The Properties of a Class of Analytic and Univalent Functions with Missing Coefficients

Li Xiao-fei*, Jie Jin and Wang An-ping

College of Technology & Engineering, Yangtze University
Jingzhou 434020, Hubei, P.R. China

Abstract
In this paper, the author consider the class \( C_n^* (\alpha, \beta, \gamma) \) consisting of analytic functions with missing coefficients. They get some properties about coefficients estimates, convex linear combinations and some distortion theorems for \( f(z) \) in the class \( C_n^* (\alpha, \beta, \gamma) \).

Keywords: Analytic and Univalent, Coefficients estimates, Convex linear combinations, Distortion theorems, Salagean operator

1. Introduction
Let \( S \) denote the class of functions of the form
\[
f(z) = z - \sum_{k=2}^{\infty} a_k z^k \quad (a_k \geq 0)
\]
which are analytic and univalent in the unit disc \( U = \{ z : |z| < 1 \} \).

We define Salagean [1] operator \( D \) like this
\[
D^0 f(z) = f(z)
\]
\[ D^1 f(z) = Df(z) = z f'(z) \]
\[ D^n f(z) = D(D^{n-1} f(z)) \quad (n = 1, 2, \cdots) \]

If the function \( f(z) \) belongs to \( S \), we get the result

\[ D^1 f(z) = z - \frac{c\beta(\alpha + \gamma)}{2^{\alpha-1}(1 + 2\alpha\beta + \beta\gamma)} z^2 - \sum_{k=3}^{\infty} k' a_k z^k \quad (1.2) \]

Let \( S_n(\alpha) \) denote the subclass of \( S \) satisfying

\[ \Re \frac{D^{n+1} f(z)}{D^n f(z)} > \alpha \quad (\alpha \geq 0) \]

The class \( S_n(\alpha) \) was studied by Owa et al. (see [2], also [3-7]).

Let \( S_n(\alpha, \beta, \gamma) \) denote the subclass of \( S \) satisfying

\[ \left| \frac{D^{n+1} f(z)}{D^n f(z)} - 1 \right| < \beta \]
\[ \alpha \left( \frac{D^{n+1} f(z)}{D^n f(z)} + \gamma \right) \]

where \( 0 \leq \alpha \leq 1, 0 < \beta \leq 1, 0 < \gamma \leq 1 \).

The class \( S_n(\alpha, \beta, \gamma) \) was studied by Lixiaofei [8]. For the class \( S_n(\alpha, \beta, \gamma) \), he showed the following lemma.

**Lemma 1.1.** A function \( f(z) \) defined by (1.2) in the class \( S_n(\alpha, \beta, \gamma) \) if and only if

\[ \sum_{k=2}^{\infty} (k - 1 + \alpha\beta k + \beta\gamma) k^n a_k \leq \beta(\alpha + \gamma) \quad (1.3) \]

The result is sharp.

In view of lemma 1.1, we can see that the function \( f(z) \) defined by (1.3) in the class \( S_n(\alpha, \beta, \gamma) \) satisfy

\[ a_k \leq \frac{\beta(\alpha + \gamma)}{k^n(k - 1 + k\alpha\beta + \beta\gamma)} \quad (1.4) \]
Let \( C_n^\ast(\alpha, \beta, \gamma) \) denote the class of \( f(z) \) in \( S_n(\alpha, \beta, \gamma) \) of the form
\[
f(z) = z - \frac{c\beta(\alpha + \gamma)}{p^n(p-1+p\alpha\beta + \beta\gamma)} z^p - \sum_{k=0}^{\infty} a_k z^k \quad (0 \leq c \leq 1, a_k \geq 0)
\] (1.5)

In this paper, we shall get some properties of the class \( C_n^\ast(\alpha, \beta, \gamma) \).

2. Properties of the class \( C_n^\ast(\alpha, \beta, \gamma) \)

First result for the class \( C_n^\ast(\alpha, \beta, \gamma) \) is contained in

**Theorem 2.1.** Let function \( f(z) \) be defined by (1.5). Then \( f(z) \in S_n^\ast(\alpha, \beta, \gamma) \) if and only if
\[
\sum_{k=p+1}^{\infty} (k-1+k\alpha\beta + \beta\gamma) k^n a_k \leq \beta (1-c)(\alpha + \gamma)
\] (2.1)

**Proof.** Putting
\[
a_p = \frac{c\beta(\alpha + \gamma)}{p^n(p-1+p\alpha\beta + \beta\gamma)} \quad (0 \leq c \leq 1)
\]
in (1.3) and simplifying we get the result. The result is sharp for the function
\[
f(z) = z - \frac{c\beta(\alpha + \gamma)}{p^n(p-1+p\alpha\beta + \beta\gamma)} z^p - \frac{\beta (1-c)(\alpha + \gamma)}{k^n(k-1+k\alpha\beta + \beta\gamma)} z^k (k \geq p+1)
\] (2.2)

**Corollary 2.1.** Let the function \( f(z) \) defined by (1.5) be in the class \( C_n^\ast(\alpha, \beta, \gamma) \). Then
\[
a_k \leq \frac{\beta (1-c)(\alpha + \gamma)}{k^n(k-1+k\alpha\beta + \beta\gamma)}
\] (2.3)

The result is sharp for the function \( f(z) \) given by (2.2).

**Theorem 2.2.** The class \( C_n^\ast(\alpha, \beta, \gamma) \) is closed under convex linear combination.

**Proof.** Let the function \( f(z) \) be defined by (1.5). Define the function \( g(z) \) by
\[
f(z) = z - \frac{c\beta(\alpha + \gamma)}{p^s(p - 1 + p\alpha\beta + \beta\gamma)}z^p - \sum_{k=p+1}^{\infty} b_k z^k
\]  \hspace{1cm} (2.4)

Assuming that \( f(z) \) and \( g(z) \) are in the class \( C_n^*(\alpha, \beta, \gamma) \), it is sufficient to prove that the function \( H(z) \) defined by

\[
H(z) = \lambda f(z) + (1 - \lambda) g(z) \quad (0 \leq \lambda \leq 1)
\]  \hspace{1cm} (2.5)

is also in the class \( C_n^*(\alpha, \beta, \gamma) \).

Since

\[
H(z) = z - \frac{c\beta(\alpha + \gamma)}{p^s(p - 1 + p\alpha\beta + \beta\gamma)}z^p - \sum_{k=p+1}^{\infty} \left[ \lambda a_k + (1 - \lambda) b_k \right] z^k
\]  \hspace{1cm} (2.6)

we observe that

\[
\sum_{k=p+1}^{\infty} \left[ (k - 1 + k\alpha\beta + \beta\gamma) k^n \left[ \lambda a_k + (1 - \lambda) b_k \right] \right] \leq \beta(1-c)(\alpha + \gamma)
\]  \hspace{1cm} (2.7)

with the aid of theorem 2.1. Hence \( H(z) \in C_n^*(\alpha, \beta, \gamma) \). This completes the proof of theorem 2.2.

**Theorem 2.3.** Let the function

\[
f_j(z) = z - \frac{c\beta(\alpha + \gamma)}{p^s(p - 1 + p\alpha\beta + \beta\gamma)}z^p - \sum_{k=p+1}^{\infty} a_{k,j} z^k
\]  \hspace{1cm} (2.8)

be in the class \( C_n^*(\alpha, \beta, \gamma) \) for every \( j = 1, 2, \cdots, m \). Then the function \( F(z) \) defined by

\[
F(z) = \sum_{j=1}^{m} d_j f_j(z) \quad (d_j \geq 0)
\]  \hspace{1cm} (2.9)

is also in the same class \( C_n^*(\alpha, \beta, \gamma) \), where

\[
\sum_{j=1}^{m} d_j = 1
\]  \hspace{1cm} (2.10)

**Proof.** Combining the definitions (2.8) and (2.9), we have

\[
F(z) = z - \frac{c\beta(\alpha + \gamma)}{2^s(1 + 2\alpha\beta + \beta\gamma)}z^2 - \sum_{k=3}^{\infty} \left[ \sum_{j=1}^{m} d_j a_{k,j} \right] z^k
\]  \hspace{1cm} (2.11)
where we have also used the relationship (2.10). Since \( f_j(z) \in C^*_n(\alpha, \beta, \gamma) \) for every \( j = 1, 2, \ldots, m \), theorem 2.1 yields

\[
\sum_{k=p+1}^{\infty} (k - 1 + k\alpha\beta + \beta\gamma)k^n a_{k,j} \leq \beta(1 - c)(\alpha + \gamma)
\]

for \( j = 1, 2, \ldots, m \). Thus we obtain

\[
\sum_{k=p+1}^{\infty} (k - 1 + k\alpha\beta + \beta\gamma)k^n d_j \left( \sum_{j=1}^{m} a_{k,j} \right) = \sum_{j=1}^{m} d_j \left( \sum_{k=p+1}^{\infty} (k - 1 + k\alpha\beta + \beta\gamma)k^n a_{k,j} \right) \leq \beta(1 - c)(\alpha + \gamma)
\]

which implies the function \( F(z) \in C^*_n(\alpha, \beta, \gamma) \).

**Theorem 2.4.** Let the function \( f(z) \) defined by (1.5) be in the class \( C^*_n(\alpha, \beta, \gamma) \). Then

\[
|z| - \frac{\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^n \leq |D^jf(z)| \leq |z| + \frac{\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^n
\]

**Proof.** We suppose that \( f(z) \) can be expressed in the form (1.5). Then we have

\[
\sum_{k=p+1}^{\infty} (k - 1 + k\alpha\beta + \beta\gamma)k^n a_k \leq \beta(1 - c)(\alpha + \gamma)
\]

from the theorem 2.1. Since

\[
p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma) \sum_{k=p+1}^{\infty} k^n a_k \leq \sum_{k=p+1}^{\infty} k^{n-1}(k - 1 + k\alpha\beta + \beta\gamma)k^n a_k
\]

\[=
\sum_{k=p+1}^{\infty} (k - 1 + k\alpha\beta + \beta\gamma)k^n a_k \leq \beta(1 - c)(\alpha + \gamma)
\]

We observe that

\[
\sum_{k=p+1}^{\infty} k^j a_k \leq \frac{\beta(1 - c)(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}
\]

So
\[ |D'f(z)| = \left| z - \frac{c\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}z^{p} - \sum_{k=p+1}^{\infty} k^{i}a_{k}z^{i} \right| \]

\[ = |z| - \frac{c\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p} - \sum_{k=p+1}^{\infty} k^{i}a_{k}|z|^{p} \]

\[ = |z| - \frac{c\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p} - \frac{\beta(1-c)(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p} \]

\[ = |z| - \frac{\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p} \]

With the same method, we have

\[ |D'f(z)| \leq |z| + \frac{\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p} \]

**Corollary 2.2.** Let the function \( f(z) \) defined by (1.5) be in the class \( C_{n}^*(\alpha, \beta, \gamma) \).

Then

\[ |z| - \frac{\beta(\alpha + \gamma)}{p^{n}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p} \leq |f(z)| \leq |z| + \frac{\beta(\alpha + \gamma)}{p^{n}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p} \]

and

\[ 1 - \frac{\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p-1} \leq |f'(z)| \leq 1 + \frac{\beta(\alpha + \gamma)}{p^{n-1}(p - 1 + p\alpha\beta + \beta\gamma)}|z|^{p-1} \]

**References**


Received: December, 2011