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On the Stability of Generalized Cauchy Linear Functional Equation

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Abstract

We investigate the following generalized Cauchy linear functional equation $f(x + y + z + a) = f(x) + f(y) + f(z)$, where a is an arbitrary number and prove the Hyers–Ulam–Rassias stability of the functional equations on Banach spaces.

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1. Introduction

The stability of functional equations originated from the following fundamental question: "when is it true that a function which approximately satisfies a functional equation ε must be close to an exact solution of ε ?" If the problem accepts a solution, we say that the equation ε is stable.

The stability problem for the functional equations was first raised by S. M. Ulam [5] in 1940. He discussed the number of unsolved problems related to the stability of functional equations. In the next year D. H. Hyer [1] gave the first affirmative answer of the Ulam's problem for additive mapping $f(x+y) = f(x) + f(y)$ on Banach spaces. A generalized version of the theorem of

Hyer [1] was given by Th. M. Rassias [6] in 1978 which allows Cauchy difference to be unbounded. The generalization given by Th. M. Rassias [6] is called the Hyers-Ulam-Rassias stability.

In1994, P. Gavruta [4] provided a further generalization of Th. M. Rassias [6] theorem in which he replaced the bound $\varepsilon(\|x\|^p + \|y\|^p)$ by a general function $\phi(x, y)$ for the existence of unique linear mapping. The Hyers-Ulam-Rassias stability of various functional equations have been extensively introduced by a number of Mathematicians. In 1999, K. W. Jun, D. S. Shin and B. D. Kim [3] proved the stability of functional equation $f(x+y) - g(x) - h(y) = 0$, which is called pexider functional equation. Later on, in 2000 this result was generalized by Y. H. Lee and K.W. Jun [7]. The functional equation

$$f(x+y+z) = f(x) + f(y) + f(z) \quad \text{for all } x, y, z \in X \tag{1.1}$$

is called the Cauchy functional equation in 3-variable. Since f is a solution of it is said to be additive or satisfies the Cauchy functional equation. The Hyers-Ulam-Rassias stability of this equation was introduced by J. R. Lee and C. Park [2] in 2009 on Banach Algebra.The functional equation

$$f(x+y+z+a) = f(x) + f(y) + f(z) \tag{1.2}$$

for all $x, y, z \in X$ and a is an arbitrary number is called the generalized Cauchy linear functional equation. A particular case of the linear functional equation (1.2) is $f(x+y+z) = f(x) + f(y) + f(z)$ at $a = 0$. In the next sections, we prove the stability problem in the sense of D. H. Hyer [1], Th. M. Rassias [6] and P. Gavruta [4] for the Generalized Cauchy functional equation (1.2) and also present some corollaries related to these results.

2. Hyers-Ulam Stability of the Cauchy functional equation (1.2)

In this section, we prove the stability of Cauchy functional equation (1.2) in the sense of D.H. Hyer [1].

Theorem 2.1: Let $f : X \to Y$ where X is an abelian group and Y be a Banach space, be a mapping. If f satisfies the functional inequality

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \leq \delta \quad \text{for all } x, y, z \in X \tag{2.1}$$

for some $\delta > 0$, then the limit

$$C(x) = \lim_{n\to\infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right) \tag{2.2}$$

exists for all $x \in X$ and there exists a unique mapping $C : X \to Y$ such that

$$C(x + y + z + a) = C(x) + C(y) + C(z) \tag{2.3}$$

and $$\|f(x) - C(x)\| \le \frac{\delta}{2} \qquad \text{for all } x, y, z \in X \tag{2.4}$$

Proof: Let us consider $x = y = z$ in (2.1), we get

$$\|f(3x + a) - 3f(x)\| \le \delta \tag{2.5}$$

replacing x with $3x+a$ in (2.5), such that

$$\|f(9x + 4a) - 3f(3x + a)\| \le \delta \tag{2.6}$$

again substituting x with $3x+a$ in (2.6), we get

$$\|f(27x + 13a) - 3f(9x + 4a)\| \le \delta \tag{2.7}$$

Now, taking induction for some positive integer n, we have

$$\left\| f(x) - \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right)\right\| \le \frac{\delta}{2}\left(1 - \frac{1}{3^n}\right) \tag{2.8}$$

The induction is true, since for n = 1, 2, 3 in (2.8) implies the inequality (2.5), (2.6) and (2.7). Now, we prove that it is true for n+1, using (2.6) and (2.8), we have

$$\left\| f(x) - \frac{1}{3^{n+1}} f\left(3^{n+1} x + \frac{(3^{n+1} - 1)a}{2}\right)\right\|$$

$$\le \frac{1}{3^{n+1}}\left\| f\left(3^{n+1} x + \frac{(3^{n+1} - 1)a}{2}\right) - 3f\left(3^n x + \frac{(3^n - 1)a}{2}\right)\right\|$$

$$+ \frac{1}{3^n}\left\| f\left(3^n x + \frac{(3^n - 1)a}{2}\right) - 3^n f(x)\right\| \le \frac{\delta}{3^{n+1}} + \frac{\delta}{2}\left(1 - \frac{1}{3^n}\right) = \frac{\delta}{2}\left(1 - \frac{1}{3^n}\right) \tag{2.9}$$

Now, we prove that the sequence $C_n(x) = \left\{\frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right)\right\}$ is a Cauchy sequence. So, for all positive integer n, such that

$$\left\|C_{n+1}(x)-C_n(x)\right\| \le \left\|\frac{1}{3^{n+1}} f\left(3^{n+1}x+\frac{(3^{n+1}-1)a}{2}\right)-\frac{1}{3^n} f\left(3^n x+\frac{(3^n-1)a}{2}\right)\right\|$$

$$\le \frac{1}{3^{n+1}}\left\|3f\left(3^n x+\frac{(3^n-1)a}{2}\right)-f\left(3^{n+1}x+\frac{(3^{n+1}-1)a}{2}\right)\right\| \le \frac{\delta}{3^{n+1}}$$

since 1/3<1, Y is a complete normed space and the limit of sequence $\{C_n(x)\}$ exists and is in Y, therefore the sequence $\{C_n(x)\}$ is a Cauchy sequence. Define a mapping $C: X \to Y$ by

$$C(x)=\lim_{n\to\infty}\frac{1}{3^n} f\left(3^n x+\frac{(3^n-1)a}{2}\right) \tag{2.10}$$

Therefore, using (2.10) inequality (2.9) implies that $\left\|f(x)-C(x)\right\| \le \delta/2$.

for all x, y, $z \in X$ and for all positive integer n, we have

$$\left\|C_n(x+y+z+a)-C_n(x)-C_n(y)-C_n(z)\right\|$$

$$=\left\|\frac{1}{3^n} f\left(3^n x+3^n y+3^n z+\frac{(3^{n+1}-2)a}{2}\right)-\frac{1}{3^n} f\left(3^n x+\frac{(3^n-1)a}{2}\right)-\frac{1}{3^n} f\left(3^n y+\frac{(3^n-1)a}{2}\right)-\frac{1}{3^n} f\left(3^n z+\frac{(3^n-1)a}{2}\right)\right\|$$

$$\le \frac{\delta}{3^n}$$

Taking limit $n\to\infty$ it shows that $C(x+y+z+a)=C(x)+C(y)+C(z)$ for all $x,y,z \in X$. Now, to prove the uniqueness of mapping $C: X \to Y$, let us consider that there exists another mapping $C^1: X \to Y$ which is the solution of inequality (2.3) and satisfies the inequality (2.4). Then, we get

$$\left\|C(x)-C^1(x)\right\| = \left\|\frac{1}{3^n} C\left(3^n x+\frac{(3^n-1)a}{2}\right)-\frac{1}{3^n} C^1\left(3^n x+\frac{(3^n-1)a}{2}\right)\right\|$$

$$\le \left\|\frac{1}{3^n} C\left(3^n x+\frac{(3^n-1)a}{2}\right)-\frac{1}{3^n} f\left(3^n x+\frac{(3^n-1)a}{2}\right)\right\|$$

$$+\left\|\frac{1}{3^n} f\left(3^n x+\frac{(3^n-1)a}{2}\right)-\frac{1}{3^n} C^1\left(3^n x+\frac{(3^n-1)a}{2}\right)\right\|$$

$$\leq \frac{\delta}{2.3^n} + \frac{\delta}{2.3^n} = \frac{\delta}{3^n}$$

Taking limit $n\to\infty$ we get $\left\| C(x) - C^1(x) \right\| = 0$ for all $x \in X$, which implies that $C(x) = C^1(x)$ for all $x \in G$. This completes the proof of theorem.

3. Th. M. Rassias Stability of the Cauchy functional equation (1.2)

In 1978, Th. M. Rassias [6] introduced the stability of Cauchy functional equation $f(x+y) = f(x) + f(y)$ on Banach spaces for unbounded Cauchy difference. In this section, using the Th. M. Rassias [6] approach we prove the stability of (1.2).

Theorem 3.1: Let $f : X \to Y$ where X be a normed space and Y be a Banach space, be a mapping. If f satisfies the functional inequality

$$\left\| f(x+y+z+a) - f(x) - f(y) - f(z) \right\| \leq \theta(\left\|x\right\|^p + \left\|y\right\|^p + \left\|z\right\|^p) \tag{3.1}$$

for some $\theta>0$, $0\leq p<1$ and for all $x, y, z \in X$, then the limit

$$C(x) = \lim_{n\to\infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n - 1)a}{2}\right) \tag{3.2}$$

exists for all $x \in X$ and there exists a unique mapping $C : X \to Y$ such that

$$C(x+y+z+a) = C(x) + C(y) + C(z) \tag{3.3}$$

and $$\left\| f(x) - C(x) \right\| \leq \theta \sum_{i=0}^{+\infty} 3^{i(p-1)} \left\| x + \frac{1}{2}\left(1 - \frac{1}{3^i}\right) \right\|^p \quad \text{for all } x, y, z \in X \tag{3.4}$$

Proof: Let $x = y = z$ in (3.1), we get

$$\left\| f(3x+a) - 3f(x) \right\| \leq 3\theta \left\|x\right\|^p \tag{3.5}$$

replacing x with $3x+a$ in (3.5), such that

$$\left\| f(9x+4a) - 3f(3x+a) \right\| \le 3\theta \left\| 3x+a \right\|^p \tag{3.6}$$

again substituting x with $3x+a$ in (3.6), we get

$$\left\| f(27x+13a) - 3f(9x+4a) \right\| \le 3\theta \left\| 9x+4a \right\|^p \tag{3.7}$$

Now, taking induction for some positive integer n, we have

$$\left\| f(x) - \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2} \right) \right\| \le \theta \sum_{i=0}^{n-1} \frac{1}{3^i} \left\| 3^i x + \left(\frac{(3^i-1)a}{2} \right) \right\|^p \tag{3.8}$$

The induction is true, since for n = 1, 2, 3 in (3.8) implies the inequality (3.5), (3.6) and (3.7). Now, we prove that it is true for n+1, using (3.6) and (3.8), we have

$$\left\| f(x) - \frac{1}{3^{n+1}} f\left(3^{n+1} x + \frac{(3^{n+1}-1)a}{2} \right) \right\|$$
$$\le \frac{1}{3^{n+1}} \left\| f\left(3^{n+1} x + \frac{(3^{n+1}-1)a}{2} \right) - 3f\left(3^n x + \frac{(3^n-1)a}{2} \right) \right\|$$
$$+ \frac{1}{3^n} \left\| f\left(3^n x + \frac{(3^n-1)a}{2} \right) - 3^n f(x) \right\| \le \frac{\theta}{3^n} \left\| 3^n x + \frac{(3^n-1)a}{2} \right\|^p + \theta \sum_{i=0}^{n-1} \frac{1}{3^i} \left\| 3^i x + \frac{(3^i-1)a}{2} \right\|^p$$
$$= \theta \sum_{i=0}^{n} \frac{1}{3^i} \left\| 3^i x + \frac{(3^i-1)a}{2} \right\|^p$$

This shows that the above inequality (3.8) is true for all $x \in X$ and for any positive integer n. Now, we prove that the sequence $C_n(x) = \left\{ \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2} \right) \right\}$ is a Cauchy sequence. So, for all positive integer n, we have

$$\left\| C_{n+1}(x) - C_n(x) \right\| \le \left\| \frac{1}{3^{n+1}} f\left(3^{n+1} x + \frac{(3^{n+1}-1)a}{2} \right) - \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2} \right) \right\|$$
$$\le \frac{1}{3^{n+1}} \left\| 3f\left(3^n x + \frac{(3^n-1)a}{2} \right) - f\left(3^{n+1} x + \frac{(3^{n+1}-1)a}{2} \right) \right\|$$

$$\leq \frac{\theta}{3^n}\left\|3^n x+\frac{(3^n-1)a}{2}\right\|^p = \theta 3^{n(p-1)}\left\|x+\frac{1}{2}\left(1-\frac{1}{3^n}\right)a\right\|^p$$

since $p<1$, $1/3<1$, Y is a complete normed space and the limit of sequence $\{C_n(x)\}$ exists and is in Y, therefore the sequence $\{C_n(x)\}$ is a Cauchy sequence for each $x\in X$. Define a mapping $C : X \to Y$ by

$$C(x)=\lim_{n\to\infty}\frac{1}{3^n}f\left(3^n x+\frac{(3^n-1)a}{2}\right) \tag{3.9}$$

therefore, using (3.9) inequality (3.8) implies that

$$\left\|f(x)-C(x)\right\|\leq\theta\sum_{i=0}^{+\infty}3^{i(p-1)}\left\|x+\frac{1}{2}\left(1-\frac{1}{3^i}\right)\right\|^p .$$

for all x, y, $z\in X$ and for all positive integer n, we have

$$\left\|C_n(x+y+z+a)-C_n(x)-C_n(y)-C_n(z)\right\|$$

$$=\left\|\frac{1}{3^n}f\left(3^n x+3^n y+3^n z+\frac{(3^{n+1}-2)a}{2}\right)-\frac{1}{3^n}f\left(3^n x+\frac{(3^n-1)a}{2}\right)-\frac{1}{3^n}f\left(3^n y+\frac{(3^n-1)a}{2}\right)-\frac{1}{3^n}f\left(3^n z+\frac{(3^n-1)a}{2}\right)\right\|$$

$$\leq\frac{\theta}{3^n}\left(\left\|3^n x+\frac{(3^n-1)a}{2}\right\|^p+\left\|3^n y+\frac{(3^n-1)a}{2}\right\|^p+\left\|3^n z+\frac{(3^n-1)a}{2}\right\|^p\right)$$

$$\leq 3^{n(p-1)}\theta\left(\left\|x+\frac{1}{2}\left(1-\frac{1}{3^n}\right)a\right\|^p+\left\|y+\frac{1}{2}\left(1-\frac{1}{3^n}\right)a\right\|^p+\left\|z+\frac{1}{2}\left(1-\frac{1}{3^n}\right)a\right\|^p\right)$$

On taking limit $n\to\infty$ it shows that $C(x+y+z+a)=C(x)+C(y)+C(z)\ \forall\ x, y, z\in X$. The uniqueness of the mapping $C : X \to Y$ can be proved by similar proof as proved in Theorem (2.1). This completes the proof of theorem.

4. Gavruta Stability of the Cauchy functional equation (1.2)

In this section, we prove the stability of Cauchy functional equation (1.2) in the sense of P. Gavruta[4].

Theorem 4.1: Let X be an abelian group and E be a Banach space and let $\phi : X \times X \to [0,+\infty)$ be a mapping satisfying

$$\sum_{i=0}^{\infty} \frac{1}{3^{i+1}} \phi\left(3^i x + \frac{(3^i-1)a}{2}, 3^i y + \frac{(3^i-1)a}{2}, 3^i z + \frac{(3^i-1)a}{2}\right) < +\infty$$

for all $x, y, z \in X$. If a function $f : X \to Y$ is a solution of the function inequality

$$\left\| f(x+y+z+a) - f(x) - f(y) - f(z) \right\| \le \phi(x, y, z) \tag{4.1}$$

for all $x, y, z \in X$, then there exists a unique mapping $C : X \to Y$ such that

$$C(x+y+z+a) = C(x) + C(y) + C(z) \tag{4.2}$$

and $\left\| f(x) - C(x) \right\| \le \sum_{i=0}^{+\infty} \frac{1}{3^{i+1}} \phi\left(3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}\right)$ $\forall x,y,z \in X$.

Proof: Let us consider $x = y = z$ in (4.1), we get

$$\left\| f(3x+a) - 3f(x) \right\| \le \phi(x, x, x) \tag{4.3}$$

replacing x with $3x+a$ in (4.3), such that

$$\left\| f(9x+4a) - 3f(3x+a) \right\| \le \phi(3x+a, 3x+a, 3x+a) \tag{4.4}$$

again substituting x with $3x+a$ in (4.4), we get

$$\left\| f(27x+13a) - 3f(9x+4a) \right\| \le \phi(9x+4a, 9x+4a, 9x+4a) \tag{4.5}$$

Now, taking induction for some positive integer n, implies

$$\left\| f(x) - \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right) \right\| \le \sum_{i=0}^{n-1} \frac{1}{3^{i+1}} \phi\left(3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}\right) \tag{4.6}$$

The induction is true, since for n = 1, 2, 3 in (4.6) the inequality (4.3), (4.4) and (4.5) holds. Now, we prove that it is true for n+1, using inequality (4.6), we have

$$\left\| f(x) - \frac{1}{3^{n+1}} f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right)\right\|$$

$$\leq \frac{1}{3^{n+1}}\left\| f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right) - 3f\left(3^{n}x + \frac{(3^{n}-1)a}{2}\right)\right\|$$

$$+\frac{1}{3^{n}}\left\| f\left(3^{n}x + \frac{(3^{n}-1)a}{2}\right) - 3^{n} f(x)\right\|$$

$$= \frac{1}{3^{n+1}}\phi\left(3^{n}x + \frac{(3^{n}-1)a}{2}, 3^{n}x + \frac{(3^{n}-1)a}{2}, 3^{n}x + \frac{(3^{n}-1)a}{2}\right)$$

$$+\sum_{i=0}^{n-1}\frac{1}{3^{i+1}}\phi\left(3^{i}x + \frac{(3^{i}-1)a}{2}, 3^{i}x + \frac{(3^{i}-1)a}{2}, 3^{i}x + \frac{(3^{i}-1)a}{2}\right)$$

$$=\sum_{i=0}^{n}\frac{1}{3^{i+1}}\phi\left(3^{i}x + \frac{(3^{i}-1)a}{2}, 3^{i}x + \frac{(3^{i}-1)a}{2}, 3^{i}x + \frac{(3^{i}-1)a}{2}\right) \tag{4.7}$$

which shows that the induction is true for n+1 also. Now, to prove that the sequence $C_n(x) = \left\{\frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right)\right\}$ is a Cauchy sequence, such that

$$\|C_{n+1}(x) - C_n(x)\| \leq \left\| \frac{1}{3^{n+1}} f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right) - \frac{1}{3^{n}} f\left(3^{n}x + \frac{(3^{n}-1)a}{2}\right)\right\|$$

$$\leq \frac{1}{3^{n+1}}\left\| 3f\left(3^{n}x + \frac{(3^{n}-1)a}{2}\right) - f\left(3^{n+1}x + \frac{(3^{n+1}-1)a}{2}\right)\right\|$$

$$= \frac{1}{3^{n+1}}\phi\left(3^{n}x + \frac{(3^{n}-1)a}{2}, 3^{n}x + \frac{(3^{n}-1)a}{2}, 3^{n}x + \frac{(3^{n}-1)a}{2}\right)$$

since 1/3<1, Y is a complete normed space and the limit of sequence $\{C_n(x)\}$ exists and is in Y therefore, the sequence $\{C_n(x)\}$ is a Cauchy sequence in Y. Define a mapping $C : X \to Y$ by

$$C(x) = \lim_{n\to\infty} \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right) \tag{4.8}$$

Therefore, using (4.8) inequality (4.7) implies

that $\|f(x) - C(x)\| \le \sum_{i=0}^{+\infty} \frac{1}{3^{i+1}} \phi\left(3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}, 3^i x + \frac{(3^i-1)a}{2}\right) \quad \forall\ x \in X.$

Now, we claim that the mapping $C : X \to Y$ satisfies the inequality (4.2) for all x, y, $z \in X$ and for all positive integer n, we have

$$\|C_n(x+y+z+a) - C_n(x) - C_n(y) - C_n(z)\|$$

$$= \left\|\frac{1}{3^n} f\left(3^n x + 3^n y + 3^n z + \frac{(3^{n+1}-2)a}{2}\right) - \frac{1}{3^n} f\left(3^n x + \frac{(3^n-1)a}{2}\right) - \frac{1}{3^n} f\left(3^n y + \frac{(3^n-1)a}{2}\right) - \frac{1}{3^n} f\left(3^n z + \frac{(3^n-1)a}{2}\right)\right\|$$

$$= \frac{1}{3^n} \phi\left(3^n x + \frac{(3^n-1)a}{2}, 3^n y + \frac{(3^n-1)a}{2}, 3^n z + \frac{(3^n-1)a}{2}\right)$$

$\Rightarrow C(x+y+z+a) = C(x) + C(y) + C(z)$ as $n\to\infty$, for all x, y, $z \in X$.

The uniqueness of the mapping $C : X \to Y$ can be obtained using the similar proof as proved in the Theorem 2.1. Hence completes the proof.

Corollary 4.1: Let $f : X \to Y$ where X be a normed space and Y be a Banach space, be a mapping. If f satisfies the functional inequality

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \le \theta(\|x\|^p \|y\|^q \|z\|^r)$$

for some $\theta > 0$ and p, q, $r \in [0, 1)$ for all x, y, $z \in X$, then there exists a unique mapping $C : X \to Y$ such that $C(x+y+z+a) = C(x) + C(y) + C(z)$ and

$$\|f(x) - C(x)\| \le \frac{\theta}{3} \sum_{i=0}^{+\infty} 3^{i(p+q+r-1)} \left\|x + \frac{1}{2}\left(1 - \frac{1}{3^i}\right)\right\|^{p+q+r} \text{ for all } x \in X.$$

Corollary 4.2: Let $f : X \to Y$ where X be a normed space and Y be a Banach space, be a mapping. If f satisfies the functional inequality

$$\|f(x+y+z+a) - f(x) - f(y) - f(z)\| \le \theta(\|x\|^p \|y\|^p \|z\|^p)$$

for some $\theta>0$ and $p\in[0, 1)$ for all x, y, $z\in X$, then there exists a unique mapping $C: X\to Y$ such that $C(x+y+z+a)=C(x)+C(y)+C(z)$ and

$$\|f(x)-C(x)\|\le\frac{\theta}{3}\sum_{i=0}^{+\infty}3^{i(3p-1)}\left\|x+\frac{1}{2}\left(1-\frac{1}{3^i}\right)\right\|^{3p} \quad \text{for all } x\in X.$$

5. Conclusion

In the stability problems of non-linear functional equations, it is important to find out the best possible approximation of the difference $f(x)-C(x)$. Therefore, we proved the stability of functional equation (1.2) and gave the improved estimation of the difference $f(x)-C(x)$ in sense of D. H. Hyer [1], Th. M. Rassias [6] and P. Gavruta [4].

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