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The Order of Starlikeness of New p-Valent Meromorphic Functions

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Abstract

In the present paper two general integral operators of meromorphic p -valent functions in the punctured open unit disk are introduced. Two subclasses of meromorphic p -valent functions are presented. The order of starlikeness of the above operators are also determined. As an application to the above operators, two p -valent meromorphic functions are defined and studied.

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1 Introduction

Let $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$, be the open unit disc in the complex plane $\mathbb{C}$, $\mathbb{U}^* = \mathbb{U}\backslash\{0\}$, the punctured open unit disk and $\mathbb{H}(\mathbb{U}) = \{f \in \mathbb{U} \longrightarrow \mathbb{C} : f \text{ is holomorphic in } \mathbb{U}\}$. For $a \in \mathbb{C}$ and $n \in \mathbb{N}$ ($\mathbb{N} = \{0, 1, 2, \dots\}$), let $\mathbb{H}[a, n] = \{f \in \mathbb{H}(\mathbb{U}), f(z) = a + a_n z^n + a_{n+1} z^{n+1} + \dots ,\ \ z \in \mathbb{U}\}$. Let Σ_p denote the class of meromorphic functions of the form

$$f(z) = \frac{1}{z^p} + \sum_{n=0}^{\infty} a_n z^n \quad (p \in \mathbb{N}^* = \mathbb{N}\backslash\{0\}), \tag{1.1}$$

which are analytic and p -valent in $\mathbb{U}^*$.

We say that a function $f \in \Sigma_p$ is the meromorphic p -valent starlike of order α $(0 \leq \alpha < p)$ and belongs to the class $f \in \Sigma_p^\star(\alpha)$, if it satisfies the inequality:

$$-\Re\left(\frac{zf'(z)}{f(z)}\right) > \alpha.$$

A function $f \in \Sigma_p$ is the meromorphic p -valent convex function of order α $(0 \leq \alpha < p)$, if f satisfies the following inequality

$$-\Re\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha,$$

and we denote this class by $\Sigma K_p(\alpha)$.

Many important properties and characteristics of various interesting subclasses of the class Σ_p of meromorphically p -valent functions were investigated extensively by (among others) Uralegaddi and Somanatha ([7] and [8]), Liu and Srivastava ([12] and [13]), Mogra ([14]and [15]), Srivastava et a1.[16], Aouf et al. ([17] and [18]), Joshi and Srivastava [19], Owa et a1. [20] and Kulkarni et al. [21].

Analogous to the integral operators defined by Breaz et a1. ([9] and [10]), Frasin [6] and Mohammed and Darus ([1], [3]) on the normalized, p -valent and meromorphic analytic functions, we now define the following two integral operators on the space meromorphic p -valent functions in the class Σ_p.

Definition 1.1. *Let $n, p \in \mathbb{N}^*, i \in \{1, 2, 3, ..., n\}, \gamma_i > 0$. We define the integral operator $\mathcal{F}_{p,\gamma_1,\dots,\gamma_n}(f_1, f_2, ..., f_n) : \Sigma_p^n \to \Sigma_p$ by*

$$\mathcal{F}_{p,\gamma_1,\dots,\gamma_n}(z) = \mathcal{I}(f_1, f_2, ..., f_n)(z) = \frac{1}{z^{p+1}} \int_0^z (u^p f_1(u))^{\gamma_1} ... (u^p f_n(u))^{\gamma_n} du. \tag{1.2}$$

Definition 1.2. *Let $n, p \in \mathbb{N}^*, i \in \{1, 2, 3, ..., n\}, \gamma_i > 0$. We define the integral operator $\mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(f_1, f_2, ..., f_n) : \Sigma_p^n \to \Sigma_p$ by*

$$\mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(z) = \mathcal{I}(f_1, f_2, ..., f_n)(z)$$

$$= \frac{1}{z^{p+1}} \int_0^z \left(\frac{-u^{p+1}}{p} f_1'(u)\right)^{\gamma_1} ... \left(\frac{-u^{p+1}}{p} f_n'(u)\right)^{\gamma_n} du. \tag{1.3}$$

For the sake of simplicity, from now on we shall write $\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)$ instead of $\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}($
$f_1, f_2, ..., f_n)(z)$ and $\mathcal{J}_{p,\gamma_1,\ldots,\gamma_n}(z)$ instead of $\mathcal{J}_{p,\gamma_1,\ldots,\gamma_n}(f_1, f_2, ..., f_n)(z)$.

If we take $p = 1$, we obtain the general integral operators $\mathcal{F}_{1,\gamma_1,\ldots,\gamma_n}(z) = \mathcal{H}(z)$ and $\mathcal{J}_{1,\gamma_1,\ldots,\gamma_n}(z) = \mathcal{H}_{\gamma_1,\ldots,\gamma_n}(z)$, introduced by the authors ([1] and [3]).

For the $f \in \Sigma_p$ $(p \in \mathbb{N})$, we introduce the following two new subclasses.

Definition 1.3. Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega_p^\star(\beta)$ $(-1 \leq \beta < p)$, if, and only if, f satisfies

$$\left|\frac{zf'(z)}{f(z)} + p\right| < -\Re\left(\frac{zf'(z)}{f(z)} + \beta\right). \tag{1.4}$$

Definition 1.4. Let a function $f \in \Sigma_p$ be analytic in $\mathbb{U}^*$. Then f is in the class $\Omega K_p(\beta)$ $(-1 \leq \beta < p)$, if, and only if, f satisfies

$$\left|\frac{zf''(z)}{f'(z)} + 1 + p\right| < -\Re\left(\frac{zf''(z)}{f'(z)} + \beta\right) - 1. \tag{1.5}$$

The following results will be useful in the sequel.

Lemma 1.1([5]). *Let $n \in \mathbb{N}^*, \alpha, \delta \in \mathbb{R}, \gamma \in \mathbb{C}$ with $\Re[\gamma - \alpha\delta] \geq 0$. If $p \in \mathbb{H}[p(0), n]$ with $p(0) \in \mathbb{R}$ and $p(0) > \alpha$, then we have*

$$\Re\left\{p(z) + \frac{zp'(z)}{\gamma - \delta p(z)}\right\} > \alpha \Longrightarrow \Re p(z) > \alpha, \;\; z \in \mathbb{U}.$$

Theorem 1.2 [**23**] *If $f \in \Sigma_p$ satisfies the inequality*

$$\left|\frac{zf'(z)}{f(z)} - \frac{zf''(z)}{f'(z)} - 1\right| < \delta, \quad 0 < \delta < 1$$

then $f \in \Sigma_p^\star\left(p(1-\delta)\right)$.

Theorem 1.3 [**23**] *If $f \in \Sigma_p$ satisfies the inequality*

$$\left|\frac{f(z)}{zf'(z)}\left(1 + \frac{zf''(z)}{f'(z)} - \frac{zf'(z)}{f(z)}\right)\right| < \mu, \;\; 0 < \mu < \frac{1}{p}$$

then $f \in \Sigma_p^\star\left(\frac{p}{1+p\mu}\right)$.

2 Starlikeness of the operator $\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)$

In this section we place conditions for the starlikeness of the integral operator $\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)$ which is defined in (1.2).

Theorem 2.1. *For $i \in \{1,...,n\}$, let $\gamma_i > 0$ and $f_i \in \Sigma_p^{\star}(\alpha_i)$ $(0 \le \alpha_i < p)$. If $0 < \sum_{i=1}^{n} \gamma_i(p-\alpha_i) \le p$, then $F_{p,\gamma_1,\ldots,\gamma_n}(z)$ is starlike by order $p - \sum_{i=1}^{n} \gamma_i(p-\alpha_i)$.*

Proof. A differentiation of $\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)$ which is defined in (1.2), we get

$$z^{p+1}\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+1)z^p\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z) = (z^p f_1(z))^{\gamma_1} \cdots (z^p f_n(z))^{\gamma_n}, \tag{2.1}$$

and

$$z^{p+1}\mathcal{F}''_{p,\gamma_1,\ldots,\gamma_n}(z) + 2(p+1)z^p\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + p(p+1)z^{p-1}\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z) =$$

$$\sum_{i=1}^{n} \gamma_i \left(\frac{z^p f_i{}'(z) + pz^{p-1} f_i(z)}{z^p f_i(z)} \right) \left[(z^p f_1(z))^{\gamma_1} \cdots (z^p f_n(z))^{\gamma_n} \right] \tag{2.2}$$

Then from (2.1) and (2.2), we obtain

$$\frac{z^{p+1}\mathcal{F}''_{p,\gamma_1,\ldots,\gamma_n}(z) + 2(p+1)z^p\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + p(p+1)z^{p-1}\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}{z^{p+1}\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+1)z^p\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}$$

$$= \sum_{i=1}^{n} \gamma_i \left(\frac{f_i{}'(z)}{f_i(z)} + \frac{p}{z} \right). \tag{2.3}$$

By multiplying (2.3)with z yield,

$$\frac{z^{p+1}\mathcal{F}''_{p,\gamma_1,\ldots,\gamma_n}(z) + 2(p+1)z^p\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + p(p+1)z^{p-1}\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}{z^{p}\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+1)z^{p-1}\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)}$$

$$= \sum_{i=1}^{n} \gamma_i \left(\frac{z f_i{}'(z)}{f_i(z)} + p \right).$$

That is equivalent to

$$\frac{z^{p+1}\mathcal{F}''_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+2)z^p\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z)}{z^{p}\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+1)z^{p-1}\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z)} + p = \sum_{i=1}^{n} \gamma_i \left(\frac{z f_i{}'(z)}{f_i(z)} + p \right). \tag{2.4}$$

And

$$-\frac{z\left(z\mathcal{F}''_{p,\gamma_1,\dots,\gamma_n}(z)+(p+2)\mathcal{F}'_{p,\gamma_1,\dots,\gamma_n}(z)\right)}{z\mathcal{F}'_{p,\gamma_1,\dots,\gamma_n}(z)+(p+1)\mathcal{F}_{p,\gamma_1,\dots,\gamma_n}(z)}=-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i{}'(z)}{f_i(z)}+p\right)+p. \tag{2.5}$$

We can write (2.5), as the following

$$\frac{-\frac{z\mathcal{F}'_{p,\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\cdots,\gamma_n}(z)}\left(\frac{z\mathcal{F}''_{p,\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{F}'_{p,\gamma_1,\cdots,\gamma_n}(z)}+p+2\right)}{\frac{z\mathcal{F}'_{p,\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\cdots,\gamma_n}(z)}+p+1}=-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i{}'(z)}{f_i(z)}+p\right)+p. \tag{2.6}$$

We define the regular function q in $\mathbb{U}$ by

$$q(z)=-\frac{z\mathcal{F}'_{p,\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\cdots,\gamma_n}(z)}, \tag{2.7}$$

and $q(0)=p$. Differentiating $q(z)$ logarithmically, we obtain

$$-q(z)+\frac{zq'(z)}{q(z)}=1+\frac{z\mathcal{F}''_{p,\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{F}'_{p,\gamma_1,\cdots,\gamma_n}(z)}. \tag{2.8}$$

From (2.6),(2.7) and (2.8) we obtain

$$q(z)+\frac{zq'(z)}{p+1-q(z)}=-\sum_{i=1}^{n}\gamma_i\left(\frac{zf_i{}'(z)}{f_i(z)}+p\right)+p. \tag{2.9}$$

Since $f_i\in\Sigma_p^{\star}(\alpha_i),\ for\ i\in\{1,\cdots,n\}$, we receive

$$\Re\left\{q(z)+\frac{zq'(z)}{p+1-q(z)}\right\}>p-\sum_{i=1}^{n}\gamma_i(p-\alpha_i). \tag{2.10}$$

It is clear that q is analytic in $\mathbb{U}$ with $q(0)=p>p-\sum\limits_{i=1}^{n}\gamma_i(p-\alpha_i)$. We also have $\Re(\gamma-\delta\alpha)>0$,(for $\gamma=p+1,\delta=1$and $\alpha=\mathrm{p}-\sum\limits_{\mathrm{i}=1}^{\mathrm{n}}\gamma_{\mathrm{i}}(\mathrm{p}-\alpha_{\mathrm{i}})$.) Since the conditions from Lemma 1.1 are met, we obtain $\Re q(z)>p-\sum\limits_{i=1}^{n}\gamma_i(p-\alpha_i)$, which is equivalent to

$$-\frac{z\mathcal{F}'_{p,\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{F}_{p,\gamma_1,\cdots,\gamma_n}(z)}>p-\sum_{i=1}^{n}\gamma_i(p-\alpha_i).$$

that is $\mathcal{F}_{p,\gamma_1,\cdots,\gamma_n}(z)$ is starlike of order $p-\sum\limits_{i=1}^{n}\gamma_i(p-\alpha_i)$.

Theorem 2.2. *For* $i \in \{1, ..., n\}$, *let* $\gamma_i > 0$ *and* $f_i \in \Omega_p^\star(\beta_i)$ $(-1 \leq \beta_i < p)$. *If* $0 < \sum\limits_{i=1}^{n} \gamma_i(p-\beta_i) \leq p$, *then* $F_{p,\gamma_1,\dots,\gamma_n}(z)$ *is starlike by order* $p - \sum\limits_{i=1}^{n} \gamma_i(p-\beta_i)$.

Proof. Using (2.9), we have

$$q(z) + \frac{zq'(z)}{p+1-q(z)} = -\sum_{i=1}^{n} \gamma_i \left(\frac{z{f_i}'(z)}{f_i(z)} + \beta_i \right) + p - \sum_{i=1}^{n} \gamma_i\,(p-\beta_i). \quad (2.11)$$

Since $f_i \in \Omega_p^\star(\beta_i)$, for $\mathrm{i} \in \{1, \cdots, \mathrm{n}\}$, we get

$$\Re\left\{ q(z) + \frac{zq'(z)}{p+1-q(z)} \right\} > \sum_{i=1}^{n} \gamma_i \left| \frac{zf_i'(z)}{f_i(z)} + p \right| + p - \sum_{i=1}^{n} \gamma_i\,(p-\beta_i). \quad (2.12)$$

Because $\sum\limits_{i=1}^{n} \gamma_i \left| \frac{zf_i'(z)}{f_i(z)} + p \right| > 0$, we obtain that

$$\Re\left\{ q(z) + \frac{zq'(z)}{p+1-q(z)} \right\} > p - \sum_{i=1}^{n} \gamma_i\,(p-\beta_i). \quad (2.13)$$

The remaining part of the proof follows the pattern of those in Theorem 2.1. The proof is complete.

3 Starlikeness of the operator $\mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(z)$

In this section we place conditions for the starlikeness of the integral operator $\mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(z)$ which is defined in (1.3).

Theorem 2.3. *For* $i \in \{1, ..., n\}$, *let* $\gamma_i > 0$ *and* $f_i \in \Sigma K_p(\alpha_i)$ $(0 \leq \alpha_i < p)$. *If* $0 < \sum\limits_{i=1}^{n} \gamma_i(p-\alpha_i) \leq p$, *then* $\mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(z)$ *is starlike by order* $p - \sum\limits_{i=1}^{n} \gamma_i(p - \alpha_i)$.

Proof. A differentiation of $\mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(z)$, which is defined in (1.3), we get

$$z^{p+1}\mathcal{J}'_{p,\gamma_1,\dots,\gamma_n}(z) + (p+1)z^p \mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(z)$$

$$= \left(\frac{-z^{p+1}}{p} f_1'(z) \right)^{\gamma_1} \cdots \left(\frac{-z^{p+1}}{p} f_n'(z) \right)^{\gamma_n}, \quad (3.1)$$

and

$$z^{p+1}\mathcal{J}''_{p,\gamma_1,\dots,\gamma_n}(z) + 2(p+1)z^p \mathcal{J}'_{p,\gamma_1,\dots,\gamma_n}(z) + p(p+1)z^{p-1}\mathcal{J}_{p,\gamma_1,\dots,\gamma_n}(z) =$$

$$\sum_{i=1}^{n}\gamma_i\left(\frac{z^{p+1}{f_i}''(z)+(p+1)z^p{f_i}'(z)}{z^{p+1}f_i'(z)}\right)\left[\left(\frac{-z^{p+1}}{p}f_1'(z)\right)^{\gamma_1}\cdots\left(\frac{-z^{p+1}}{p}f_n'(z)\right)^{\gamma_n}\right] \tag{3.2}$$

Then from (3.1) and (3.2), we obtain

$$\frac{z^{p+1}\mathcal{J}''_{p},_{\gamma_1,\ldots,\gamma_n}(z)+2(p+1)z^p\mathcal{J}'_{p},_{\gamma_1,\ldots,\gamma_n}(z)+p(p+1)z^{p-1}\mathcal{J}_{p,\gamma_1,\ldots,\gamma_n}(z)}{z^p\mathcal{J}'_{p},_{\gamma_1,\ldots,\gamma_n}(z)+(p+1)z^{p-1}\mathcal{J}_{p,\gamma_1,\ldots,\gamma_n}(z)}$$

$$=\sum_{i=1}^{n}\gamma_i\left(\frac{z{f_i}''(z)}{{f_i}'(z)}+p+1\right). \tag{3.3}$$

That is equivalent to

$$-\frac{z\left(z\mathcal{J}''_{p},_{\gamma_1,\ldots,\gamma_n}(z)+(p+2)\mathcal{J}'_{p},_{\gamma_1,\ldots,\gamma_n}(z)\right)}{z\mathcal{J}'_{p},_{\gamma_1,\ldots,\gamma_n}(z)+(p+1)\mathcal{J}_{p,\gamma_1,\ldots,\gamma_n}(z)}=-\sum_{i=1}^{n}\gamma_i\left(\frac{z{f_i}''(z)}{{f_i}'(z)}+p+1\right)+p. \tag{3.4}$$

We can write (3.4), as the following

$$\frac{-\frac{z\mathcal{J}'_{p},_{\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{J}_{p,\gamma_1,\cdots,\gamma_n}(z)}\left(\frac{z\mathcal{J}''_{p},_{\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{J}'_{p},_{\gamma_1,\cdots,\gamma_n}(z)}+p+2\right)}{\frac{z\mathcal{J}'_{p},_{\gamma_1,\cdots,\gamma_n}(z)}{\mathcal{J}_{p,\gamma_1,\cdots,\gamma_n}(z)}+p+1}=-\sum_{i=1}^{n}\gamma_i\left(\frac{z{f_i}''(z)}{{f_i}'(z)}+p+1\right)+p. \tag{3.5}$$

Define the regular function q in $\mathbb{U}$ by

$$q(z)=-\frac{z\mathcal{J}'_{p},_{\gamma_1},\cdots,_{\gamma_n}(z)}{\mathcal{J}_{p},_{\gamma_1},\cdots,_{\gamma_n}(z)}, \tag{3.6}$$

and $q(0)=p$. Differentiating $q(z)$ logarithmically, we obtain

$$-q(z)+\frac{zq'(z)}{q(z)}=1+\frac{z\mathcal{J}''_{p},_{\gamma_1},\cdots,_{\gamma_n}(z)}{\mathcal{J}'_{p},_{\gamma_1},\cdots,_{\gamma_n}(z)}. \tag{3.7}$$

From (3.5),(3.6) and (3.7) we obtain

$$q(z)+\frac{zq'(z)}{p+1-q(z)}=-\sum_{i=1}^{n}\gamma_i\left(\frac{z{f_i}''(z)}{{f_i}'(z)}+p+1\right)+p. \tag{3.8}$$

Since $f_i\in\Sigma K_p(\alpha_i),\ for\ i\in\{1,\cdots,n\}$, we receive

$$\Re\left\{q(z)+\frac{zq'(z)}{p+1-q(z)}\right\}>p-\sum_{i=1}^{n}\gamma_i(p-\alpha_i). \tag{3.9}$$

The remaining part of the proof follows the pattern of those in Theorem 2.1.

Theorem 2.4. *For $i \in \{1, ..., n\}$, let $\gamma_i > 0$ and $f_i \in \Omega K_p(\beta_i)$ $(-1 \leq \beta_i < p)$. If $0 < \sum_{i=1}^{n} \gamma_i(p - \beta_i) \leq p$, then $\mathcal{J}_{p,\gamma_1, \ldots, \gamma_n}(z)$ is starlike by order $p - \sum_{i=1}^{n} \gamma_i(p - \beta_i)$.*

Other work that we can look at regarding these operators is on superordination-preseving integral operator[22] and related to it (see [2],[4], [11]).

4 Application

As an application to the integral operators $\mathcal{F}_{p,\gamma_1, \ldots, \gamma_n}(z)$ and $\mathcal{J}_{p,\gamma_1, \ldots, \gamma_n}(z)$, we define the following two p -valent meromorphic functions.

Definition 4.1. *Let $\mathcal{F}_{p,\gamma_1, \ldots, \gamma_n}(z)$ be the integral operator defined as in (1.2). We define the following function,*

$$\Phi(z) = z\mathcal{F}'_{p,\gamma_1, \ldots, \gamma_n}(z) + (p+1)\mathcal{F}_{p,\gamma_1, \ldots, \gamma_n}(z). \tag{4.1}$$

Definition 4.2 *Let $\mathcal{J}_{p,\gamma_1, \ldots, \gamma_n}(z)$ be the integral operator defined as in (1.3). We define the following function,*

$$\Upsilon(z) = z\mathcal{J}'_{p,\gamma_1, \ldots, \gamma_n}(z) + (p+1)\mathcal{J}_{p,\gamma_1, \ldots, \gamma_n}(z). \tag{4.2}$$

Next, we study some orders of starlikeness of $\Phi(z)$ and $\Upsilon(z)$.

Theorem 4.1. *For $i \in \{1, ..., n\}$, let $\gamma_i \in \mathbb{R}$, $\gamma_i > 0$ and $f_i \in \Sigma_p^{\star}(\alpha_i)$ $(0 \leq \alpha_i < p)$. If $0 < \sum_{i=1}^{n} \gamma_i(p - \alpha_i) \leq p$, then $\Phi(z)$ given by (4.1) is starlike by order $p - \sum_{i=1}^{n} \gamma_i(p - \alpha_i)$.*

Proof. A successive differentiation of $\mathcal{F}_{p,\gamma_1, \ldots, \gamma_n}(z)$ which is defined in (1.2), we get

$$z^p\Phi(z) = z^{p+1}\mathcal{F}'_{p,\gamma_1,\ldots,\gamma_n}(z) + (p+1)z^p\mathcal{F}_{p,\gamma_1,\ldots,\gamma_n}(z) = (z^p f_1(z))^{\gamma_1} \ldots (z^p f_n(z))^{\gamma_n}. \tag{4.3}$$

Using (4.3), we get

$$\frac{z\Phi'(z)}{\Phi(z)} = \sum_{i=1}^{n} \gamma_i \frac{z f_i{}'(z)}{f_i(z)} + p\sum_{i=1}^{n} \gamma_i - p. \tag{4.4}$$

Since $f_i \in \Sigma_p^\star(\alpha_i)$ we receive

$$-\Re\frac{z\Phi'(z)}{\Phi(z)} = \sum_{i=1}^{n}\gamma_i\Re\left\{-\frac{zf_i{}'(z)}{f_i(z)}\right\} - p\sum_{i=1}^{n}\gamma_i + p$$

$$> p - \sum_{i=1}^{n}\gamma_i\left(p-\alpha_i\right).$$

But by the hypothesis, $0 \le p - \sum_{i=1}^{n}\gamma_i\left(p-\alpha_i\right) < 1$. Thus $\Phi(z)$ is starlike by order $p - \sum_{i=1}^{n}\gamma_i(p-\alpha_i)$.

Theorem 4.2. *For $i \in \{1, ..., n\}$, let $\gamma_i \in \mathbb{R}$, $\gamma_i > p$ and $\sum_{i=1}^{n}\gamma_i \le n+1$. If $f_i \in \Sigma_p^\star\left(\frac{p}{\gamma_i}\right)$, then $\Phi(z)$ belong to $\Sigma^\star(0)$.*

Now, adopting the same technique used in Theorem 4.1 and applying Theorem 1.2 and Theorem 1.3, one can prove

Theorem 4.3. *For $i \in \{1, ..., n\}$, let $\gamma_i > 0$, $0 < \delta_i < 1$, $f_i \in \Sigma_p$, and*

$$\left|\frac{zf_i'(z)}{f_i(z)} - \frac{zf_i''(z)}{f_i'(z)} - 1\right| < \delta_i.$$

If $0 < \sum_{i=1}^{n}\gamma_i\delta_i \le 1$, then $\Phi(z)$ is starlike by order $p - p\sum_{i=1}^{n}\gamma_i\delta_i$.

Theorem 4.4. *For $i \in \{1, ..., n\}$, let $\gamma_i > 0$, $0 < \mu_i < \frac{1}{p}$, $f_i \in \Sigma_p$ and*

$$\left|\frac{f_i(z)}{zf_i'(z)}\left(1 + \frac{zf_i''(z)}{f_i'(z)} - \frac{zf_i'(z)}{f_i(z)}\right)\right| < \mu_i.$$

If $0 < \sum_{i=1}^{\infty}\frac{\gamma_i\mu_i}{\frac{1}{p}+\mu_i} \le 1$ then $\Phi(z)$ is starlike by order $p - p\sum_{i=1}^{\infty}\frac{\gamma_i\mu_i}{\frac{1}{p}+\mu_i}$.

Finally, we introduce the following results for the function $\Upsilon(z)$.

Theorem 4.5. *For $i \in \{1, ..., n\}$, let $\gamma_i \in \mathbb{R}$, $\gamma_i > 0$ and $f_i \in \Sigma K_p(\alpha_i)$ $(0 \le \alpha_i < p)$. If $0 < \sum_{i=1}^{n}\gamma_i(p-\alpha_i) \le p$, then $\Upsilon(z)$ given by (4.2) is starlike by order $p - \sum_{i=1}^{n}\gamma_i(p-\alpha_i)$.*

Theorem 4.6. *For $i \in \{1, ..., n\}$, let $\gamma_i \in \mathbb{R}$, $\gamma_i > p$ and $\sum\limits_{i=1}^{n} \gamma_i \leq n+1$. If $f_i \in \Sigma K_p\left(\frac{p}{\gamma_i}\right)$, then $\Upsilon(z)$ belong to $\Sigma^{\star}(0)$.*

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