On Quasi $p$-Normal Spaces

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Abstract

The aim of this paper is to study a weaker version of $p$-normality called quasi $p$-normality, which lies between $\pi p$-normality and mild $p$-normality. We show that this property is a topological property and it is a hereditary property only with respect to closed domain subspaces. Some properties, examples and various characterizations of this property are presented. Also, we establish various preservation theorems of quasi $p$-normality under continuous and some generalized sense of continuous mappings.

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1 Introduction and Preliminary

Throughout this paper, a space $X$ always means a topological space on which no separation axioms are assumed, unless explicitly stated. For a subset $A$ of a space $X$, $X \setminus A$, $\overline{A}$ and int$(A)$ denote to the complement, the closure and the interior of $A$ in $X$, respectively. A subset $A$ of a space $X$ is said to be regularly-open or an open domain if it is the interior of its own closure, or equivalently if it is the interior of some closed set, see [11]. A set $A$ is said to be a regularly-closed or a closed domain if its complement is an open domain. A subset $A$ of a space $X$ is called a $\pi$-closed if it is a finite intersection of closed domain sets, see [29]. A subset $A$ is called a $\pi$-open if its complement is a $\pi$-closed. Two sets $A$ and $B$ of a space $X$ are said to be separated if there exist two disjoint open sets $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$, [5, 7, 19]. A subset $A$ of a space $X$ is said to be pre-open (briefly $p$-open), [14], if $A \subseteq \text{int}(\overline{A})$. A subset $A$ of a space $X$ is said to be a semi-open if $A \subseteq \text{int}(\overline{A})$, see [3]. A space

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X is called a \textit{p-normal}, see [20], if any two disjoint closed subsets \( A \) and \( B \) of \( X \) can be separated by two disjoint \( p \)-open subsets. A space \( X \) is called an \textit{almost \( p \)-normal}, see [16], if any two disjoint closed subsets \( A \) and \( B \) of \( X \), one of which is closed domain, can be separated by two disjoint \( p \)-open subsets. A space \( X \) is called a \textit{mildly \( p \)-normal}, see [16], if any pair of disjoint closed domain subsets \( A \) and \( B \) of \( X \), can be separated by two disjoint \( p \)-open subsets. A space \( X \) is said to be a \textit{\( \pi p \)-normal}, see [27], if any pair of disjoint \( \pi \)-closed subsets \( A \) and \( B \) of \( X \), one of which is \( \pi \)-closed, can be separated by two disjoint \( p \)-open subsets. A space \( X \) is said to be a \textit{\( \pi \)-normal}, [10], if any pair of disjoint closed subsets \( A \) and \( B \) of \( X \), one of which is \( \pi \)-closed, can be separated by two disjoint open subsets. The complement of \( p \)-open (resp. semi-open) set is called \( p \)-closed (resp. semi-closed). The intersection of all \( p \)-closed sets containing \( A \) is called \textit{pre-closure} of \( A \), see [13], and denoted by \( pcl(A) \). Dually, the \textit{pre-interior} of \( A \) denoted by \( pint(A) \), is defined to be the union of all \( p \)-open sets contained in \( A \). A subset \( A \) is said to be a \textit{\( p \)-neighborhood} of \( x \), [16], if there exists a \( p \)-open set \( U \) such that \( x \in U \subseteq A \).

In this paper, we show that quasi \( p \)-normality is a topological property and it is a hereditary property only with respect to closed domain subspaces. Some properties, examples, characterizations and preservation theorems of this property are presented.

\section{Main Results}

First, we give the definition of quasi \( p \)-normality.

\begin{definition}
A space \( X \) is said to be a quasi \( p \)-normal if for every pair of disjoint \( \pi \)-closed subsets \( A \) and \( B \) of \( X \), there exist disjoint \( p \)-open subsets \( U \) and \( V \) of \( X \) such that \( A \subseteq U \) and \( B \subseteq V \).
\end{definition}

Clearly, every normal space is \( \pi \)-normal as well as \( p \)-normal and we have:

\[
\text{\( p \)-normal} \implies \text{\( \pi \text{\( p \)} \)-normal} \implies \text{almost \( p \)-normal} \implies \text{mildly \( p \)-normal} \\
\text{\( p \)-normal} \implies \text{\( \pi \text{\( p \)} \)-normal} \implies \text{quasi \( p \)-normal} \implies \text{mildly \( p \)-normal}
\]

None of the above implications is reversible as the following examples show.

\begin{example}
Consider the Example 2.2 in [27]. Observe that the topology \( T = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\} \) on the set \( X = \{a, b, c\} \) is quasi \( p \)-normal but not \( p \)-normal space. Also, the particular point topology on \( \mathbb{R} \), see Example 2.12 in [27], is quasi \( p \)-normal space but not \( p \)-normal.
\end{example}

\begin{example}
Quasi \( p \)-normality does not imply almost \( p \)-regularity. Consider the Example 2.3. in [27]. Observe that the topology \( T = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\} \) on the set \( X = \{a, b, c\} \) is quasi \( p \)-normal but not almost \( p \)-regular space.
\end{example}
Example 2.4 The co-finite topology on $\mathbb{R}$ and the Niemytzki plane topology are quasi $p$-normal spaces because they are $\pi p$-normal, see [27], but they are not normal.

Example 2.5 The rational sequence topology is an almost $p$-normal and not $\pi p$-normal space, see [27]. We proved that the rational sequence topology is not quasi-normal in [26] and showed that every $p$-open set in this space is an open in [27]. Thus, it is easy to observe that the rational sequence topology is not quasi $p$-normal.

Until now, we do not know a Tychonoff quasi $p$-normal space, which is not $\pi p$-normal (or not almost $p$-normal). Now, we need to recall the following definitions.

Definition 2.6 A subset $A$ of a space $X$ is called:

(a) generalized closed (briefly $g$-closed), [12], if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and $U$ is open.

(b) strongly generalized closed (briefly $g^*$-closed), [23], if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open.

(c) $\pi$-generalized closed (briefly $\pi g$-closed), [4], if $\overline{A} \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open.

(d) generalized pre-closed, [15], (briefly $gp$-closed) if $p\operatorname{cl}(A) \subseteq U$, whenever $A \subseteq U$ and $U$ is open.

(e) strongly generalized pre-closed, [28], (briefly $g^*p$-closed) if $p\operatorname{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $g$-open.

(f) $\pi$-generalized pre-closed, [21], (briefly $\pi gp$-closed) if $p\operatorname{cl}(A) \subseteq U$ whenever $A \subseteq U$ and $U$ is $\pi$-open.

The complement of $g$-closed (resp. $g^*$-closed, $\pi g$-closed, $gp$-closed, $g^*p$-closed, $\pi gp$-closed) is called $g$-open (resp. $g^*$-open, $\pi g$-open, $gp$-open, $g^*p$-open, $\pi gp$-open). From the above definitions we have:

\[
\text{closed} \implies g^*\text{-closed} \implies g\text{-closed} \implies \pi g\text{-closed} \\
\text{closed} \implies p\text{-closed} \implies g^*p\text{-closed} \implies gp\text{-closed} \implies \pi gp\text{-closed}
\]

None of the above implications is reversible. The following theorem is useful for giving some characterizations of quasi $p$-normal spaces.

Theorem 2.7 For a space $X$, the following are equivalent:

(a) $X$ is quasi $p$-normal.
(b) For every pair of \( \pi \)-open subsets \( U \) and \( V \) of \( X \) whose union is \( X \), there exist \( p \)-closed subsets \( G \) and \( H \) of \( X \) such that \( G \subseteq U \), \( H \subseteq V \) and \( G \cup H = X \).

(c) For any \( \pi \)-closed set \( A \) and each \( \pi \)-open set \( B \) such that \( A \subseteq B \), there exists a \( p \)-open set \( U \) such that \( A \subseteq U \subseteq p\text{cl}(U) \subseteq B \).

(d) For every pair of disjoint \( \pi \)-closed subsets \( A \) and \( B \) of \( X \), there exist \( p \)-open subsets \( U \) and \( V \) of \( X \) such that \( A \subseteq U \), \( B \subseteq V \) and \( p\text{cl}(U) \cap p\text{cl}(V) = \emptyset \).

Proof. (a) \( \Rightarrow \) (b). Let \( U \) and \( V \) be any \( \pi \)-open subsets of a quasi \( p \)-normal space \( X \) such that \( U \cup V = X \). Then, \( X \setminus U \) and \( X \setminus V \) are disjoint \( \pi \)-closed subsets of \( X \). By quasi \( p \)-normality of \( X \), there exist disjoint \( p \)-closed subsets \( U_1 \) and \( V_1 \) of \( X \) such that \( X \setminus U \subseteq U_1 \) and \( X \setminus V \subseteq V_1 \). Let \( G = X \setminus U_1 \) and \( H = X \setminus V_1 \). Then, \( G \) and \( H \) are \( p \)-closed subsets of \( X \) such that \( G \subseteq U \), \( H \subseteq V \) and \( G \cup H = X \).

(b) \( \Rightarrow \) (c). Let \( A \) be a \( \pi \)-closed and \( B \) be a \( \pi \)-open subset such that \( A \subseteq B \). Then, \( X \setminus A \) and \( B \) are \( \pi \)-open subsets of \( X \) whose union is \( X \). Then by (b), there exist \( p \)-closed sets \( G \) and \( H \) such that \( G \subseteq X \setminus A \), \( H \subseteq B \) and \( G \cup H = X \). Then, \( A \subseteq X \setminus G \), \( X \setminus B \subseteq X \setminus H \) and \( (X \setminus G) \cap (X \setminus H) = \emptyset \). Let \( U = X \setminus G \) and \( V = X \setminus H \). Then, \( U \) and \( V \) are disjoint \( p \)-open sets such that \( A \subseteq U \subseteq X \setminus V \subseteq B \). Since \( X \setminus V \) is \( p \)-closed, then we have \( p\text{cl}(U) \subseteq X \setminus V \). Thus, \( A \subseteq U \subseteq p\text{cl}(U) \subseteq B \).

(c) \( \Rightarrow \) (d). Let \( A \) and \( B \) be any disjoint \( \pi \)-closed subsets of \( X \). Then, \( A \subseteq X \setminus B \), where \( X \setminus B \) is \( \pi \)-open. By (c), there exists a \( p \)-open subset \( U \) of \( X \) such that \( A \subseteq U \subseteq p\text{cl}(U) \subseteq X \setminus B \). Let \( V = X \setminus p\text{cl}(U) \). Then, \( V \) is \( p \)-open subset of \( X \). Thus, we obtain \( A \subseteq U \), \( B \subseteq V \) and \( p\text{cl}(U) \cap p\text{cl}(V) = \emptyset \).

(d) \( \Rightarrow \) (a). It is obvious. \( \square \)

Now, we prove the following result.

**Theorem 2.8** The image of a quasi \( p \)-normal space under an open continuous injective function is a quasi \( p \)-normal.

Proof. Let \( X \) be a quasi \( p \)-normal space and let \( f : X \longrightarrow Y \) be an open continuous injective function. We need to show that \( f(X) \) is a quasi \( p \)-normal. Let \( A \) and \( B \) be any two disjoint \( \pi \)-closed sets in \( f(X) \). Since the inverse image of a \( \pi \)-closed set under an open continuous function is a \( \pi \)-closed, see Proposition 2.1 in [25], we have \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint \( \pi \)-closed sets in \( X \). By quasi \( p \)-normality of \( X \), there exist \( p \)-open subsets \( U \) and \( V \) of \( X \) such that \( f^{-1}(A) \subseteq U \), \( f^{-1}(B) \subseteq V \) and \( U \cap V = \emptyset \). Since \( f \) is an open continuous injective function, we have \( A \subseteq f(U) \), \( B \subseteq f(V) \) and \( f(U) \cap f(V) = \emptyset \). By the Proposition 3.2 in [27], we obtain \( f(U) \) and \( f(V) \) are disjoint \( p \)-open sets in \( f(X) \) such that \( A \subseteq f(U) \) and \( B \subseteq f(V) \). Hence, \( f(X) \) is quasi \( p \)-normal.
From the above Theorem, we obtain the following corollary.

**Corollary 2.9** Quasi $p$-normality is a topological property.

The following result shows that quasi $p$-normality is a hereditary property with respect to closed domain subspaces.

**Theorem 2.10** Quasi $p$-normality is a hereditary with respect to closed domain subspaces.

**Proof.** Let $M$ be a closed domain subspace of a quasi $p$-normal space $X$. Let $A$ and $B$ be any disjoint $\pi$-closed sets in $M$. Since $M$ is a closed domain subspace of $X$, then by the Proposition 2.1 in [25] we have $A$ and $B$ are disjoint $\pi$-closed subsets of $X$. By quasi $p$-normality of $X$, there exist disjoint $p$-open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. By the Lemma 3.9 in [27], we obtain $U \cap M$ and $V \cap M$ are disjoint $p$-open sets in $M$ such that $A \subseteq U \cap M$ and $B \subseteq V \cap M$. Hence, $M$ is quasi $p$-normal. □

Since every closed-and-open (clopen) subset is a closed domain, then we have the following corollary.

**Corollary 2.11** Quasi $p$-normality is a hereditary with respect to clopen subspaces.

**Definition 2.12** A space $X$ is called weakly $p$-regular, [16], if for each $x \in X$ and for each open domain subset $U$ of $X$ such that $x \in U$, there exists a $p$-open subset $V$ of $X$ such that $x \in V \subseteq p\operatorname{cl}(V) \subseteq U$.

Observe that for each $\pi$-open subset $U$ of a space $X$ such that $x \in U$, there exists an open domain $D$ in $X$ such that $D \subseteq U$ and $x \in D$. Thus, we have the following theorem that gives a useful characterization of weakly $p$-regular spaces and it can be proved easily.

**Theorem 2.13** A space $X$ is a weakly $p$-regular if and only if for each $\pi$-open subset $U$ of $X$ and each $x \in X$ with $x \in U$, there exists a $p$-open subset $V$ of $X$ such that $x \in V \subseteq p\operatorname{cl}(V) \subseteq U$.

Recall that a space $X$ is called sub-maximal, [17], if every dense subset of $X$ is an open subset. In a sub-maximal space, every $p$-open subset is an open.

**Definition 2.14** A space $X$ is called a $p_1$-paracompact, [14], if every $p$-open cover of $X$ has a locally finite open refinement.
Clearly, every $p_1$-paracompact space is a paracompact. Now, we prove the following result, which is analogous to the Theorem 5.5 in [16].

**Theorem 2.15** Every weakly $p$-regular $p_1$-paracompact space is $\pi p$-normal (hence a quasi $p$-normal).

**Proof.** Let $X$ be a weakly $p$-regular $p_1$-paracompact space. Since $X$ is $p_1$-paracompact, then it is sub-maximal and paracompact, see Theorem 5.5 in [16]. Since $X$ is sub-maximal and weakly $p$-regular, then it is a weakly regular. In view of that fact that every weakly regular paracompact space is $\pi$-normal, see [24], we obtain $X$ is $\pi$-normal. Hence, $X$ is $\pi p$-normal. Therefore, $X$ is quasi $p$-normal. □

Since every almost $p$-regular is a weakly $p$-regular, we have the following corollary.

**Corollary 2.16** Every almost $p$-regular $p_1$-paracompact space is a $\pi p$-normal (hence quasi $p$-normal).

The following result is useful for giving some other characterizations of quasi $p$-normal spaces.

**Theorem 2.17** For a space $X$, the following are equivalent:

(a) $X$ is quasi $p$-normal.

(b) For any disjoint $\pi$-closed subsets $A$ and $B$ of $X$, there exist disjoint gp-open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

(c) For any disjoint $\pi$-closed subsets $A$ and $B$ of $X$, there exist disjoint $\pi gp$-open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

(d) For every $\pi$-closed set $A$ and every $\pi$-open set $B$ such that $A \subseteq B$, there exists a gp-open subset $V$ of $X$ such that $A \subseteq V \subseteq \text{p cl}(V) \subseteq B$.

(e) For every $\pi$-closed set $A$ and every $\pi$-open set $B$ such that $A \subseteq B$, there exists a $\pi gp$-open subset $V$ of $X$ such that $A \subseteq V \subseteq \text{p cl}(V) \subseteq B$.

**Proof.** (a) $\implies$ (b). Let $X$ be a quasi $p$-normal space. Let $A$ and $B$ be any disjoint $\pi$-closed subsets of $X$. By quasi $p$-normality of $X$, there exist disjoint $p$-open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. Thus, $U$ and $V$ are disjoint gp-open subsets of $X$ such that $A \subseteq U$ and $B \subseteq V$.

(b) $\implies$ (c). Suppose (b) holds. Let $A$ and $B$ be disjoint $\pi$-closed subsets of $X$. By (b), there exist disjoint gp-open subsets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. Since every gp-open set is a $\pi gp$-open, then $U$ and $V$ are disjoint $\pi gp$-open subsets of $X$ such that $A \subseteq U$ and $B \subseteq V$. 

(c) \implies (d). Suppose (c) holds. Let \( A \) be a \( \pi \)-closed and \( B \) be a \( \pi \)-open subset of \( X \) such that \( A \subseteq B \). Then, \( A \cap X \setminus B = \emptyset \). Thus, \( A \) and \( X \setminus B \) are disjoint \( \pi \)-closed subsets of \( X \). By (c), there exists disjoint \( \pi gp \)-open subsets \( U \) and \( V \) of \( X \) such that \( A \subseteq U \) and \( X \setminus B \subseteq V \). Therefore, we have \( A \subseteq \text{p int}(U), X \setminus B \subseteq \text{p int}(V) \) and \( \text{p int}(U) \cap \text{p int}(V) = \emptyset \). Let \( G = \text{p int}(U) \). Then, \( G \) is a \( p \)-open subset of \( X \) and hence \( gp \)-open such that \( A \subseteq G \subseteq \text{p cl}(G) \subseteq B \).

(d) \implies (e). Suppose (d) holds. Let \( A \) be a \( \pi \)-closed and \( B \) be a \( \pi \)-open subset of \( X \) such that \( A \subseteq B \). By (d), there exists a \( gp \)-open subset \( V \) of \( X \) such that \( A \subseteq V \subseteq \text{p cl}(V) \subseteq B \). Therefore, \( V \) is a \( \pi gp \)-open subset of \( X \) such that \( A \subseteq V \subseteq \text{p cl}(V) \subseteq B \).

(e) \implies (a). Suppose (e) holds. Let \( A \) and \( B \) be disjoint \( \pi \)-closed subsets of \( X \). Then, we have \( A \subseteq X \setminus B \) where \( X \setminus B \) is \( \pi \)-open. By (e), there exists a \( \pi gp \)-open subset \( V \) of \( X \) such that \( A \subseteq V \subseteq \text{p cl}(V) \subseteq X \setminus B \). Then, we obtain \( A \subseteq \text{p int}(V) \subseteq V \subseteq \text{p cl}(V) \subseteq X \setminus B \). Let \( G = \text{p int}(V) \) and \( H = X \setminus \text{p cl}(V) \). Then, \( G \) and \( H \) are disjoint \( p \)-open subsets of \( X \) such that \( A \subseteq G \) and \( B \subseteq H \). Hence, \( X \) is quasi \( p \)-normal. \( \square \)

The following definitions are in [1], [2], [9], [16], [17], [18] and [22].

**Definition 2.18** A function \( f : X \longrightarrow Y \) is said to be:

(a) almost continuous (resp. rc-continuous) if \( f^{-1}(F) \) is a closed (resp. closed domain) set in \( X \) for each closed domain subset \( F \) of \( Y \).

(b) \( \pi \)-continuous if \( f^{-1}(F) \) is \( \pi \)-closed set in \( X \) for each closed subset \( F \) of \( Y \).

(c) almost closed (resp. rc-preserving) function if \( f(U) \) is closed (resp. closed domain) set in \( Y \) for each closed domain subset \( U \) of \( X \).

(d) weakly open if for each open subset \( U \) of \( X \), \( f(U) \subseteq \text{int}(f(U)) \).

(e) pre \( gp \)-continuous if \( f^{-1}(F) \) is \( gp \)-closed in \( X \) for every \( p \)-closed subset \( F \) of \( Y \).

(f) \( R \)-map (resp. completely continuous) if \( f^{-1}(V) \) is open domain in \( X \) for every open domain (resp. open) subset \( V \) of \( Y \).

(g) pre \( gp \)-closed if \( f(F) \) is \( gp \)-closed set in \( Y \) for every \( p \)-closed subset \( F \) of \( X \).

(h) almost pre-irresolute if for each \( x \in X \) and each \( p \)-neighborhood \( V \) of \( f(x) \) in \( Y \), \( \text{p cl}(f^{-1}(V)) \) is a \( p \)-neighborhood of \( x \) in \( X \).

(i) \( Mp \)-closed (\( Mp \)-open) if \( f(U) \) is \( p \)-closed (resp. \( p \)-open) set in \( Y \) for each \( p \)-closed (resp. \( p \)-open) set \( U \) in \( X \).
The following lemma is in [17].

**Lemma 2.19** If a function \( f : X \rightarrow Y \) is weakly open continuous function, then \( f \) is \( Mp \)-open and \( R \)-map.

Clearly, every pre-irresolute function is an almost pre-irresolute and we have:

\[ \pi \text{-continuous} \Rightarrow \text{continuous} \Rightarrow p \text{-continuous} \Rightarrow gp \text{-continuous} \]

Next, we prove the invariance of quasi \( p \)-normality in the following.

**Theorem 2.20** If \( f : X \rightarrow Y \) is an \( Mp \)-open \( rc \)-continuous and almost pre-irresolute function from a quasi \( p \)-normal space \( X \) onto a space \( Y \), then \( Y \) is quasi \( p \)-normal.

**Proof.** Let \( A \) be a \( \pi \)-closed and \( B \) be a \( \pi \)-open subsets of \( Y \) such that \( A \subseteq B \). Then by \( rc \)-continuity of \( f \), \( f^{-1}(A) \) is \( \pi \)-closed and \( f^{-1}(B) \) is \( \pi \)-open subsets of \( X \) such that \( f^{-1}(A) \subseteq f^{-1}(B) \). Since \( X \) is quasi \( p \)-normal, then by the Theorem 2.7 there exists a \( p \)-open subset \( V \) of \( X \) such that \( f^{-1}(A) \subseteq V \subseteq p\text{cl}(V) \subseteq f^{-1}(B) \). Since \( f \) is \( Mp \)-open and an almost pre-irresolute surjection, it follows that \( f(V) \) is \( p \)-open subset of \( Y \) and \( A \subseteq f(V) \subseteq p\text{cl}(f(V)) \subseteq B \). Hence by the Theorem 2.7, \( Y \) is quasi \( p \)-normal. \( \square \)

**Theorem 2.21** If \( f : X \rightarrow Y \) is a weakly open \( \pi \)-continuous almost pre-irresolute surjection and \( X \) is quasi \( p \)-normal, then \( Y \) is quasi \( p \)-normal.

**Proof.** Let \( A \) be a \( \pi \)-closed subset of \( Y \) and let \( B \) be a \( \pi \)-open subsets of \( Y \) such that \( A \subseteq B \). By \( \pi \)-continuity of \( f \), we have \( f^{-1}(A) \) is \( \pi \)-closed and \( f^{-1}(B) \) is \( \pi \)-open subset of \( X \) such that \( f^{-1}(A) \subseteq f^{-1}(B) \). By quasi \( p \)-normality of \( X \), there exists a \( p \)-open subset \( U \) of \( X \) such that \( f^{-1}(A) \subseteq U \subseteq p\text{cl}(U) \subseteq f^{-1}(B) \). Then, \( f(f^{-1}(A)) \subseteq f(U) \subseteq f(p\text{cl}(U)) \subseteq f(f^{-1}(B)) \). Since \( f \) is a weakly open continuous almost pre-irresolute surjection, then by the Lemma 2.19, we have \( f \) is \( Mp \)-open and \( R \)-map. Thus, we have \( f(U) \) is a \( p \)-open subset of \( Y \) such that \( A \subseteq f(U) \subseteq p\text{cl}(f(U)) \subseteq B \). Hence by the Theorem 2.7, \( Y \) is quasi \( p \)-normal. \( \square \)

**Theorem 2.22** If \( f : X \rightarrow Y \) is a \( \pi \)-continuous, weakly open pre \( gp \)-closed surjection and \( X \) is quasi \( p \)-normal, then \( Y \) is \( \pi p \)-normal.

**Proof.** Let \( A \) and \( B \) be any disjoint closed subsets of \( Y \) such that \( A \) is \( \pi \)-closed. Since \( f \) is \( \pi \)-continuous surjection, then \( f^{-1}(A) \) and \( f^{-1}(B) \) are disjoint \( \pi \)-closed subsets of \( X \). Since \( X \) is quasi \( p \)-normal, then there exist disjoint \( p \)-open subsets \( U \) and \( V \) of \( X \) such that \( f^{-1}(A) \subseteq U \) and \( f^{-1}(B) \subseteq V \). Since \( f \) is a weakly open continuous surjection, then by the Lemma 2.19, we have \( f \) is \( Mp \)-open and \( R \)-map. Thus, \( f(U) \) and \( f(V) \) are disjoint \( p \)-open subsets of \( Y \).
such that \( A \subseteq f(U) \) and \( B \subseteq f(V) \). Hence, \( Y \) is \( \pi p \)-normal. \( \Box \)

The following theorems can be proved easily by using arguments similar to those in Theorem 2.20 and the Theorem 2.21.

**Theorem 2.23**  The following statements are true:

(a) If \( f : X \rightarrow Y \) is \( rc \)-continuous, \( Mp \)-closed map from a quasi \( p \)-normal space \( X \) onto a space \( Y \), then \( Y \) is quasi \( p \)-normal.

(b) If \( f : X \rightarrow Y \) is an \( R \)-map \( pre \ gp \)-closed surjection and \( X \) is quasi \( p \)-normal, then \( Y \) is quasi \( p \)-normal.

(c) If \( f : X \rightarrow Y \) is a completely continuous \( pre \ gp \)-closed surjection and \( X \) is quasi \( p \)-normal, then \( Y \) is \( p \)-normal.

(d) If \( f : X \rightarrow Y \) is almost continuous \( pre \ gp \)-closed surjection and \( X \) is \( p \)-normal, then \( Y \) is quasi \( p \)-normal.

(e) If \( f : X \rightarrow Y \) is \( \pi \)-continuous weakly open \( pre \ gp \)-closed surjection and \( X \) is quasi \( p \)-normal, then \( Y \) is \( p \)-normal.

(f) If \( f : X \rightarrow Y \) is \( pre \ gp \)-continuous \( rc \)-preserving injection and \( Y \) is quasi \( p \)-normal, then \( X \) is quasi \( p \)-normal.

(g) If \( f : X \rightarrow Y \) is \( pre \ gp \)-continuous almost closed injection and \( Y \) is \( p \)-normal, then \( X \) is quasi \( p \)-normal.

### 3 Conclusion

We used generalized closed (open) sets to obtain various characterizations and preservation theorems of quasi \( p \)-normality. Some properties, examples and results on quasi \( p \)-normal spaces were given.

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**References**


On quasi $p$-normal spaces


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