The Order of Starlikeness of New $p$-Valent Meromorphic Functions

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Abstract

In the present paper two general integral operators of meromorphic $p$-valent functions in the punctured open unit disk are introduced. Two subclasses of meromorphic $p$-valent functions are presented. The order of starlikeness of the above operators are also determined. As an application to the above operators, two $p$-valent meromorphic functions are defined and studied.

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1 Introduction

Let $U = \{ z \in \mathbb{C} : |z| < 1 \}$, be the open unit disc in the complex plane $\mathbb{C}$, $U^* = U \setminus \{0\}$, the punctured open unit disk and $\mathbb{H}(U) = \{ f \in U \rightarrow \mathbb{C} : f$ is holomorphic in $U \}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \ldots \}$), let $\mathbb{H}[a, n] = \{ f \in \mathbb{H}(U), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \ldots, z \in \mathbb{U}\}$. Let $\Sigma_p$ denote the class of meromorphic functions of the form

\[ f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n \quad (p \in \mathbb{N}^* = \mathbb{N}\setminus\{0\}). \] (1.1)
which are analytic and $p$-valent in $\mathbb{U}^*$. 

We say that a function $f \in \Sigma_p$ is the meromorphic $p$-valent starlike of order $\alpha$ $(0 \leq \alpha < p)$ and belongs to the class $f \in \Sigma_p^*(\alpha)$, if it satisfies the inequality:

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha.$$ 

A function $f \in \Sigma_p$ is the meromorphic $p$-valent convex function of order $\alpha$ $(0 \leq \alpha < p)$, if $f$ satisfies the following inequality

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,$$

and we denote this class by $\Sigma K_p(\alpha)$.

Many important properties and characteristics of various interesting subclasses of the class $\Sigma_p$ of meromorphically $p$-valent functions were investigated extensively by (among others) Uralegaddi and Somanatha ([7] and [8]), Liu and Srivastava ([12] and [13]), Mogra ([14] and [15]), Srivastava et al. [16], Aouf et al. ([17] and [18]), Joshi and Srivastava [19], Owa et al. [20] and Kulkarni et al. [21].

Analogous to the integral operators defined by Breaz et al. ([9] and [10]), Frasin [6] and Mohammed and Darus ([1], [3]) on the normalized, $p$-valent and meromorphic analytic functions, we now define the following two integral operators on the space meromorphic $p$-valent functions in the class $\Sigma_p$.

**Definition 1.1.** Let $n, p \in \mathbb{N}^*$, $i \in \{1, 2, 3, ..., n\}$, $\gamma_i > 0$. We define the integral operator $\mathcal{F}_{p;\gamma_1, \ldots, \gamma_n} (f_1, f_2, ..., f_n): \Sigma_p^n \to \Sigma_p$ by

$$\mathcal{F}_{p;\gamma_1, \ldots, \gamma_n} (z) = I(f_1, f_2, ..., f_n)(z) = \frac{1}{z^{p+1}} \int_0^z \left( u^p f_1(u) \right)^{\gamma_1} \cdots \left( u^p f_n(u) \right)^{\gamma_n} du.$$ 

(1.2)

**Definition 1.2.** Let $n, p \in \mathbb{N}^*$, $i \in \{1, 2, 3, ..., n\}$, $\gamma_i > 0$. We define the integral operator $\mathcal{J}_{p;\gamma_1, \ldots, \gamma_n} (f_1, f_2, ..., f_n): \Sigma_p^n \to \Sigma_p$ by

$$\mathcal{J}_{p;\gamma_1, \ldots, \gamma_n} (z) = I(f_1, f_2, ..., f_n)(z)$$

$$= \frac{1}{z^{p+1}} \int_0^z \left( \frac{-u^{p+1}}{p} f_1'(u) \right)^{\gamma_1} \cdots \left( \frac{-u^{p+1}}{p} f_n'(u) \right)^{\gamma_n} du.$$ 

(1.3)
For the sake of simplicity, from now on we shall write \( F_{p, \gamma_1}, \ldots, \gamma_n \) (\( f_1, f_2, \ldots, f_n \))(z) instead of \( F_{p, \gamma_1, \ldots, \gamma_n} \) and \( J_{p, \gamma_1, \ldots, \gamma_n} \) instead of \( J_{p, \gamma_1, \ldots, \gamma_n} (f_1, f_2, \ldots, f_n)(z) \).

If we take \( p = 1 \), we obtain the general integral operators \( F_{1, \gamma_1, \ldots, \gamma_n} (z) = H(z) \) and \( J_{1, \gamma_1, \ldots, \gamma_n} (z) = H_{\gamma_1, \ldots, \gamma_n} (z) \), introduced by the authors ([1] and [3]).

For the \( f \in \Sigma_p \) (\( p \in \mathbb{N} \)), we introduce the following two new subclasses.

**Definition 1.3.** Let a function \( f \in \Sigma_p \) be analytic in \( \mathbb{U}^* \). Then \( f \) is in the class \( \Omega_{\star}^{p}(\beta) (-1 \leq \beta < p) \), if, and only if, \( f \) satisfies

\[
\left| \frac{zf'(z)}{f(z)} + p \right| < -\Re \left( \frac{zf'(z)}{f(z)} + \beta \right). \tag{1.4}
\]

**Definition 1.4.** Let a function \( f \in \Sigma_p \) be analytic in \( \mathbb{U}^* \). Then \( f \) is in the class \( \Omega K_{p}(\beta) (-1 \leq \beta < p) \), if, and only if, \( f \) satisfies

\[
\left| \frac{zf''(z)}{f'(z)} + 1 + p \right| < -\Re \left( \frac{zf''(z)}{f'(z)} + \beta \right) - 1. \tag{1.5}
\]

The following results will be useful in the sequel.

**Lemma 1.1([5]).** Let \( n \in \mathbb{N}^*, \alpha, \delta \in \mathbb{R}, \gamma \in \mathbb{C} \) with \( \Re[\gamma - \alpha \delta] \geq 0 \). If \( p \in \mathbb{H}[p(0), n] \) with \( p(0) \in \mathbb{R} \) and \( p(0) > \alpha \), then we have

\[
\Re \left\{ p(z) + \frac{zp'(z)}{\gamma - \delta p(z)} \right\} > \alpha \implies \Re p(z) > \alpha, \quad z \in \mathbb{U}.
\]

**Theorem 1.2 [23]** If \( f \in \Sigma_p \) satisfies the inequality

\[
\left| \frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1 \right| < \delta, \quad 0 < \delta < 1
\]

then \( f \in \Sigma_p^{\star} (p(1 - \delta)) \).

**Theorem 1.3 [23]** If \( f \in \Sigma_p \) satisfies the inequality

\[
\left| \frac{f(z)}{zf'(z)} \left( 1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)} \right) \right| < \mu, \quad 0 < \mu < \frac{1}{p}
\]

then \( f \in \Sigma_p^{\star} \left( \frac{p}{1+p^\mu} \right) \).
2 Starlikeness of the operator $F_{p, \gamma_1, \ldots, \gamma_n}(z)$

In this section we place conditions for the starlikeness of the integral operator $F_{p, \gamma_1, \ldots, \gamma_n}(z)$ which is defined in (1.2).

**Theorem 2.1.** For $i \in \{1, \ldots, n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma^*(\alpha_i) \ (0 \leq \alpha_i < p)$. If $0 < \sum_{i=1}^{n} \gamma_i (p - \alpha_i) \leq p$, then $F_{p, \gamma_1, \ldots, \gamma_n}(z)$ is starlike by order $p - \sum_{i=1}^{n} \gamma_i (p - \alpha_i)$.

**Proof.** A differentiation of $F_{p, \gamma_1, \ldots, \gamma_n}(z)$ which is defined in (1.2), we get

$$z^{p+1} F'_{p, \gamma_1, \ldots, \gamma_n}(z) + (p+1) z^p F'_{p, \gamma_1, \ldots, \gamma_n}(z) = (z^p f_1(z))^{\gamma_1} \cdots (z^p f_n(z))^{\gamma_n},$$

(2.1)

and

$$z^{p+1} F''_{p, \gamma_1, \ldots, \gamma_n}(z) + 2(p+1) z^p F'_{p, \gamma_1, \ldots, \gamma_n}(z) + p(p+1) z^{p-1} F_{p, \gamma_1, \ldots, \gamma_n}(z) =$$

$$\sum_{i=1}^{n} \gamma_i \left( \frac{z^p f_i'(z) + p z^{p-1} f_i(z)}{z^p f_i(z)} \right) [(z^p f_1(z))^{\gamma_1} \cdots (z^p f_n(z))^{\gamma_n}]$$

(2.2)

Then from (2.1) and (2.2), we obtain

$$\frac{z^{p+1} F''_{p, \gamma_1, \ldots, \gamma_n}(z) + 2(p+1) z^p F'_{p, \gamma_1, \ldots, \gamma_n}(z) + p(p+1) z^{p-1} F_{p, \gamma_1, \ldots, \gamma_n}(z)}{z^{p+1} F'_{p, \gamma_1, \ldots, \gamma_n}(z) + (p+1) z^p F_{p, \gamma_1, \ldots, \gamma_n}(z)} =$$

$$\sum_{i=1}^{n} \gamma_i \left( \frac{f_i'(z)}{f_i(z)} + \frac{p}{z} \right).$$

(2.3)

By multiplying (2.3) with $z$ yield,

$$\frac{z^{p+1} F''_{p, \gamma_1, \ldots, \gamma_n}(z) + 2(p+1) z^p F'_{p, \gamma_1, \ldots, \gamma_n}(z) + p(p+1) z^{p-1} F_{p, \gamma_1, \ldots, \gamma_n}(z)}{z^p F'_{p, \gamma_1, \ldots, \gamma_n}(z) + (p+1) z^{p-1} F_{p, \gamma_1, \ldots, \gamma_n}(z)} =$$

$$\sum_{i=1}^{n} \gamma_i \left( \frac{z f_i'(z)}{f_i(z)} + p \right).$$

(2.4)

That is equivalent to

$$\frac{z^{p+1} F''_{p, \gamma_1, \ldots, \gamma_n}(z) + (p+2) z^p F'_{p, \gamma_1, \ldots, \gamma_n}(z)}{z^p F'_{p, \gamma_1, \ldots, \gamma_n}(z) + (p+1) z^{p-1} F_{p, \gamma_1, \ldots, \gamma_n}(z)} + p = \sum_{i=1}^{n} \gamma_i \left( \frac{z f_i'(z)}{f_i(z)} + p \right).$$

(2.4)
And
\[ -\frac{z\left(z \mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+2)\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)\right)}{z\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+1)\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)} = -\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} + p\right) + p. \]  
(2.5)

We can write (2.5), as the following
\[ -\frac{z\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)} \left(\frac{z\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)} + p + 2\right) = -\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} + p\right) + p. \]  
(2.6)

We define the regular function \( q \) in \( \mathbb{U} \) by
\[ q(z) = -\frac{z\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}, \]  
(2.7)

and \( q(0) = p \). Differentiating \( q(z) \) logarithmically, we obtain
\[ -q(z) + \frac{zq'(z)}{q(z)} = 1 + \frac{z\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}. \]  
(2.8)

From (2.6), (2.7) and (2.8) we obtain
\[ q(z) + \frac{zq'(z)}{p + 1 - q(z)} = -\sum_{i=1}^{n} \gamma_i \left(\frac{zf_i'(z)}{f_i(z)} + p\right) + p. \]  
(2.9)

Since \( f_i \in \Sigma^*_p(\alpha_i) \), for \( i \in \{1, \ldots, n\} \), we receive
\[ \Re \left\{ q(z) + \frac{zq'(z)}{p + 1 - q(z)} \right\} > p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i). \]  
(2.10)

It is clear that \( q \) is analytic in \( \mathbb{U} \) with \( q(0) = p > p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i) \). We also have \( \Re(\gamma - \delta \alpha) > 0 \), for \( \gamma = p + 1, \delta = 1 \) and \( \alpha = p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i) \). Since the conditions from Lemma 1.1 are met, we obtain \( \Re q(z) > p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i) \), which is equivalent to
\[ -\frac{z\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)} > p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i). \]

that is \( \mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z) \) is starlike of order \( p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i) \).
Theorem 2.2. For \( i \in \{1, \ldots, n\} \), let \( \gamma_i > 0 \) and \( f_i \in \Omega_p^* (\beta_i) \) \((-1 \leq \beta_i < p)\). If \( 0 < \sum_{i=1}^{n} \gamma_i (p - \beta_i) \leq p \), then \( I_{p, \gamma_1, \ldots, \gamma_n} (z) \) is starlike by order \( p - \sum_{i=1}^{n} \gamma_i (p - \beta_i) \).

Proof. Using (2.9), we have

\[
q(z) + \frac{zq'(z)}{p + 1 - q(z)} = - \sum_{i=1}^{n} \gamma_i \left( \frac{zf_i'(z)}{f_i(z)} + \beta_i \right) + p - \sum_{i=1}^{n} \gamma_i (p - \beta_i). \tag{2.11}
\]

Since \( f_i \in \Omega_p^* (\beta_i) \), for \( i \in \{1, \ldots, n\} \), we get

\[
\Re \left\{ \frac{zq'(z)}{p + 1 - q(z)} \right\} > \sum_{i=1}^{n} \gamma_i \left| \frac{zf_i'(z)}{f_i(z)} + p \right| + p - \sum_{i=1}^{n} \gamma_i (p - \beta_i). \tag{2.12}
\]

Because \( \sum_{i=1}^{n} \gamma_i \left| \frac{zf_i'(z)}{f_i(z)} + p \right| > 0 \), we obtain that

\[
\Re \left\{ \frac{zq'(z)}{p + 1 - q(z)} \right\} > p - \sum_{i=1}^{n} \gamma_i (p - \beta_i). \tag{2.13}
\]

The remaining part of the proof follows the pattern of those in Theorem 2.1. The proof is complete.

3 Starlikeness of the operator \( I_{p, \gamma_1, \ldots, \gamma_n} (z) \)

In this section we place conditions for the starlikeness of the integral operator \( I_{p, \gamma_1, \ldots, \gamma_n} (z) \) which is defined in (1.3).

Theorem 2.3. For \( i \in \{1, \ldots, n\} \), let \( \gamma_i > 0 \) and \( f_i \in \Sigma K_p (\alpha_i) \) \((0 \leq \alpha_i < p)\). If \( 0 < \sum_{i=1}^{n} \gamma_i (p - \alpha_i) \leq p \), then \( I_{p, \gamma_1, \ldots, \gamma_n} (z) \) is starlike by order \( p - \sum_{i=1}^{n} \gamma_i (p - \alpha_i) \).

Proof. A differentiation of \( I_{p, \gamma_1, \ldots, \gamma_n} (z) \), which is defined in (1.3), we get

\[
z^{p+1} I_{p, \gamma_1, \ldots, \gamma_n}' (z) + (p + 1)z^p I_{p, \gamma_1, \ldots, \gamma_n} (z) = \left( \frac{-z^{p+1}}{p} f_i'(z) \right)^{\gamma_i} \cdots \left( \frac{-z^{p+1}}{p} f_n'(z) \right)^{\gamma_n}. \tag{3.1}
\]

and

\[
z^{p+1} I_{p, \gamma_1, \ldots, \gamma_n}'' (z) + 2(p + 1)z^p I_{p, \gamma_1, \ldots, \gamma_n}' (z) + p(p + 1)z^{p-1} I_{p, \gamma_1, \ldots, \gamma_n} (z) =
\]
From (3.5), (3.6) and (3.7) we obtain

That is equivalent to

Then from (3.1) and (3.2), we obtain

\[
\sum_{i=1}^{n} \gamma_i \left( \frac{z^{p+1} f_i''(z)}{z^{p+1} f_i'(z)} + (p+1)z^{p} f_i'(z) \right) \left[ \left( -\frac{z^{p+1}}{p} f_i'(z) \right)^\gamma_1 \cdots \left( -\frac{z^{p+1}}{p} f_n'(z) \right)^\gamma_n \right] = \sum_{i=1}^{n} \gamma_i \left( \frac{z f_i''(z)}{f_i'(z)} + p + 1 \right).
\]  
(3.2)

That is equivalent to

\[
\frac{z^{p+1} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}''(z) + 2(p+1)z^{p} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}'(z) + p(p+1)z^{p-1} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z)}{z^{p} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}'(z) + (p+1)z^{p-1} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z)} = -\sum_{i=1}^{n} \gamma_i \left( \frac{z f_i''(z)}{f_i'(z)} + p + 1 \right) + p.
\]  
(3.3)

We can write (3.4), as the following

\[
\frac{z \left( \frac{z^{p+1} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}''(z)}{\mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}'(z)} + p + 2 \right)}{z^{p+1} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}'(z) + (p+1) \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z)} = -\sum_{i=1}^{n} \gamma_i \left( \frac{z f_i''(z)}{f_i'(z)} + p + 1 \right) + p.
\]  
(3.4)

Define the regular function \( q \) in \( U \) by

\[
q(z) = -\frac{z \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}'(z)}{\mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z)},
\]  
(3.6)

and \( q(0) = p \). Differentiating \( q(z) \) logarithmically, we obtain

\[
-q(z) + \frac{z q'(z)}{q(z)} = 1 + \frac{z \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}''(z)}{\mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}'(z)}.
\]  
(3.7)

From (3.5), (3.6) and (3.7) we obtain

\[
q(z) + \frac{z q'(z)}{p+1-q(z)} = -\sum_{i=1}^{n} \gamma_i \left( \frac{z f_i''(z)}{f_i'(z)} + p + 1 \right) + p.
\]  
(3.8)

Since \( f_i \in \Sigma K_p(\alpha_i) \), for \( i \in \{1, \ldots, n\} \), we receive

\[
\Re \left\{ q(z) + \frac{z q'(z)}{p+1-q(z)} \right\} > p - \sum_{i=1}^{n} \gamma_i(p-\alpha_i).
\]  
(3.9)
The remaining part of the proof follows the pattern of those in Theorem 2.1.

**Theorem 2.4.** For \( i \in \{1, ..., n\} \), let \( \gamma_i > 0 \) and \( f_i \in \Omega K_p(\beta_i) \) \((-1 \leq \beta_i < p)\). If \( 0 < \sum_{i=1}^{n} \gamma_i(p - \beta_i) \leq p \), then \( \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z) \) is starlike by order \( p - \sum_{i=1}^{n} \gamma_i(p - \beta_i) \).

Other work that we can look at regarding these operators is on superordination-preserving integral operator[22] and related to it (see [2],[4], [11]).

**4 Application**

As an application to the integral operators \( \mathcal{F}_{p, \gamma_1, \ldots, \gamma_n}(z) \) and \( \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z) \), we define the following two \( p \)-valent meromorphic functions.

**Definition 4.1.** Let \( \mathcal{F}_{p, \gamma_1, \ldots, \gamma_n}(z) \) be the integral operator defined as in (1.2). We define the following function,

\[
\Phi(z) = z^{p'} \mathcal{F}_{p, \gamma_1, \ldots, \gamma_n}(z) + (p + 1) \mathcal{F}_{p, \gamma_1, \ldots, \gamma_n}(z)
\tag{4.1}
\]

**Definition 4.2.** Let \( \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z) \) be the integral operator defined as in (1.3). We define the following function,

\[
\Upsilon(z) = z^{p'} \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z) + (p + 1) \mathcal{J}_{p, \gamma_1, \ldots, \gamma_n}(z)
\tag{4.2}
\]

Next, we study some orders of starlikeness of \( \Phi(z) \) and \( \Upsilon(z) \).

**Theorem 4.1.** For \( i \in \{1, ..., n\} \), let \( \gamma_i \in \mathbb{R} \), \( \gamma_i > 0 \) and \( f_i \in \Sigma_p^*(\alpha_i) \) \((0 \leq \alpha_i < p)\). If \( 0 < \sum_{i=1}^{n} \gamma_i(p - \alpha_i) \leq p \), then \( \Phi(z) \) given by (4.1) is starlike by order \( p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i) \).

**Proof.** A successive differentiation of \( \mathcal{F}_{p, \gamma_1, \ldots, \gamma_n}(z) \) which is defined in (1.2), we get

\[
z^{p} \Phi(z) = z^{p+1} \mathcal{F}_{p, \gamma_1, \ldots, \gamma_n}(z) + (p + 1)z^{p} \mathcal{F}_{p, \gamma_1, \ldots, \gamma_n}(z) = (z^p f_1(z))^\gamma_1 \ldots (z^p f_n(z))^\gamma_n.
\tag{4.3}
\]

Using (4.3), we get

\[
\frac{z \Phi'(z)}{\Phi(z)} = \sum_{i=1}^{n} \gamma_i \frac{z f_i'(z)}{f_i(z)} + p \sum_{i=1}^{n} \gamma_i - p.
\tag{4.4}
\]
Since \( f_i \in \Sigma_{p}^{*}(\alpha_i) \) we receive

\[
-\Re \frac{z\Phi'(z)}{\Phi(z)} = \sum_{i=1}^{n} \gamma_i \Re \left\{ -\frac{zf_i'(z)}{f_i(z)} \right\} - p \sum_{i=1}^{n} \gamma_i + p
\]

\[
> p - \sum_{i=1}^{n} \gamma_i (p - \alpha_i).
\]

But by the hypothesis, \( 0 \leq p - \sum_{i=1}^{n} \gamma_i (p - \alpha_i) < 1 \). Thus \( \Phi(z) \) is starlike by order \( p - \sum_{i=1}^{n} \gamma_i (p - \alpha_i) \).

**Theorem 4.2.** For \( i \in \{1, ..., n\} \), let \( \gamma_i \in \mathbb{R}, \gamma_i > p \) and \( \sum_{i=1}^{n} \gamma_i \leq n + 1 \). If \( f_i \in \Sigma_{p}^{*} \left( \frac{\alpha_i}{\gamma_i} \right) \), then \( \Phi(z) \) belong to \( \Sigma^{*}(0) \).

Now, adopting the same technique used in Theorem 4.1 and applying Theorem 1.2 and Theorem 1.3, one can prove

**Theorem 4.3.** For \( i \in \{1, ..., n\} \), let \( \gamma_i > 0, \ 0 < \delta_i < 1, \ f_i \in \Sigma_{p} \), and

\[
\left| \frac{zf_i'(z)}{f_i(z)} - \frac{zf_i''(z)}{f_i'(z)} - 1 \right| < \delta_i.
\]

If \( 0 < \sum_{i=1}^{n} \gamma_i \delta_i \leq 1 \), then \( \Phi(z) \) is starlike by order \( p - p \sum_{i=1}^{n} \gamma_i \delta_i \).

**Theorem 4.4.** For \( i \in \{1, ..., n\} \), let \( \gamma_i > 0, \ 0 < \mu_i < \frac{1}{p} \), \( f_i \in \Sigma_{p} \) and

\[
\left| \frac{f_i(z)}{zf_i'(z)} \left( 1 + \frac{zf_i''(z)}{f_i'(z)} - \frac{zf_i'(z)}{f_i(z)} \right) \right| < \mu_i.
\]

If \( 0 < \sum_{i=1}^{\infty} \frac{\gamma_i \mu_i}{p + \mu_i} \leq 1 \) then \( \Phi(z) \) is starlike by order \( p - p \sum_{i=1}^{\infty} \frac{\gamma_i \mu_i}{p + \mu_i} \).

Finally, we introduce the following results for the function \( \Upsilon(z) \).

**Theorem 4.5.** For \( i \in \{1, ..., n\} \), let \( \gamma_i \in \mathbb{R}, \gamma_i > 0 \) and \( f_i \in \Sigma K_{p}(\alpha_i) \) (\( 0 \leq \alpha_i < p \)). If \( 0 < \sum_{i=1}^{n} \gamma_i (p - \alpha_i) \leq p \), then \( \Upsilon(z) \) given by (4.2) is starlike by order \( p - \sum_{i=1}^{n} \gamma_i (p - \alpha_i) \).
Theorem 4.6. For $i \in \{1, ..., n\}$, let $\gamma_i \in \mathbb{R}$, $\gamma_i > p$ and $\sum_{i=1}^{n} \gamma_i \leq n + 1$. If $f_i \in \Sigma K_p \left( \frac{p}{\gamma_i} \right)$, then $\Upsilon(z)$ belong to $\Sigma^*(0)$.

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References


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