that the integral equation (4) has a positive solution $u \in C[0,1]$. Based on this criterion we can prove the following theorem.

Theorem 4.1 let f be a monotonic nondecreasing function in u. If the assumptions of Theorem 2.1 are satisfied , then there exist maximal and minimal solutions of the integral equation (4) on $[0,1]$.

Proof Consider the fractional-order integral equation

$$u_{\epsilon}(t)\;=\;\epsilon\;+\;\int_0^1\;G(t,s)\;f(s,u(s))\;ds,\quad\epsilon\;>\;0. \tag{9}$$

In the view of Theorem 2.1, it is clear that equation (9) has at least one positive solution $u(t) \in C[0,1]$. Now, let ϵ_1 and ϵ_2 be such that $0<\epsilon_2<\epsilon_1\le\epsilon$. Then, we have $u_{\epsilon_2}(0) \;<\; u_{\epsilon_1}(0)$ (from (3)-(5), we have $u_{\epsilon_2}(0) \;=\; \epsilon_2 \;<\; \epsilon_1 \;=\; u_{\epsilon_1}(0)$). We can prove

$$u_{\epsilon_2}(t)\;<\;u_{\epsilon_1}(t)\;\;\text{for all}\;\;t\;\in\;[0,1]. \tag{10}$$

To prove conclusion (10), we assume that it is false, then there exist a t_1 such that

$$u_{\epsilon_2}(t_1)\;=\;u_{\epsilon_1}(t_1)\;\;\text{and}\;\;u_{\epsilon_2}(t)\;<\;u_{\epsilon_1}(t)\;\;\text{for all}\;\;t\;\in\;[0,t_1).$$

Since f is monotonic nondecreasing in u, it follows that $f(t,u_{\epsilon_2}(t)) \leq f(t,u_{\epsilon_1}(t))$ and consequently, using equation (9), we obtain

$$\begin{aligned} u_{\epsilon_2}(t_1) &= \epsilon_2 + \int_0^1 G(t_1,s)\ f(s,u_{\epsilon_2}(s))\ ds \\ &< \epsilon_1 + \int_0^1 G(t_1,s)\ f(s,u_{\epsilon_1}(s))\ ds = u_{\epsilon_1}(t_1). \end{aligned}$$

Which contradict the fact that $u_{\epsilon_2}(t_1) = u_{\epsilon_1}(t_1)$. Hence the inequality (10) is true.

From the hypothesis, it follows as in the proof of Theorem 2.1 that the family of functions $\{u_\epsilon\}$ is relatively compact on $[0,1]$, hence, we can extract a uniformly convergent subsequence $\{u_{\epsilon p}\}$, that is, there exists a decreasing sequence $\{\epsilon_p\}$ such that $\epsilon_p \to 0$ as $p \to \infty$ and $\lim_{p\to\infty} u_{\epsilon p}(t)$ exists uniformly in $t \in [0,1]$, we denote this limiting value by $n(t)$.

Obviously, the uniform continuity of f and the equation

$$u_{\epsilon_p}(t) = \epsilon_p + \int_0^1 G(t,s)\ f(s,u_{\epsilon_p}(s))\ ds,\ t \in [0,1],$$

yields n is a solution of equation (4). Finally, we show that the solution n is the maximal solution of equation (4). To achieve this goal, let u be any solution of (4) existing on the interval $[0,1]$. Then

$$u(t) < \epsilon + \int_0^1 G(t,s)\ f(s,u(s))\ ds = u_\epsilon(t),\ t \in [0,1].$$

Since the maximal solution is unique (see [14]), it is clear that $u_\epsilon(t)$ tends to $n(t)$ uniformly in $t \in [0,1]$ as $\epsilon \to 0$. Which proves the existence of maximal solution to the integral equation (4). A similar argument holds for the minimal solution. ■

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Received: January, 2012