Int. Journal of Math. Analysis, Vol. 6, 2012, no. 32, 1571 - 1582

The Fuzzy Normal Subhypergroup Based on Fuzzy Space

Karema S. Abdulmula* and Abdul Razak Salleh

School of Mathematical Sciences, Faculty of Science and Technology
Universiti Kebangsaan Malaysia
43600 UKM Bangi, Selangor Darul Ehsan, Malaysia
* karema1979saa@yahoo.com
aras@ukm.my

Abstract

In this paper we continue the study of fuzzy hypergroups by introducing the notion of fuzzy normal subhypergroup based on fuzzy space as a generalisation of fuzzy normal subgroup. and to find out the relationships of fuzzy normal subhypergroups and the classical fuzzy normal subhypergroups.

Keywords: Fuzzy space, Fuzzy hypergroup, Fuzzy subhypergroup Fuzzy normal subhypergroup

1 Introduction

B. Davvaz in Davvaz (2003) presented a new and natural interpretation of sub-hypergroups in a partially ordered algebra. He also in B. Davvaz (1999) introduced the concept of fuzzy sub-hypergroup of a hypergroup as well as the fuzzy H_v-group. He and Corsini in Davvaz B and Corsini P (2007) introduced the notion of a fuzzy and anti fuzzy n-ary sub-hypergroup of an n-ary hypergroup.

The fuzzy normal subgroups were studied by Wangming Wu Wangming (1986, 1988). Salleh in Abdul Razak (1996) gave us a new approach of the notion of fuzzy normal subgroups to continue the theory of fuzzy groups obtained by Dib Dib (1994). After that, the fuzzy normal subgroups were introduced by Dib and Hassan Dib and Hassan (1998), in a similar manner to Salleh, which depends on the concept of fuzzy space which serves as the concept of the universal set in the ordinary algebra. Fathi in Fathi M and Salleh A. R. (2010) introduced the concept of fuzzy hypergroups and fuzzy subhypergroups.

Our interest in this paper is to define fuzzy normal subhypergroup in a fuzzy hypergroup defined in Fathi M and Salleh A. R. (2010), and to find out the relationship between this fuzzy normal subhypergroup and the classical fuzzy normal subhypergroups.

2 Preliminaries

In this section we recall some definitions and results which will be used in the paper.

Definition 2.1 (*Dib* [6]) *A fuzzy space* (X, I) *is the set of all ordered pairs* $(x, I); x \in X$,

$$(X, I) = \{(x, I) : x \in X\}, \text{ where } (x, I) = \{(x, r) : r \in I\}.$$

The ordered pair (x, I) *is called a fuzzy element in the fuzzy space* (X, I).

Definition 2.2 (*Dib* [6]) *A* fuzzy subspace U *of the fuzzy space* (X, I) *is the collection of all ordered pairs* (x, u_x), *where* $x \in U_\circ$ *for some* $U_\circ \in X$ *and* u_x *is a subset of* I, *which contains at least one element beside the zero element. If it happens that* $x \notin U_\circ$, *then* $u_x = 0$. *An empty fuzzy subspace is defined as* $\{(x, \varnothing_x) : x \in U_\circ\}$.

For a fuzzy subset A of X, A induces the following fuzzy subspace $H(A)$ (called the induced fuzzy subspace by A) of X:

$$H(A) = \{(x, [0, A(x)]), x \in A_\circ\}$$

where $A_\circ = \{x \in X, A(x) \neq 0\}$ is the support of A.

Definition 2.3 (*Dib* [6]) *A fuzzy binary operation* $\boldsymbol{F} = (F, f_{xy})$ *on the fuzzy space* X *is a fuzzy function* $\boldsymbol{F}$ *from* $X \times X$ *to* X *with comembership functions* $f_{xy} : I \Box I \longrightarrow I$ *that satisfy*

1. $f_{xy}(r, s) = 0$ *iff* $r = 0$ *or* $s = 0$,
2. f_{xy} *are onto , i.e.,*$f_{xy}(I \Box I) = I$ *for all* $(x, y) \in X \times X$.

Recall that the action of the fuzzy function $\Diamond = (\triangle, \triangledown_{xy})$ on fuzzy elements of the fuzzy space (H, I) can be symbolized as follows:

$$(x, I) \Diamond (y, I) = (x \triangle y, \triangledown_{xy}(I \Box I)) = (\triangle\ (x, y), I).$$

Definition 2.4 (*Dib* [7]) *Let* $\langle (X, I), \boldsymbol{F} \rangle$ *be a fuzzy group having the fuzzy sub-group* $U = \{(x, u_x); x \in U_\circ\}$. *Contrary to the ordinary case, the fuzzy elements* (x, u_x) *of the fuzzy sub-group* U *are not necessary associative with fuzzy elements* (x, I) *of the fuzzy group* $\langle (X, I), \boldsymbol{F} \rangle$ *in the usual sense. That is,*

$$\alpha \boldsymbol{F} (\beta \boldsymbol{F} \gamma) \neq (\alpha \boldsymbol{F} \beta) \boldsymbol{F} \gamma,$$

where α, β *and* γ *are some fuzzy elements of* U *or* (X, I) *such that one or two of* α, β *or* γ *belong to* U.

Definition 2.5 (*Dib* [7]) *The fuzzy sub-group U of the fuzzy group $((X,I),\underline{F})$ is called a* fuzzy normal sub-group *if*

(1) *U is associative in $((X,I),\underline{F})$.*

(2) *$(x,I)U = U(x,I)$ such that $x \in X$.*

Definition 2.6 (*Fathi* [5]) *Let (H,I) be a non-empty fuzzy space. A* fuzzy hyperstructure (hypergroupoid)*, denoted by $\langle (H,I),\Diamond\rangle$ is a fuzzy space together with a fuzzy function having onto co-membership functions (referred to as a* fuzzy hyperoperation*) $\Diamond : (H,I)\times(H,I) \to P^*(H,I)$, where $P^*(H,I)$ denotes the set of all nonempty fuzzy subspaces of (H,I) and*

$$\Diamond = (\triangle, \nabla_{xy}) \text{ with } \triangle\colon H \times H \to P^*(H) \text{ and } \nabla_{xy} : I \times I \to I.$$

A fuzzy hyperoperation $\Diamond = (\triangle, \nabla_{xy})$ on (H,I) is said to be uniform *if the associated co-membership functions ∇_{xy} are identical, i.e.,$\nabla_{xy} = \nabla$ for all $x, y \in H$. A* uniform fuzzy hyperstructure *$\langle (H,I),\Diamond\rangle$is a fuzzy hyperstructure $\langle (H,I);\Diamond\rangle$ with uniform fuzzy hyperoperation.*

Theorem 2.7 (*Fathi* [5]) *To each fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ there is an associated ordinary hyperstructure $\langle H,\triangle\rangle$ which is isomorphic to the fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ by the correspondence $(x,I) \leftrightarrow x$.*

Definition 2.8 (*Fathi* [5]) *A fuzzy hypergroup is a fuzzy hyperstrucure $\langle (H,I),\Diamond\rangle$ satisfying the following axioms:*

(i) $((x,I)\Diamond(y,I))\,\Diamond(z,I) = (x,I)\Diamond\,((y,I)\Diamond(z,I))$,
for all $(x,I),(y,I),(z,I)$ in (H,I),

(ii) $(x,I)\Diamond(H,I) = (H,I)\Diamond(x,I) = (H,I)$,
for all (x,I) in (H,I).

Definition 2.9 (*Fathi* [5]) *A fuzzy H_v-group is a fuzzy hyperstructure $\langle (H,I),\Diamond\rangle$ satisfying the following conditions:*

(i) $((x,I)\Diamond(y,I))\,\Diamond(z,I) \cap (x,I)\Diamond\,((y,I)\Diamond(z,I)) \neq \phi$,
for all $(x,I),(y,I),(z,I) \in (H,I)$,

(ii) $(x,I)\Diamond(H,I) = (H,I)\Diamond(x,I) = (H,I)$,
for all $(x,I) \in (H,I)$.

Definition 2.10 (*Fathi* [5]) *Let $\langle (H,I),\Diamond\rangle$ be a fuzzy hypergroup (fuzzy H_v-group) and let $U = \{(x,u_x) : \ x \in U_\circ\}$be a fuzzy subspace of (H,I). Then $\langle U;\Diamond\rangle$ is called a fuzzy subhypergroup (fuzzy H_v-subgroup) of the fuzzy hypergroup $\langle (H,I),\Diamond\rangle$ if $\Diamond$ is closed on the fuzzy subspace U and $\langle U,\Diamond\rangle$ satisfies the conditions of an ordinary hypergroup (H_v-group).*

Theorem 2.11 (*Fathi* [5]) $\langle H_\circ(A), \diamondsuit\rangle, \langle \underline{H}(A), \diamondsuit\rangle$ *and* $\langle \overline{H}(A), \diamondsuit\rangle$ *are fuzzy subhypergroups (fuzzy* H_v*-subgroups) of the fuzzy hypergroup (fuzzy* H_v*-group)* $\langle (H,I), \diamondsuit\rangle$ *iff*

(1) $(A_\circ, \triangle)$ *is an ordinary subhypergroup (*H_v*-subgroup) of the ordinary hypergroup (*H_v*-subgroup)* $(H, \triangle)$.

(2) $\bigtriangledown_{xy}(A(x), A(y)) = \inf\{A(z) \ : \ z \in x \triangle y\}$, *for all* $x, y \in A_\circ$.

3 Associative fuzzy subhypergroups

In this section we will introduce the notion of associative fuzzy subhypergroup and then define the normal subhypergroup based on associative fuzzy subhypergroup to obtain interesting results regarding fuzzy hypergroups and related fuzzy normal subhypergroups.

Let $\langle (H,I), \boldsymbol{F}\rangle$ be a fuzzy hypergroup having fuzzy subhypergroup $U = \{(x, u_x) : x \in U_\circ\}$. Contrary to the ordinary case, the fuzzy elements (x, u_x) of the fuzzy subhypergroup U are not necessarily associative with fuzzy elements (x, I) of the fuzzy hypergroup $\langle (H,I), \boldsymbol{F}\rangle$. That is

$$\alpha\boldsymbol{F}(\beta\boldsymbol{F}\gamma) \neq (\alpha\boldsymbol{F}\beta)\boldsymbol{F}\gamma,$$

where α, β and γ are some fuzzy elements of U or (H, I) such that one or two sets of these fuzzy elements subset to U.

Example 3.1 Let $H = \{-1, 1, -i, i\}$. Define the fuzzy binary hyperoperation $\boldsymbol{F} = (F, f_{xy})$ on (H, I) such that $F : H \times H \to P^*(H)$ with $F(x,y) = xFy = \{xy\}$ for all $x, y \in H$, and the comembership functions have the following forms*:*

$$f_{11}(r,s) = \begin{cases} \frac{rs}{\sqrt{\alpha}} & \text{if } rs < \alpha \\ 1 + \frac{rs-1}{1+\sqrt{\alpha}} & \text{if } rs \geq \alpha \end{cases},$$

$$f_{-11}(r,s) = f_{1-1}(r,s) = \begin{cases} \frac{rs}{\sqrt{\alpha}} & \text{if } rs < \sqrt{\alpha\beta} \\ 1 + \frac{1-\sqrt{\beta}}{1-\sqrt{\alpha\beta}}(rs-1) & \text{if } rs \geq \sqrt{\alpha\beta} \end{cases},$$

$$f_{-1-1}(r,s) = \begin{cases} \frac{\sqrt{\alpha}}{\beta} rs & \text{if } rs \le \beta \\ 1 + \frac{1-\sqrt{\alpha}}{1-\beta}(rs-1) & \text{if } rs > \beta \end{cases},$$

and the other comembership functions are defined by the product rs, where α, β are given fixed real numbers satisfying $0 < \beta < \alpha < 1$.

Clearly $\langle (H, I), \boldsymbol{F} \rangle$ defines a fuzzy hypergroup. Also the fuzzy subspace $U = \{(1, [0, \sqrt{\alpha}]), (-1, [0, \sqrt{\beta}])\}$ together with the fuzzy binary hyperoperation $\boldsymbol{F}$ define a fuzzy subhypergroup of $\langle (H, I), \boldsymbol{F} \rangle$.

Now, we notice that

$$\big((1, I)\boldsymbol{F}(1, [0, \sqrt{\alpha}])\big)\, \boldsymbol{F}(-1, [0, \sqrt{\beta}]) =$$
$$\left\{ \left(-1, \left[0, 1 + \frac{1-\sqrt{\beta}}{1-\sqrt{\alpha\beta}} \left(\frac{\sqrt{\alpha\beta}+\beta}{1+\sqrt{\alpha}} - 1 \right) \right] \right) \right\},$$

On the other hand

$$(1, I)\boldsymbol{F}\Big((1, [0, \sqrt{\alpha}])\boldsymbol{F}(-1, [0, \sqrt{\beta}])\Big) =$$
$$\left\{ \left(\{-1\}, \left[0, 1 + \frac{1-\sqrt{\beta}}{1-\sqrt{\alpha\beta}} \left(\frac{1-\sqrt{\beta}}{1-\sqrt{\alpha\beta}} (\sqrt{\alpha\beta} - 1) \right) \right] \right) \right\}.$$

That is, the set of fuzzy elements of U are not associative with the set of fuzzy elements of (H, I).

Definition 3.2 *A fuzzy subhypergroup* $(U, \mathbf{F})$ *of a fuzzy hypergroup* $\langle (H, I), \mathbf{F} \rangle$ *is said to be associative in* $\langle (H, I), \mathbf{F} \rangle$ *if the fuzzy elements of* U *are associative with fuzzy elements of* $\langle (H, I), \boldsymbol{F} \rangle$*, i.e.*

$$\alpha \boldsymbol{F}(\beta \boldsymbol{F} \gamma) = (\alpha \boldsymbol{F} \beta) \boldsymbol{F} \gamma,$$

for arbitrary choices of fuzzy elements α, β, γ *of* U *and* (H, I).

Example 3.3 Let $H = \{-1, 1, -i, i\}$. Define the fuzzy binary hyperoperation $\boldsymbol{F} = (F, f_{xy})$ on (H, I) such that $F : H \times H \to P^*(H)$ is define as $\mathbf{F}(x, y) = \{xy\}$, and the comembership functions are for all $x, y \in H$ by

$$f_{xy}(r, s) = r \wedge s.$$

Obviously the fuzzy subspace $U = \{(1, [0, \frac{1}{4}]), (-1, [0, \frac{1}{8})\}$ defines an associative fuzzy subhypergroup of $\langle (H, I), \boldsymbol{F} \rangle$ under $\boldsymbol{F}$.

The above definitions and examples have the following interesting results.

Theorem 3.4 *Let $\langle(H,I),\boldsymbol{F}\rangle$ with $\boldsymbol{F}=\big(F,\underline{f}\big)$ be a uniform fuzzy hypergroup such that $f(r,1)=f(1,r)=r$. Then every fuzzy sub-hypergroup U of the fuzzy hypergroup $\langle(H,I),\boldsymbol{F}\rangle$ is an associative fuzzy sub-hypergroup in $\langle(H,I),\boldsymbol{F}\rangle$.*

Proof Let $(U,\boldsymbol{F})$ be a fuzzy sub-hypergroup of the fuzzy hypergroup $\langle(H,I),\boldsymbol{F}\rangle$. Consider the fuzzy elements $(x,I),(y,I)$ and (z,I) of U or (H,I) such that one or two of these fuzzy elements belong to U.
Now using the properties of $\boldsymbol{F}$ and the associativity of F we have

$$\begin{aligned}(x,I)\boldsymbol{F}\left((y,I)\boldsymbol{F}(z,I)\right) &= (x,I)\boldsymbol{F}\left\{(yFz,\boldsymbol{F}\left(I\times I\right)\right\}\\ &= (x,I)\boldsymbol{F}\left\{(yFz,I)\right\}\\ &= \left\{xF(yFz),\boldsymbol{F}\left(I\times I\right)\right\}\\ &= ((xFy)Fz,I) = \left((x,I)\boldsymbol{F}(y,I)\right)\boldsymbol{F}(z,I)\end{aligned}$$

which proves the associativity of fuzzy members of U with fuzzy members of (H,I) under $\boldsymbol{F}$.

Corollary 3.5 *Let $f:I\times I\longrightarrow I$ be a t-norm function and $\boldsymbol{F}=(F,f)$ be a uniform fuzzy binary hyperoperation. If $((H,I),\boldsymbol{F})$ be a fuzzy hypergroup, then every fuzzy sub-hypergroup induced by a fuzzy subset A of H is an associative in $((H,I),\boldsymbol{F})$.*

4 Fuzzy normal sub-hypergroup

Before introducing the fuzzy normal subhypergroup we will define now the notions of left (right) coset of fuzzy subhypergroup.

Definition 4.1 *If $(U,\mathbf{F})$ where $U=\{(z,u_z):z\in U_\circ\}$ is a fuzzy subhypergroup of the fuzzy hypergroup $\langle(H,I),\mathbf{F}\rangle$ then for every fuzzy element (x,I) of (H,I), the fuzzy subspace defined by*

$$(x,I)U=(x,I)\mathbf{F}U=\{(xFz,f_{xz}(I,u_z))\}$$

is called a left coset of the fuzzy subhypergroup $(U,\mathbf{F})$. Similarly, a right coset of the of fuzzy subhypergroup $(U,\mathbf{F})$ is defined by the fuzzy subspace

$$U(x,I)=U\mathbf{F}(x,I)=\{(zFx,f_{zx}(u_z,I))\}.$$

Theorem 4.2 *For any associative fuzzy subhypergroup $(U,\boldsymbol{F})$ of the fuzzy hypergroup $\langle(H,I),\mathbf{F}\rangle$ the following hold*

(a) *$(x,I)U=(h,\underline{I}_h)U$ for every fuzzy element $(h,\underline{I}_h)\in(x,I)U$ with $\underline{I}_h$ denotes the possible membership values of h.*

(b) *There is a one-to-one correspondence between any two left (right) cosets of the fuzzy subhypergroup* $(U, \boldsymbol{F})$.

(c) *There is a one-to-one correspondence between the family of right cosets and the family of left cosets of the fuzzy subhypergroup* $(U, \boldsymbol{F})$.

(d) *Any two right cosets (left cosets) of the fuzzy subhypergroup* $(U, \boldsymbol{F})$ *are either identical or disjoint fuzzy subspaces.*

Proof (a) Let (h, I_h) be any fuzzy element in $(x, I)U$. Then $(h, I_h) = (x, I)(y, u_y)$ for some $y \in U_\circ$. If (z, u_z) is an arbitrary element of U then

$$\begin{aligned}
(x, I)(z, u_z) &= (x, I)\left((y, u_y)(y^{-1}, u_{y^{-1}})\right)(z, u_z) \\
&= ((x, I)(y, u_y))\left((y^{-1}, u_{y^{-1}})(z, u_z)\right) \\
&\in (h, \underline{I}_h)U.
\end{aligned}$$

(b) Let $(x, I)U$ and $(y, I)U$ be any two left cosets of the fuzzy subhypergroup U. Then $(x, I)(z, u_z) \leftrightarrow (y, I)(z, u_z)$ is the required one-to-one correspondence between $(x, I)U$ and $(y, I)U$. For the right cosets we prove similarly.

(c) Let $\{(x, I)U : x \in H\}$ and $\{U(x, I) : x \in H\}$ denote the family of left and right cosets respectively of the fuzzy sub-hypergroup U. Then the required one-to-one correspondence is defined by

$$(x, I)U \leftrightarrow U(x, I).$$

(d) Let $(x, I)U$ and $(y, I)U$ be any two intersecting left cosets of the fuzzy subhypergroup U. Then there exist $\alpha, \beta \in U_\circ$ such that

$$(x, I)(\alpha, u_\alpha) = (y, I)(\beta, u_\beta).$$

Choose any fuzzy element $(x, I)(z, u_z) \in (x, I)U$ then

$$\begin{aligned}
(x, I)(z, u_z) &= (x, I)\left((\alpha, u_\alpha)(\alpha^{-1}, u_{\alpha^{-1}})\right)(z, u_z) \\
&= ((x, I)(\alpha, u_\alpha))\left((\alpha^{-1}, u_{\alpha^{-1}})(z, u_z)\right) \\
&= ((y, I)(\beta, u_\beta))\left((\alpha^{-1}, u_{\alpha^{-1}})(z, u_z)\right) \\
&= (y, I)\left((\beta, u_\beta)\left((\alpha^{-1}, u_{\alpha^{-1}})(z, u_z)\right)\right) \in (y, I)U.
\end{aligned}$$

That is $(x, I)U \subset (y, I)U$. Similarly, we can show that $(y, I)U \subset (x, I)U$ and also we can show the same result for the right cosets of U which proves (d).

Definition 4.3 *A fuzzy subhypergroup* U *of the fuzzy hypergroup* $\langle (H, I), \mathbf{F} \rangle$ *is called a* fuzzy normal subhypergroup *if*

(1) U *is associative in* $\langle (H, I), \mathbf{F} \rangle$,

(2) $(x, I)U = U(x, I)$ *for all* $x \in H$.

Example 4.4 *1. Let* $\langle (H, I), \boldsymbol{F} \rangle$ *be as defined in Example 3.3. Then one can easily check that the fuzzy subspace* $U = \{(1, [0, \frac{1}{2}]), (-1, [0, \frac{1}{2}])\}$ *together with the fuzzy binary hyperoperation* $\boldsymbol{F}$ *define a fuzzy normal subhypergroup of* $\langle (H, I), \boldsymbol{F} \rangle$.

2. Let $H = S_3$ *be the set of all permutations on* $\{1, 2, 3\}$. *Define the fuzzy binary hyperoperation* $\boldsymbol{F} = (F, f_{xy})$ *over the fuzzy space* (H, I) *where* F *is the set of ordinary composition of permutations and* $f_{xy}(r, s) = r \wedge s$ *for all* $x, y \in H$. *Consider the fuzzy subspace* $U = \{(\epsilon, [0, \frac{1}{2}]), (\gamma, [0, \frac{1}{2}])\}$ *where* ϵ *denotes the identity permutation and* $\gamma = (12)$. *One can easily investigate that* $(U, \boldsymbol{F})$ *is an associative fuzzy subgroup which is not normal.*

Theorem 4.5 *A fuzzy sub-hypergroup* $(U, \boldsymbol{F})$ *where* $U = \{(z, u_z) : z \in U_\circ\}$ *of the fuzzy hypergroup* $\langle (H, I), \boldsymbol{F} \rangle$ *is a fuzzy normal subhypergroup iff*

(i) $(U_\circ, F)$ *is an ordinary normal sub-hypergroup of the ordinary hypergroup* (H, F).

(ii) $f_{xz}(I, u_z) = f_{z'x}(u_{z'}, I), xFz = z'Fx$ *where* $x \in H$ *and* $z, z' \in U_\circ$.

Proof. Assume $U = \{(z, u_z) : z \in U_\circ\}$ is a fuzzy normal subhypergroup of $\langle (H, I), \boldsymbol{F} \rangle$. From (the correspondence theorem), we have $(U_\circ, F)$ is an ordinary normal subhypergroup of the ordinary hypergroup (H, F). Using the normality of U we have $(x, I)U = U(x, I)\ for\ x \in H$. That is,

$$\{(xFz, f_{xz}(I, u_z)) : z \in U_\circ\} = \{(zFx, f_{xz}(u_z.I)) : z \in U_\circ\}.$$

Therefore for every $z \in U_\circ$, there exists $z' \in U_\circ$, such that $xFz = z'Fx$. In other words $xFU_\circ = U_\circ Fx$. Hence $U_\circ$ is an ordinary normal sub-hypergroup of the ordinary hypergroup (H, F) which proves (i). The other part of the proof is direct.

Theorem 4.6 *Every fuzzy normal subhypergroup* U *of* $\langle (H, I), \boldsymbol{F} \rangle$ *defines an equivalence relation* $\Re$ *on the fuzzy space* (H, I) *given by* $(x, I)\Re(y, I) \Leftrightarrow (x, I)U = U(y, I)$.

The equivalence relation $\Re$ *on the fuzzy space* (H, I) *induces an equivalence relation on* H *by the correspondence* $(x, I) \leftrightarrow x$. *That is,*
$(x, I)\Re(y, I) \Leftrightarrow x\Re y$, *which is equivalent to* $xU_\circ = U_\circ y$.

Proof. The relation $(x, I)\Re(y, I)$ is an equivalence relation. It is followed from the normality of the fuzzy subhypergroup U. Moreover, it is clear that $(x, I)U = U(y, I)$ leads to $xU_o = U_oy$, which defines an equivalence relation on H.

For every $x, y \in H$, and from the normality of the fuzzy subhypergroup $U = \{(z, u_z); z \in Uo\}$ we have

$$\begin{aligned} ((x,I)\boldsymbol{F}(z_1,u_{z_1}))\,\boldsymbol{F}\,((y,I)\boldsymbol{F}(z_2,u_{z_2})) &= (x,I)\boldsymbol{F}\,((z_1,u_{z_1})\boldsymbol{F}(y,I))\,\boldsymbol{F}(z_2,u_{z_2}) \\ &= (x,I)\boldsymbol{F}\,((y,I)\boldsymbol{F}(z_3,u_{z_3}))\,\boldsymbol{F}(z_2,u_{z_2}) \\ &= ((x,I)\boldsymbol{F}(y,I))\,\boldsymbol{F}((z_3,u_{z_3})\boldsymbol{F}(z_2,u_{z_2})) \\ &= (xFy,I)\boldsymbol{F}(z_4,u_{z_4}), \end{aligned}$$

where $z_1, z_2, z_3, z_4 \in Uo$. Therefore, it follows that

$$((x,I)U)\,\boldsymbol{F}\,((y,I)U) = (xFy,I)\,U, ...(*).$$

Therefore, the fuzzy binary hyperoperation $\boldsymbol{F}$ of the fuzzy hypergroup $\langle (H, I), \boldsymbol{F}\rangle$ induces a binary operation defined by the operation $(*)$ on the family of cosets of the fuzzy normal subhypergroup U. This family of cosets together with the induced binary operation forms an ordinary hypergroup called the factor hypergroup of $\langle (H, I), \boldsymbol{F}\rangle$ modulo U and denoted by $\langle (H, I), \boldsymbol{F}\rangle/U$.

Theorem 4.7 *If $U = \{(z, u_z) : z \in U_o\}$ is a fuzzy normal subhypergroup of the fuzzy hypergroup $\langle (H, I), \boldsymbol{F}\rangle$, then the factor hypergroup $\langle (H, I), \boldsymbol{F}\rangle/U$ is isomorphic to the quotient hypergroup $(H, F)/U_o$ by the correspondence*

$$(x, I)U_o \leftrightarrow xU_o.$$

Now we well define fuzzy normal subgroups induced by fuzzy subsets

Let $((H, I), \boldsymbol{F})$ be a fuzzy hypergroup and let A be a fuzzy subset of H. It was shown in Fathi M and Salleh A. R. (2010) that A induces the fuzzy sub-hypergroups $H_o(A), \underline{H}(A)$ and $\overline{H}(A)$, iff

1. (A_o, F) is an ordinary sub-hypergroup of (H, F),
2. $f_{xy}\,(A(x), A(y)) = A\,(xFy)\,, x, y \in A_o$.

Definition 4.8 *The fuzzy subset A is called associative in the fuzzy hypergroup $\langle (H, I), \mathbf{F}\rangle$ if the fuzzy subspace $H_\circ(A) = \{(x, \{0, A(x)\})\,; A(x) \neq 0\}$ is associative in $\langle (H, I), \mathbf{F}\rangle$, then $\underline{H}(A)$ and $\overline{H}(A)$ are associative in $\langle (H, I), \mathbf{F}\rangle$.*

It is easy to show that $H_\circ(A)$ if the fuzzy subspace is associative in $\langle (H,I), \boldsymbol{F}\rangle$, then $\underline{H}(A)$ and $\overline{H}(A)$ are associative in $\langle (H,I), \boldsymbol{F}\rangle$.

For fuzzy subspaces, which are induced by a fuzzy subset A, Theorem 4.5, can be reformulated as follows.

Theorem 4.9 *Let A be a fuzzy subset of H, then the fuzzy subspace $H_\circ(A)$, $\big(\underline{H}(A)$ or $\overline{H}(A)\big)$, is a fuzzy normal subhypergroup in the fuzzy hypergroup $\langle (H,I), \mathbf{F}\rangle$, iff*

1. *(A_o, F) is an ordinary normal subhypergroup of (H,F)*
2. *$f_{xy}\,(1, A(z)) = f_{zx}\,(A(z'), 1)$ if $xFz = z'Fx$, for all $x \in H$ and $z, \acute{z} \in A_\circ$.*
3. *A is associative in $\langle (H,I), \boldsymbol{F}\rangle$.*
4. *$f_{xy}\,(A(x), A(y)) = A\,(xFy)\,, forall\ x, y \in A_0$.*

Remarks.

(i) Not every fuzzy subhypergroup of an abelian fuzzy hypergroup $\langle (H,I), \boldsymbol{F}\rangle$ is normal which is different from the ordinary case.

(ii) The fuzzy hypergroup $\langle (H,I), \boldsymbol{F}\rangle$ is abelian iff (H,F) is an abelian hypergroup.

(iii) If $H_\circ(A)$ is a fuzzy normal subhypergroup in $\langle (H,I), \boldsymbol{F}\rangle$, then the family of cosets: $(x,I)\,H_o(A) = \{(xFy, \{0, f\,(1, A(y))\}) : y \in A_\circ\}$ for all $x \in H$ of the fuzzy subhypergroup $H_o(A)$ with respect to the binary hyperoperation,

$$(x,I)H_o(A)\underline{F}(y,I)H_o(A) = \{xFy, I\}\,H_o(A),$$

is a hypergroup which is isomorphic to the quotient hypergroup $(H,F)/A_o$.

Now we introduce relationship between the introduced and the classical fuzzy normal subhypergroups as follows:

Let $\langle (H,I), \boldsymbol{F}\rangle, \boldsymbol{F} = (F, \underline{f})$ be a uniform fuzzy hypergroup with a t-norm comembership function $\underline{f}$. If the fuzzy subset A induces fuzzy normal subhypergroups, then using Theorem 4.9, we have,

1. (A_o, F) is an ordinary normal subhypergroup of (H,F).
2. $A\,(xFy) = \underline{f}\,(A(x), A(y))\,,\ for\ all\ x, y \in A_o \cdots\cdots\cdots\cdots\cdots (\star)$.

3. $A(z) = A(z')$ for all $xFz = z'Fx,\ x \in H,\ z, z' \in A_o$.

   (1) and (2) lead A to be a classical fuzzy subhypergroup of (H, F) and (3) can be written as:

4. $A(z) = A(xFzFx^{-1}),\ x \in H,\ z \in A_o$, which gives,

$$A(xFzFx^{-1}) \geq A(z),\ for\ all\ x, z \in H.$$

Therefore, A is a classical fuzzy normal sub-hypergroup. We notice that (4) can also be written as

$$A\left(xFz\right) = A\left(x^{-l}F(xFz)Fx\right) = A\left(zFx\right),$$

which means that A is a classical fuzzy normal sub-hypergroup.
From the above discussion we conclude that

Theorem 4.10 *Let $\langle(H, I), \mathbf{F}\rangle, \mathbf{F} = (F, \underline{f})$ be a uniform fuzzy hypergroup and let f is a t-norm. Then every fuzzy subset A of H which induces fuzzy normal subhypergroups is a classical fuzzy normal sub-hypergroup of (H, F).*

Proof. If the fuzzy subsets A induces fuzzy normal sub-hypergroups of the fuzzy normal hypergroup $\langle(H, I), \boldsymbol{F}\rangle$, then by Theorems4.9 and ($\star$) the co-membership function f satisfies

$$f_{xy}\left(A(x), A(y)\right) = A(xFy)\ for\ all\ x, y \in A_\circ$$

for all $A(x) \neq 0$ and$A(y) \neq 0$. Therefore, if the fuzzy subset A induces fuzzy normal sub-hypergroup, then A satisfies the inequality

$$f_{xy}\left(A(x), A(y)\right) \geq A(xFy)\ for\ all\ x, y \in A_\circ$$

Therefore, A is a classical fuzzy normal sub-hypergroup.

Theorem 4.11 *If (U,F)is an ordinary normal subhypergroup of the ordinary hypergroup (H, F). Then every fuzzy subset A of H for which $A_\circ = A$ induces an fuzzy normal subhypergroup of fuzzy hypergroup $((H, I), \boldsymbol{G})$, where $\boldsymbol{G} = (G, \underline{g}_{xy})$ and $G = F$, are suitable comembership functions.*

Corollary 4.12 *Every classical fuzzy normal sub-hypergroup A of hypergroup (H, F) induces a fuzzy normal sub-hypergroups relative to some fuzzy hypergroup $\langle(H, I), \boldsymbol{F}\rangle$.*

5 Conclusion

In this paper, we have generalized the study initiated in Dib and Hassan (1998) about fuzzy normal subgroup to the context of fuzzy normal subhypergroup. We define the notion of a fuzzy normal subhypergroup using the notion of a fuzzy space.

6 Acknowledgements

The authors would like to acknowledge the financial support received from Universiti Kebangsaan Malaysia under the research grant UKM-ST-06-FRGS0104-2009.

References

Abdul Razak S. 1996. Properties of fuzzy homomorphism of fuzzy groups (in Malay). *Proc.* 7^{th} *National Symposium of Mathematical Sciences. Institut Teknologi Mara:* **80**: 346-351.

Davvaz B. 1999. Fuzzy H_v-groups. *Fuzzy Sets and Systems* **101**(1): 191-195.

Davvaz B. 2003. A new interpretation of subhypergroups of a hypergroup. *J. Korea Soc. Math. Educ. Ser. B: Pure Appl. Math.* **10**(3): 163-169

Davvaz B and Corsini P. 2007. Fuzzy n-ary hypergroups. *Journal of Intelligent and Fuzzy Systems* **18**(4): 377-382.

Mohammed Fathi and Abdul Razak Salleh. 2010. Fuzzy hypergroups (H_v-groups) based on fuzzy fuzzy space. *Technical Report. School of Mathematical Sciences, Universiti Kebangsaan Malaysia.*

Dib K.A. 1994. On fuzzy spaces and fuzzy group theory. *Information Sciences* **80**: 253-282.

Dib K.A. and Hassan A. M. 1998. The fuzzy normal subgroup theory. *Fuzzy Sets and Sestems* **98**: 393-402.

Wangming Wu. 1981. Normal fuzzy subgroups. *Fuzzy Math* **1**(1): 21-30.

Wangming Wu. 1988. Fuzzy congruences and normal fuzzy subgroups. *Fuzzy Math.* **3**: 9-20.

Received: January, 2012