Abstract
In this paper, we prove a fixed point theorem for six mappings in a Menger space using the notion of weak compatibility. This is a generalization and an improvement of a Theorem of B.D.Pant and Sunny Chauhan[2].

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1 Introduction

The concept of a metric space enabled a number of generalizations. An interesting one is Menger space initiated by Menger[3]. Schweizer and Sklar[4] had
a critical look on this and proved fundamental results on this.

The concept of semi-compatibility is introduced by Singh and Jain[5]. Jungck and Rhoades[1] introduced weak-compatibility. It is observed that semi compatibility pair of self maps is weakly compatible but the converse is false [5].

2 Preliminaries

We follow the standard definitions and results given in [4]. In fact, we mainly use the following results in the subsequent section.

Result 2.1.[4]
Let \( \{x_n\} (n = 0, 1, 2, \ldots) \) be a sequence in a Menger space \((X, F, T)\) with continuous t-norm \(T\) and \(T(a, a) \geq a\) for all \(a \in [0, 1]\). If there is a constant \(k \in (0, 1)\) such that
\[
F_{x_n,x_{n+1}}(kt) \geq F_{x_{n-1},x_n}(t)
\]
for all \(t > 0\) and \(n = 0, 1, 2, \ldots\), then \(\{x_n\}\) is a Cauchy sequence in \(X\).

Observation 2.2.
Min norm \(\ast\) is a continuous t-norm with \(\ast(a, a) \geq a\).

Result 2.3.[5]
Let \((X, F, T)\) be a Menger space. If there is a \(k \in (0, 1)\) such that
\[
F_{x,y}(kt) \geq F_{x,y}(t)
\]
for all \(x, y \in X\) and \(t > 0\), then \(y = x\).

3 Main Result

Now, we state below the Theorem of [2].

Theorem 3.1. (Theorem 3.1 of [2]) Let \(A, B, S, T, L\) and \(M\) be self maps on a complete Menger space \((X, F, T)\) with continuous t-norm \(T\) such that \(T(a, b) = \min\{a, b\}\), for all \(a, b \in [0, 1]\) and satisfying the following:
(a) $AB(X) \subseteq M(X)$ and $ST(X) \subseteq L(X)$;
(b) $M(X)$ and $L(X)$ are complete subspaces of $X$;
(c) either $AB$ or $ST$ is continuous;
(d) The pairs $\{AB, L\}$ is semi compatible and $\{ST, M\}$ is weakly compatible;
(e) For all $x, y \in X$, for all $u > 0$, $k \in (0, 1)$,
   \[
   F_{ABx,STy}^3(ku) \geq Min\left\{ F_{Lx,My}^3(u), F_{ABx,Lx}^3(u), F_{STy,My}^3(u),
                       F_{ABx,My}^2(2u), F_{STy,Lx}(2u), F_{STy,My}(u) \right\}.
   \]

Then $AB$, $ST$, $L$ and $M$ have a unique common fixed point in $X$.

**Remark 3.2.** The term $F_{STy,My}^2(u)$ in the R.H.S. of the inequality (e) in the above Theorem is unnecessary, since it is $\geq F_{STy,My}^3(u)$.

Now, we prove the following generalization of the above theorem. Further, our conditions in the Theorem are more realistic than theirs. In fact, our result is also valid with their conditions. We further illustrate our result by an example.

**Theorem 3.3.** Let $A$, $B$, $S$, $T$, $L$ and $M$ be self mappings on a Menger space $(X, F, *)$ where $*$ is the min t-norm and satisfying:

(3.3.1) $AB(X) \subseteq M(X)$ and $ST(X) \subseteq L(X)$;
(3.3.2) Either $AB(X)$ or $M(X)$ or $ST(X)$ or $L(X)$ is a complete subspace of $X$;
(3.3.3) The pairs $\{ST, M\}$ and $\{AB, L\}$ are weakly compatible;
(3.3.4) $ST = TS$ and $AB = BA$;
(3.3.5) 'either $MT = TM$ or $MS = SM$' and 'either $LA = AL$ or $LB = BL$';
(3.3.6) there is a $k \in (0, 1)$ such that
   \[
   F_{ABx,STy}^m(ku) \geq F_{ABx,Lx}^m(u) \ast F_{STy,My}^m(u) \ast F_{Lx,My}^m(u) \ast
                       F_{ABx,My}^m(\alpha u) \ast F_{STy,Lx}^m((2 - \alpha)u)
   \]
   for all $x, y \in X$, for all $u > 0$, for all $\alpha \in (0, 2)$ and for some positive integer $m$. 

Then $A, B, S, T, L$ and $M$ have a unique common fixed point in $X$.

**Proof:** Let $x_0 \in X$. By (3.3.1) we construct sequences $\{x_n\}$ and $\{y_n\}$ in $X$ such that

$$ABx_{2n} = M_{2n+1} = y_{2n} (\text{say})$$

and $STx_{2n+1} = Lx_{2n+2} = y_{2n+1} (\text{say}), \text{for } n = 0, 1, 2, \ldots$

As in the proof of Theorem (3.1) of [2], using Result (2.1), it can be shown that $\{y_n\}$ is a Cauchy sequence in $X$ and so the subsequences $\{y_{2n}\}$ and $\{y_{2n+1}\}$ are also Cauchy in $X$.

**Case 1:** Suppose that either $AB(X)$ or $M(X)$ is a complete subspace of $X$.

Since $\{y_{2n}\} \subset AB(X) \subset M(X)$, there is a $z \in X$ such that $y_{2n} \to z$ as $n \to \infty (\Rightarrow y_{2n+1} \to z$ as $n \to \infty$).

Clearly, $z \in M(X)$. So, there is a $v \in X$ such that $z = Mv$.

Taking $x = x_{2n}(n \geq 1), y = v$ and $\alpha = 1$ in (3.3.6), we get that

$$F_{y_{2n},STv}(ku) \geq F_{y_{2n},y_{2n-1}}^m(u) * F_{STv,z}^m(u) * F_{y_{2n},z}^m(u) * F_{STv,y_{2n-1}}^m(u).$$

Now, as $n \to \infty$, we get that

$$F_{z,STv}^m(ku) \geq F_{STv,z}^m(u) \geq F_{STv,z}^m(u)$$

and this is true for all $u > 0. \Rightarrow STv = z (= Mv)$.

Since $\{ST, M\}$ is weakly compatible, follows that $MSTv = STMv$; i.e., $Mz = STz$.

Taking $x = x_{2n}, y = Mz, \alpha = 1$ and using the fact that $STz = Mz$ in (3.3.6), we get that

$$F_{y_{2n},Mz}^m(ku) \geq F_{y_{2n},y_{2n-1}}^m(u) * F_{Mz,Mz}^m(u) * F_{y_{2n-1},Mz}^m(u) * F_{y_{2n},Mz}^m(u) * F_{Mz,y_{2n}}^m(u).$$

Now, as $n \to \infty$, we get that

$$F_{z,Mz}^m(ku) \geq F_{z,Mz}^m(u) \geq F_{z,Mz}^m(u)$$

and this is true for all $u > 0. \Rightarrow (STz =) Mz = z$.

Since $ST = TS$, we have $ST(Tz) = TS(Tz) = T(STz) = Tz$.

Now, taking $x = x_{2n}(n \geq 1), y = Tz$ and $\alpha = 1$ in (3.3.6), we get that

$$F_{y_{2n},Tz}^m(ku) \geq F_{y_{2n},y_{2n-1}}^m(u) * F_{Tz,Mz}^m(Tz) * F_{y_{2n-1},Mz}^m(Tz) * F_{y_{2n},M(Tz)}^m(Tz) * F_{Tz,y_{2n-1}}^m(u).$$

Now, suppose that $MT = TM$; so we have

$$F_{y_{2n},Tz}^m(ku) \geq F_{y_{2n},y_{2n-1}}^m(u) * F_{Tz,Tz}^m(u) * F_{y_{2n-1},Tz}^m(u) * F_{y_{2n},Tz}^m(u) * F_{Tz,y_{2n-1}}^m(u).$$
As \( n \to \infty \), we get that \( F_{x,Tz}^m(ku) \geq F_{z,Tz}^m(u) \) and this is true for all \( u > 0 \). So \( Tz = z \). Thus \( Mz = Sz = Tz = z \).

Similar is the case when \( MS = SM \). There we first show that \( Sz = z \).

Since \( ST(X) \subseteq L(X) \), there is a \( w \in X \) such that \( z = Lw \).

Similarly, by taking \( x = w, y = x_{2n+1} \) and \( \alpha = 1 \) in (3.3.6), we get that

\[
F_{ABw,y_{2n+1}}^m(ku) \geq F_{ABw,z}^m(u) * F_{y_{2n+1},y_{2n}}^m(u) * F_{z,y_{2n}}^m(u) * F_{ABw,y_{2n}}^m(u) * F_{y_{2n+1},z}^m(u).
\]

Now, as \( n \to \infty \), we get that \( F_{ABw,z}^m(ku) \geq F_{ABw,z}^m(u) \), for all \( u > 0 \).

\[ \Rightarrow ABw = z (= Lw). \]

Since \( \{AB, L\} \) is weakly compatible, \( AB(Lw) = L(ABw) = Lz; \)
i.e, \( ABz = Lz \).

Now, taking \( x = z, y = x_{2n+1}, \alpha = 1 \) in (3.3.6) and using \( ABz = Lz \), we get that

\[
F_{Lz,y_{2n+1}}^m(ku) \geq F_{Lz,Lz}^m(u) * F_{y_{2n+1},y_{2n}}^m(u) * F_{Lz,y_{2n}}^m(u) * F_{Lz,y_{2n}}^m(u) * F_{y_{2n+1},Lz}^m(u).
\]

Now, as \( n \to \infty \), we get that \( F_{Lz,z}^m(ku) \geq F_{Lz,z}^m(u) \), for all \( u > 0 \).

\[ \Rightarrow Lz = z. \text{Thus} (ABz =) Lz = z. \]

Since \( AB = BA \), we have \( AB(Az) = A(BA)z = A(ABz) = Az. \)

Suppose \( LA = AL \), so \( L(Az) = (LA)z = (AL)z = A(Lz) = Az \).

Now, taking \( x = Az, y = x_{2n+1}, \alpha = 1 \) in (3.3.6) and using \( AB(Az) = Az \), we get that

\[
F_{Az,y_{2n+1}}^m(ku) \geq F_{Az,Az}^m(u) * F_{y_{2n+1},y_{2n}}^m(u) * F_{Az,y_{2n}}^m(u) * F_{Az,y_{2n}}^m(u) * F_{y_{2n+1},Az}^m(u).
\]

Now, as \( n \to \infty \), we get that \( F_{Az,z}^m(ku) \geq F_{Az,z}^m(u) \), for all \( u > 0 \). \( \Rightarrow Az = z. \)

Since \( AB = BA \), we have \( z = (AB)z = B(Az) = Bz \). Thus \( Az = Bz = Lz = z \).

Hence, \( Az = Bz = Lz = Mz = Sz = Tz = z. \)

Similar is the case when \( LB = BL \). There we first show that \( Bz = z. \)

**Case 2 :** Suppose that either \( ST(X) \) or \( L(X) \) is a complete subspace of \( X \).
Here we first get that \( Az = Bz = Lz = z \) and then \( Mz = Sz = Tz = z. \)

Hence, \( Az = Bz = Lz = Mz = Sz = Tz = z. \) i.e, \( z \) is a common fixed point for \( A, B, S, T, L \) and \( M \).

Uniqueness follows trivially by taking \( \alpha = 1. \)
Example 3.4. \((X,F,*)\) is a Menger space, where \(X = [0, \infty)\) with the usual metric and \(F : \mathbb{R} \to [0, 1]\) is defined by
\[
F_{x,y}(u) = \frac{u}{u + |x - y|}
\]
for all \(x, y \in X\) and \(*\) is the min t-norm, i.e, \(a*b = \min\{a, b\}\) for all \(a, b \in [0, 1]\). Let \(A, B, S, T, L\) and \(M\) be the self maps on \(X\), defined by
\[
A(x) = \begin{cases} 
0 & \text{if } x \leq 9, \\
1 & \text{if } x > 9.
\end{cases}
\]
\[
S(x) = \begin{cases} 
0 & \text{if } x \leq 9, \\
4 & \text{if } x > 9.
\end{cases}
\]
\(Bx = Tx = Mx = x\) and \(Lx = x^2\) for all \(x \in X\). Then, clearly \(A, B, S, T, L\) and \(M\) satisfy the hypothesis of Theorem (3.3) with \(k \in \left[\frac{8}{9}, 1\right) \subset (0, 1)\).

For, when \(x, y \in [0, 9]\), \(F_{ABx,STy}(ku) = 1\).
When \(x \in [0, 9], y \in (9, \infty)\), \(F_{ABx,STy}(ku) = \frac{u}{u + \frac{k}{x}}\) and \(F_{ABx,My}(au) = \frac{u}{u + \frac{k}{x}}\).
When \(x \in (9, \infty), y \in [0, 9]\), \(F_{ABx,STy}(ku) = \frac{u}{u + x}\) and \(F_{ABx,La}(u) = \frac{u}{u + x^2 - 1}\).
When, \(x, y \in (9, \infty)\), \(F_{ABx,STy}(ku) = \frac{u}{u + x}\) and \(F_{ABx,La}(u) = \frac{u}{u + x^2 - 1}\).

So, in (3.3.6), L.H.S \(\geq\) R.H.S when \(k \geq \max\{\frac{8}{9}, \frac{1}{80}, \frac{3}{80}\}\), that is \(k \geq \frac{8}{9}\).
Clearly 0 is the unique common fixed point of \(A, B, S, T, L\) and \(M\).

(Observe that neither \(AB\) nor \(ST\) is continuous on \(X\).)

References


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