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Fixed Points of Occasionally Weakly Biased Mappings Under Contractive Conditions

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Abstract

Some common fixed point theorems due to Pant and Pant (R. P. Pant and V. Pant, Common fixed points under strict contractive conditions, J. Math. Anal. Appl., **248**(2000), 327-332), and Aamri and Moutawakil (M. Aamri and D. El Moutawakil, Some new common fixed point theorems under strict contractive conditions, J. Math. Anal. Appl., **270**(2002), 181-188) are extended to occasionally weakly biased mappings pair. Illustrative examples are also provided to justify the improvement of our results.

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1 Introduction

In 1982, Sessa [13] initiated the concept of weakly commuting maps as a generalization of commuting maps [4]. The study of common fixed points of compatible mappings is an active area of many authors since Jungck [5] introduced the notion of compatible mappings. Pant [8, 9] studied common fixed points of non-compatible mappings and introduced R-weakly commuting mappings. Non-compatible mappings can be extended to the class of non-expansive or Lipshitz type of mappings. Work along these lines has been recently initiated by Pant [8, 9]. Aamri and Matouwakil[1] introduced the concept of property (E.A)(or tangential maps, also see Sastry and Murthy [12]) and thus generalized the notion of non-compatible mappings. In [2] Al-Thagafi and Shahzad introduced the concept of occasionally weakly compatible mappings (woc) which is more general than the concept of weakly compatible mappings. Recently Hussain et al.[3] introduced the concept of occasionally weakly biased mappings in order to generalize weakly biased[6] (respectively, occasionally weakly compatible mappings[2]).

In this paper we extend Theorems 2.2 and 2.3 of Pant and Pant [10], and Theorems 1 and 2 of Aamri and Moutawakil [1] to occasionally weakly biased mappings.

2 Preliminary

Let f and g be two self mappings of a metric space (X, d). We denote $C(f, g) = \{u \in X : fu = gu\}$. The following definitions are given in details in Sessa[13], Jungck[5], Pant[8, 9], Aamri and Moutawakik[1], Pant[10], Pathak[11],Jungck and Rhoades[7], Jungck and Pathak[6], Thagafi and N. Shazad[2], and Hussain et al.[3] respectively.

Definition 2.1 *Mappings f and g are called weakly commuting if*

$$d(fgx, gfx) \leq d(fx, gx), \forall x \in X.$$

Definition 2.2 *Mappings f and g are called compatible mappings if*

$$\lim_{n\to\infty} d(fgx_n,\, gfx_n) = 0,$$

whenever, $\{x_n\}$ is a sequence in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$.

Note that mappings f and g are non-compatible(Pant[8, 9]) if there exists at least one sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$ for some $t \in X$ but $\lim_{n\to\infty} d(fgx_n, gfx_n)$ is either non-zero or non-existent.

Definition 2.3 *Mappings f and g are said to satisfy property (E.A) if there exists a sequence $\{x_n\}$ in X such that $\lim_{n\to\infty} fx_x = \lim_{n\to\infty} gx_n = t$, for some $t \in X$.*

Definition 2.4 *Mappings f and g are said called R-weakly commuting at a point x in X if $d(fgx, gfx) \leq Rd(fx, gx)$, for some $R > 0$.*

Definition 2.5 *Mappings f and g are called pointwise R-weakly commuting on X if given x in X there exists $R > 0$ such that*

$$d(fgx, gfx) \leq Rd(fx, g).$$

Definition 2.6 *Mappings f and g are called R-weakly commuting of type (A_f), if there exists a positive real number R such that*

$$d(fgx, ggx) \leq Rd(fx, gx), \forall x \in X.$$

Definition 2.7 *Mappings f and g are called R-weakly commuting of type (A_g), if there exists a positive real number R such that*

$$d(ffx, gfx) \leq Rd(fx, gx), \forall x \in X.$$

Definition 2.8 *Mappings f and g are called weakly compatible if $fgx = gfx$, $\forall x \in C(f, g)$.*

Definition 2.9 *Mappings f and g are called weakly g-biased if $d(gfx, gx) \leq d(fgx, fx)$, $\forall x \in C(f, g)$.*

Definition 2.10 *Mappings f and g are called occasionally weakly compatible(owc) if $fgx = gfx$,for some $x \in C(f, g)$.*

Definition 2.11 *Mappings f and g are called weakly occasionally g-biased if $d(gfx, gx) \leq d(fgx, fx)$, for some $x \in C(f, g)$.*

It may be noted that weakly commuting, R-weakly commuting, R-weakly commuting of type (A_f) or (A_g), weakly compatible and owc for a pair of self mappings agree only at their coincidence points i.e. commuting at their coincidence points. However, the notion of weakly biased/occasionally weakly biased does not assure the commutativity at the coincidence points. Therefore, the study of common fixed points for weakly biased and occasionally weakly biased are equally interesting. In [14], it has shown that weakly compatible implies weakly biased in the strong sense but the converse is not true in general. Further, in [3], Hussain et al. has shown that woc and non-trivial weakly g-biased are occasionally weakly g-biased pair but the converse does not hold in general.

Example 2.12 *Let $f, g : X \to X$, where $X = R$, the set of real numbers with usual metric d on X. Define $f = 1 - 2x$ and $g = 2x$. Clearly, f and g are not commuting for all $x \in X$. But $C(f, g) = \{\frac{1}{4}\}$ and $f\frac{1}{4} = g\frac{1}{4} = \frac{1}{2} \Rightarrow d(gf\frac{1}{4}, g\frac{1}{4}) \leq d(fg\frac{1}{4}, f\frac{1}{4})$ i.e. f and g are occasionally weakly g-biased mappings pair, however $0 = fg\frac{1}{4} \neq gf\frac{1}{4} = 1$. The mappings f and g are neither weakly commuting, R-weakly commuting, weakly compatible nor owc but weakly biased and hence occasionally weakly g-biased.*

Example 2.13 *Let $X = R$ with usual metric d and $M = [0, 1] \subset X$. Define $f, g : M \to M$ as $fx = 1 - \frac{x}{2}$ and $gx = \frac{x}{2}$ and $f0 = g0 = \frac{2}{3}$. Then, $C(f, g) = \{0, 1\}$. $fg0 = f\frac{2}{3} = \frac{2}{3} \neq gf0 = \frac{1}{3}$ and $fg1 = \frac{3}{4} \neq gf1 = \frac{1}{4}$. Moreover $|gf0 - g0| = \frac{1}{3} \nleq |fg0 - f0| = 0$ and $|gf1 - g1| = \frac{1}{4} \leq |fg1 - f1| = \frac{1}{4}$. The mappings f and g are occasionally weakly g-biased but neither weakly g-biased(respectively, occasionally weakly compatible, R-weakly commuting) nor weakly compatible.*

3 Main Results

Following theorem extend Theorem 1 of Aamri and Moutawakil [1], and Theorem 2.2 of Pant and Pant[10] .

Theorem 3.1 *Let f and g be two self mappings of a metric space (X, d) and gX is closed in X such that*
(i) f and g satisfy property(E.A);
(ii) for every $x \neq y \in X$

$$d(fx, fy) < max\{d(gx, gy), [d(fx, gx) + d(fy, gy)]/2,$$
$$[d(fy, gx) + d(fx, gy)]/2\}$$

then, $C(f, g) \neq \phi$. If f and g are occasionally weakly g-biased then, f and g have a unique common fixed point.

Proof. Since f and g satisfy property (E.A), there exists a sequence in X such that $fx_n, gx_n \to t$ for some $t \in X$. Closeness of gX and $t \in X$, there exists $u \in X$ such that $t = gu$. We claim that $fu = gu$. If $fu \neq gu$, by (ii), we obtain

$$d(fx_n, fu) < max\{d(gx_n, gu), [d(fx_n, gx_n) + d(fu, gu)]/2,$$
$$[d(fu, gx_n) + d(fx_n, gu)]/2\}$$

On letting $n \to \infty$, we obtain

$$d(gu, fu) < \tfrac{1}{2}d(fu, gu)$$

which is a contradiction and hence $fu = gu$. Therefore, $C(f,g) \neq \phi$. Since f and g are occasionally weakly g-biased mappings, there exists a point $v \in C(f,g)$ such that $fv = gv$. Also, $fv = gv$ yields $ffv = fgv$ and $gfv = ggv$. Now we show that $ffv = fv$, otherwise by (ii) and occasionally weakly g-biased, we obtain

$$\begin{aligned} d(ffv, fv) &< max\{d(gfv, gv), [d(ffv, gfv) + d(fv, gv)]/2, \\ &\qquad [d(fv, gfv) + d(ffv, gv)]/2\} \\ &\leq max\{d(gfv, gv), [d(ffv, fv) + d(gfv, gv)]/2, \\ &\qquad \tfrac{1}{2}[d(gfv, gv) + d(ffv, fv)]\} \\ &\leq max\,\{d(ffv, fv), d(ffv, fv), d(ffv, fv)\} \\ &= d(ffv, fv), \end{aligned}$$

a contradiction. Therefore, $ffv = fv$. By occasionally weakly g-biased pair, we obtain

$$d(gfv, gv) \leq d(fgv, fv) = d(ffv, fv) = 0,$$

which in turn gives $gfv = fv$. Therefore, fv is a common fixed point of f and g. The uniqueness of common fixed point is immediately followed from (ii). This completes the proof.

Now we give the following example to verify the validity of above theorem.

Example 3.2 *Let $X = R$ and $M = [0, 1) \subset R$ with usual metric d. Define $f, g : X \to X$ by $fx = 1/2, x \leq 1/2, fx = 3/5, x > 1/2$ and $gx = 1/3 + x, x < 1/2, g1/2 = 1/2, gx = 2/5, x > 1/2$. Clearly, $gX = [1/3, 5/6]$ is closed in X. Mappings f and g satisfy property (E.A). For this, taking $\{x_n\}$ be a sequence in X where $x_n > 0$, $n = 1, 2, 3, ...$ such that $x_n \to 1/6$ as $n \to \infty$. Consequently, $fx_n, gx_n \to 1/2$ as $n \to \infty$. One can verify that the contractive condition (ii) holds for every $x \neq y \in X$. Also, we have $C(f,g) = \{1/6, 1/2\}$ and $f1/6 = 1/2 = g1/6 \Rightarrow d(gf1/6, g1/6) \leq d(fg1/6, f1/6)$ i.e. f and g are occasionally weakly g-biased mappings pair. Thus, all the conditions of theorem are satisfied and $1/2$ is the unique common fixed point of f and g.*

Let $F : R^+ \to R^+$ satisfy the following conditions:
(i) F is nondecreasing on R^+;
(ii) $0 < F(t) < t, \forall t > 0$.

Proposition 3.3 *Let A, B, S, T be self mappings of a metric space (X, d) satisfying the following conditions*

(i) $BX \subseteq SX$ and TX is closed in X,

(ii) the pair (B, T) satisfy property (E.A) and

(iii) for every $x, y \in X$

$$d(Ax, By) \leq F(max\{d(Sx, Ty), [d(Ax, Sx) + d(By, Ty)]/2,$$
$$[d(Ay, Ty) + d(By, Sx)]/2\}),$$

then, $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$.

Proof. Suppose the pair (B, T) satisfies property (E.A), there exists a sequence $\{x_n\}$ in X such that $Bx_n, Tx_n \to t \in X$ for some $t \in X$. Since, $BX \subseteq SX$, for every x_n there exists y_n in X such that $Bx_n = Sy_n$. Therefore, $\lim_{n\to\infty} Sy_n \to t$.

We claim that $\lim_{n\to\infty} Ay_n \to t$. Since $d(Ay_n, t) \leq d(Ay_n, Bx_n) + d(Bx_n, t)$. In order to show $\lim_{n\to\infty} Ay_n \to t$, it is sufficient to show that $\lim_{n\to\infty} d(Ay_n, Bx_n) = 0$. If not, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ in X, a real number $\epsilon > 0$ such that for some positive integer $k \geq n$, $\lim_{n\to\infty} d(Ay_{n_k}, Bx_{n_k}) > \epsilon$. By (ii), we obtain

$$d(Ay_{n_k}, Bx_{n_k}) \leq F(max\{d(Sy_{n_k}, Tx_{n_k}), [d(Ay_{n_k}, Sx_{n_k}) + d(Bx_{n_k}, Tx_{n_k})]/2,$$
$$[d(Ay_{n_k}, Tx_{n_k}) + d(Bx_{n_k}, Sy_{n_k})]/2\}).$$

On letting $k \to \infty$, we obtain

$$\epsilon < \lim_{k\to\infty} d(Ay_{n_k}, Bx_{n_k}) < \epsilon$$

which is a contradiction. Hence, $\lim_{k\to\infty} Ay_{n_k} \to t$. Since TX is closed in X and $Tx_n \to t \in X$, there exists $v \in X$ such that $Tv = t$. If $Bv \neq Tv$ then, by (ii), we obtain

$$d(Ay_n, Bv) \leq F(max\{d(Sy_n, Tv), [d(Ay_n, Sy_n) + d(Bv, Tv)]/2,$$
$$[d(Ay_n, Tv) + d(Bv, Sy_n)]/2\}).$$

Again, by taking $n \to \infty$, one can obtain

$$d(Tv, Bv) = d(t, Bv)$$

$$\leq F(max\{0, d(Bv, Tv)/2, d(Bv, Tv)/2\})$$

$$\leq F(d(Tv, Bv)/2)$$

$$< d(Tv, Bv),$$

a contradiction and hence $Tv = Bv$. This shows that $C(B, T) \neq \phi$.

Since $BX \subseteq SX$ and $Bv = t$, there exists $u \in X$ such that $Bv = Su$. Therefore, $Su = Bv = Tv = t$. We show that $Au = Su$. For this, by (iii), we obtain

$$d(Au, Bv) \leq F(max\{d(Su, Tv), [d(Au, Su) + d(Bv, Tv)]/2,$$

$$[d(Au, Tv) + d(Bv, Su)]/2\})$$

$$\leq F(d(Au, Su)/2)$$

$$< d(Au, Su),$$

a contradiction, if $As \neq Su$. Thus, $C(A, S) \neq \phi$. This completes the proof.

Remark 3.4 *It remains true if we replace the conditions (i) and (ii) of above proposition by the following conditions*
(i) $AX \subseteq TX$ *and* SX *is closed in* X*;*
(ii) (A, S) *satisfy property (E.A).*

Following theorem extend Theorem 2 of Aamri and Moutawakil [1].

Theorem 3.5 *In addition to all the hypothesis of Proposition 2.3 on* A, B, S *and* T*, if the mappings pairs* (A, S) *and* (B, T) *are occasionally weakly* S*- and* T*- biased, then* A, B, S *and* T *have a unique common fixed point in* X*.*

Proof. By hypothesis of Proposition 2.3, we have $C(A, S) \neq \phi$ and $C(B, T) \neq \phi$. Since (A, S) and (B, T) are occasionally weakly S-and T biased mappings pairs, there exist $w, z \in X$ such that $Aw = Sw$ and $Bz = Tz$. Also from $Aw = Sw$ and $Bz = Tz$, we have $AAw = ASw, SAw = SSw, BBz = BTz$, and $TBz = TTz$. Now, we claim that $Aw = Bz$ otherwise by (iii) of Proposition 2.3, we obtain

$$d(Aw, Bz) \leq F(max\{d(Sw, Tz), [d(Aw, Sw) + d(Bz, Tz)]/2,$$

$$[d(Aw, Tz) + d(Bz, Sw)]/2\})$$

$$\leq F(d(Aw, Bz))$$

$$< d(Aw, Bz)$$

a contradiction and hence $Au = Bz$. Therefore, $Aw = Sw = Bz = Tz$. We show that $AAw = Aw$. If not, then by (iii) of Proposition 2.3 and occasionally weakly S-biased of the pair (A, S), we obtain

$$\begin{aligned}
d(AAw, Aw) &= d(AAw, Bz) \\
&\leq F(max\{d(SAw, Tz), [d(AAw, SAw) + d(Bz, Tz)]/2, \\
&\qquad [d(AAw, Tz) + d(Bz, SAw)]/2\}) \\
&\leq F(d(ASw, Aw)) \\
&< d(ASw, Aw),
\end{aligned}$$

a contradiction and hence $AAw = Aw$. Since (A, S) is occasionally weakly S-biased mappings pair, so that $d(SAw, Sw) \leq d(ASw, Aw) = 0$. Consequently, $SAw = Aw$. Further, we show that $BAw = Aw$. For this by (iii) of Prop. 2.3 and occasionally weakly T-biased of the pair (B, T), we obtain

$$\begin{aligned}
d(Aw, BAw) &= d(Aw, BBz) \\
&\leq F(max\{d(Sw, TBz), [d(Aw, Sw) + d(BBz, TBz)]/2, \\
&\qquad [d(Aw, TBz) + d(BBz, Sw)]/2\}) \\
&= F(max\{d(Tz, TBz), [d(BBz, Bz) + d(TBz, Tz)]/2, \\
&\qquad [d(Tz, TBz) + d(BBz, Bz)]/2\}) \\
&\leq F(max\{d(Bz, BTz), d(BBz, Bz), d(BBz, Bz)\}) \\
&= F(d(BAwz, Aw)) \\
&< d(BAwz, Aw),
\end{aligned}$$

a contradiction, if $BAw \neq Aw$. Also, T-weakly biased of the pair (B, T), one can obtain

$$d(TAw, Aw) = d(TBz, Tz) \leq d(BTz, Bz) = d(BAw, Aw) = 0.$$

Therefore, $TAw = Aw$. Thus, $AAw = BAw = SAw = TAw = Aw$ i.e. Aw is a common fixed point of A, B, S and T. The Uniqueness of common fixed point is immediately followed. This completes the proof.

The following example illustrate the validity of above theorem.

Example 3.6 *Let $X = [0, 1)$ with usual metric d. For the sake of simplicity, we define $A = B, S = T : X \to X$ as $Ax = Bx = (1+x)/3, x < 1/3, Ax = Bx = 1/3, x \geq 1/3$ and $Sx = Tx = 1/3 + x, x < 1/3, S1/3 = T1/3 = 1/3, Sx = Tx = 2/3, x > 1/3$. Clearly, $AX = [1/3, 4/9), TX = [1/3, 2/3]$. Since $A = B, S = T$ so $BX \subseteq SX$ i.e. $AX \subseteq TX$ and TX is closed in X. Mappings A and S satisfy property (E.A). To verify this, let $\{x_n\}$ be a sequence in X such that $x_n \to 0$ as $n \to \infty$ then $Ax_n, Sx_n \to 1/3 \in X$. One can also verify that A and S satisfy the contractive condition of the theorem for every $x, y \in X$. Also, $C(A, S) = \{0, 1/3\}$ and $A0 = 1/3 = S0$ which implies that A and S are occasionally weakly S-biased mappings. Thus, all the conditions of theorem are satisfied and 1/3 is the unique common point.*

Remark 3.7 *It may be pointed out the following observations:*
(i) Theorem 2.1 remains true if one replace the following contractive condition in lieu of the existing one

$$d(fx, fy) < max\{d(gx, gy), k[d(fx, gx) + d(fy, gy)]/2,$$
$$k[d(fy, gx) + d(fx, gy)]/2\}, 1/2 \leq k < 1$$

which in turn to give a good variant of Theorem 2.1 of Pant and Pant [10].
(ii) Theorem 2.5 remains true if we replace either of the following contractive conditions for every $x \neq y \in X$ in lieu of the existing one

(a) $$d(Ax, By) < max\{d(Sx, Ty), [d(Ax, Sx) + d(By, Ty)]/2,$$
$$[d(Ay, Ty) + d(By, Sx)]/2\});$$

(b) $$d(Ax, By) < max\{d(Sx, Ty), k[d(Ax, Sx) + d(By, Ty)]/2,$$
$$k[d(Ay, Ty) + d(By, Sx)]/2\}), 1/2 \leq k < 1.$$

From the above we notice that contractive conditions of Theorems 2.1 and 2.3 of Pant and Pant [10] are not applicable in Theorem 3.5.

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