$\pi p$-Normal Topological Spaces

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Abstract

The main aim of this work is to introduce a weaker version of $p$-normality called $\pi p$-normality, which lies between $p$-normality and almost $p$-normality. We prove that $p$-normality is a topological property and it is a hereditary property with respect to closed domain subspaces. Some basic properties, examples, characterizations and preservation theorems of this property are presented.

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1 Introduction and Preliminary

Throughout this paper, a space $X$ always means a topological space on which no separation axioms are assumed, unless explicitly stated. We will denote an ordered pair by $(x, y)$ and the set of real numbers by $\mathbb{R}$. For a subset $A$ of a space $X$, $X \setminus A$, $\overline{A}$ and $\text{int}(A)$ denote to the complement, the closure and the interior of $A$ in $X$, respectively. If $M$ is a subspace of $X$ and $A \subseteq M$, then $\overline{A}^X$, $\overline{A}^M$ and $\text{int}_X(A)$, $\text{int}_M(A)$ denote to the closure (resp. the interior) of $A$ in $X$ and in $M$, respectively. A subset $A$ of a space $X$ is said to be regularly-open or an open domain if it is the interior of its own closure, or equivalently if it is the interior of some closed set, [7]. A set $A$ is said to be a closed domain if its complement is an open domain. A subset $A$ of a space $X$ is called a $\pi$-closed if it is a finite intersection of closed domain subsets and $A$ is called a $\pi$-open if its complement is a $\pi$-closed, [17]. Two sets $A$ and $B$ of a space $X$ are said to be separated if there exist two disjoint open sets $U$ and $V$ in $X$ such that $A \subseteq U$ and $B \subseteq V$, [2, 4, 11]. A space $X$ is called an almost normal if any

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two disjoint closed subsets $A$ and $B$ of $X$, one of which is closed domain, can be separated, [15]. A space $X$ is called a $\pi$-normal if any two disjoint closed subsets $A$ and $B$ of $X$, one of which is $\pi$-closed, can be separated, [6]. Also, we need to recall the following definitions.

**Definition 1.1** [8], A subset $A$ of a space $X$ is said to be:

1. pre-open (briefly $p$-open) if $A \subseteq \text{int}(\overline{A})$.
2. semi-open if $A \subseteq \text{int}(A)$.
3. $\alpha$-open if $A \subseteq \text{int(\text{int}(A))}$.

The complement of $p$-open (resp. semi-open, $\alpha$-open) set is called $p$-closed (resp. semi-closed, $\alpha$-closed). The intersection of all $p$-closed sets containing $A$ is called $p$-closure of $A$ and denoted by $p\text{cl}(A)$. Dually, the $p$-interior of $A$, denoted by $p\text{int}(A)$, is defined to be the union of all $p$-open sets contained in $A$. A subset $A$ is said to be a $p$-neighborhood of $x$ if there exists a $p$-open set $U$ such that $x \in U \subseteq A$.

**Definition 1.2** A space $X$ is said to be a $p$-normal, [12] (resp. mildly $p$-normal, [8]) if any disjoint closed (resp. closed domain) sets $A$ and $B$ of $X$ can be separated by two disjoint $p$-open sets $U$ and $V$ of $X$.

**Definition 1.3** A space $X$ is said to be an almost $p$-normal if any disjoint closed sets $A$ and $B$ of $X$, one of which is closed domain, can be separated by two disjoint $p$-open sets $U$ and $V$, see [8].

Clearly that:

closed domain $\implies$ $\pi$-closed $\implies$ closed $\implies$ $\alpha$-closed $\implies$ $p$-closed

None of the above implications is reversible.

In this paper, we introduce a weaker version of $p$-normality called $\pi p$-normality. The importance of this property is that it behaves slightly different from $p$-normality and almost $p$-normality. In fact, there are many $\pi p$-normal spaces which are not $p$-normal. We present some characterizations and preservation theorems of $\pi p$-normality. Also, we show that it is a topological property and a hereditary property with respect to closed domain subspaces. Some basic properties are given.

## 2 Main Results

First, we begin with the definition of $\pi p$-normality.
**Definition 2.1** A space $X$ is said to be $\pi p$-normal if for every pair of disjoint closed sets $A$ and $B$ of $X$, one of which is $\pi$-closed, there exist disjoint $p$-open sets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$.

Clearly, every $\pi$-normal space is $\pi p$-normal and every $\pi p$-normal space is almost $p$-normal. Thus, we have:

$$
\text{normal} \implies \text{p-normal} \implies \text{\pi p-normal} \implies \text{almost p-normal}
$$

$$
\text{normal} \implies \text{\pi-normal} \implies \text{\pi p-normal}
$$

None of the above implications is reversible.

**Example 2.2** Consider the topology $\mathcal{T} = \{X, \emptyset, \{a\}, \{a, b\}, \{a, c\}\}$ on the set $X = \{a, b, c\}$. Then, $X$ is $\pi p$-normal space because the only $\pi$-closed sets in $X$ are $X$ and $\emptyset$. But it is not $p$-normal, since the pair of disjoint closed sets $\{b\}$ and $\{c\}$ have no disjoint $p$-open subsets containing them.

**Example 2.3** $\pi p$-Normality does not imply almost $p$-regularity as the following example shows. Let $X = \{a, b, c\}$ and $\mathcal{T} = \{X, \emptyset, \{a\}, \{b\}, \{a, b\}\}$. Then, $X$ is $p$-normal (hence $\pi p$-normal). But it is not almost $p$-regular (hence not $p$-regular), since for the closed domain set $\{a, c\}$ and the point $b \notin \{a, c\}$. There do not exist disjoint $p$-open subsets containing them.

Before giving counterexamples about the other implications, we present the following Lemmas:

**Lemma 2.4** Let $D$ be a dense subset of a space $X$, then $D$ is $p$-open.

*Proof.* Let $D$ be a dense subset of a space $X$. Then $\overline{D} = X$. Thus, $\text{int}(\overline{D}) = \text{int}(X) = X$. Therefore, $D \subseteq \text{int}(\overline{D})$. Hence, $D$ is $p$-open. □

**Lemma 2.5** If $D$ and $E$ are disjoint dense subsets of a space $X$, then $D$ and $E$ are disjoint $p$-open subsets.

**Lemma 2.6** Let $D$ be a dense subset of a space $X$. For any closed subset $A$ of $X$, the set $G = D \cup A$ is $p$-open.

*Proof.* Let $D$ be a dense subset of a space $X$. Then $\overline{D} = X$. Let $A$ be a closed subset of $X$ and $G = D \cup A$. Then, $\overline{G} = (\overline{D} \cup \overline{A}) = \overline{D} \cup \overline{A} = X \cup A = X$. Thus, $\text{int}(\overline{G}) = \text{int}(X) = X$. Hence, $G \subseteq \text{int}(\overline{G})$. Therefore, $G$ is $p$-open subset of $X$. □

**Lemma 2.7** Let $D$ be a dense subset of a space $X$. For any closed subset $A$ of $X$, the subset $H = D \setminus A$ is $p$-open.
Proof. Let $A$ be a closed and $D$ be a dense subset of $X$. Let $H = D \setminus A$. Then, we have $H = D \setminus A = D \cap X \setminus A = X \setminus A = X \setminus \text{int}(A)$ as $D$ is dense and $X \setminus A$ is open. Therefore, $\text{int}(H) = \text{int}(X \setminus \text{int}(A)) = X \setminus \text{int}(A)$. Since $\text{int}(A) \subseteq A$, then $X \setminus A \subseteq X \setminus \text{int}(A) = \text{int}(H)$. Thus, $H = D \setminus A \subseteq X \setminus A \subseteq X \setminus \text{int}(A) = \text{int}(H)$. Therefore, $H \subseteq \text{int}(H)$. Hence, $H$ is $p$-open. □

The following lemma can be proved easily.

**Lemma 2.8** Let $D$ be a dense subset of a space $X$. For any two disjoint closed subsets $A$ and $B$ of $X$, the sets $U = (D \setminus A) \cup B$ and $V = (D \setminus B) \cup A$ are $p$-open subsets.

The following lemma is obvious by using the Lemma 2.8.

**Lemma 2.9** If $D$ and $E$ are disjoint dense subsets of a space $X$, then $X$ is $p$-normal.

**Corollary 2.10** The co-finite topology on $\mathbb{R}$ is $p$-normal space.

Recall that, the co-finite topology on $\mathbb{R}$ is denoted by $\mathcal{CF}$ and defined as: $U \in \mathcal{CF}$ if and only if $U = X$ or $\mathbb{R} \setminus U$ is finite. In this topology, observe that $\mathbb{Q} = \mathbb{R} = \mathbb{P}$. Thus, $\mathbb{Q}$ and $\mathbb{P}$ are disjoint dense subsets of $(\mathbb{R}, \mathcal{CF})$. Hence, by Lemma 2.9, $(\mathbb{R}, \mathcal{CF})$ is $p$-normal (hence $\pi p$-normal). It is well known that $(\mathbb{R}, \mathcal{CF})$ is not normal, see [16]. Therefore, the co-finite topology on $\mathbb{R}$ is an example of $\pi p$-normal space but not normal.

**Corollary 2.11** The Niemytzki plane topology is $p$-normal space.

In the Niemytzki Plane Topology $X = L \cup P$, see [16], let $D = \{\langle x, y \rangle : \langle x, y \rangle \in \mathbb{Q}^2, y \geq 0\}$ and $E = \{\langle x, y \rangle : \langle x, y \rangle \in \mathbb{P}^2, y > 0\}$. Then, $\overline{D} = \overline{E} = X$ and $D \cap E = \emptyset$. Therefore, $D$ and $E$ are disjoint dense subsets of $X$. By Lemma 2.9, $X$ is $p$-normal and hence $\pi p$-normal. But the Niemytzki plane is not $\pi$-normal, see [13]. Therefore, the Niemytzki plane topology is an example of a $\pi p$-normal Tychonoff space but not a $\pi$-normal.

**Example 2.12**. The particular point topology on $X = \mathbb{R}$ is denoted by $\mathcal{T}_{\sqrt{2}}$ and defined as: $U \in \mathcal{T}_{\sqrt{2}}$ if and only if $U = \emptyset$ or $\sqrt{2} \in U$. Since the particular point topology $(\mathbb{R}, \mathcal{T}_{\sqrt{2}})$ is $\pi$-normal, then it is $\pi p$-normal. In fact, the only $\pi$-closed subsets of $X$ are $X$ and $\emptyset$. Now, let $A \subseteq \mathbb{R}$. Observe that:

$$
\overline{A} = \begin{cases}
\mathbb{R} & \text{if } \sqrt{2} \in A \\
A & \text{if } \sqrt{2} \notin A
\end{cases}
$$

and so,

$$
\text{int}(\overline{A}) = \begin{cases}
\mathbb{R} & \text{if } \sqrt{2} \in A \\
\emptyset & \text{if } \sqrt{2} \notin A
\end{cases}
$$
Therefore, the only p-open subsets of \( X \) are those which are open. Thus, any two disjoint closed subsets of \( X \) can not be separated by two disjoint p-open subsets. Hence, \( X \) is not p-normal. Observe that the particular point topology is an example of \( \pi p \)-normal space but not p-normal.

Now, we show that the Rational Sequence topology is an example of an almost p-normal space but not a \( \pi p \)-normal. First, we recall its definition.

**Example 2.13** Let \( X = \mathbb{R} \). For each \( x \in \mathbb{P} \), where \( \mathbb{P} \) is the irrational numbers, fix a sequence \( \{x_n\}_{n \in \mathbb{N}} \subset \mathbb{Q} \), such that \( x_n \to x \), where the convergence is taken in \((\mathbb{R}, \mathcal{U})\). Let \( A_n(x) \) denote the \( n^{\text{th}} \)-tail of the sequence, where \( A_n(x) = \{x_j : j \geq n\} \). For each \( x \in \mathbb{P} \), let \( \mathcal{B}(x) = \{U_n(x) : n \in \mathbb{N}\} \), where \( U_n(x) = A_n(x) \cup \{x\} \). For each \( x \in \mathbb{Q} \), let \( \mathcal{B}(x) = \{\{x\}\} \). Then \( \{\mathcal{B}(x)\}_{x \in \mathbb{R}} \) is a neighborhood system. The unique topology on \( \mathbb{R} \) generated by \( \{\mathcal{B}(x)\}_{x \in \mathbb{R}} \) is called the Rational Sequence topology on \( \mathbb{R} \) and denoted by \( RS \).

It is well known that the Rational Sequence topology is a Tychonoff, first countable and not normal, see [16]. Also, we proved that it is almost normal and not \( \pi \)-normal in [14]. Now, we give the following lemma.

**Lemma 2.14** Every p-open subset of the Rational Sequence topology is an open subset.

**Proof.** Let \( A \) be a p-open subset of \( X \), then \( A \subseteq \text{int}(\overline{A}) \). \( A \) can not be in \( \mathbb{P} \) (i.e. \( A \not\subseteq \mathbb{P} \)). In fact, if \( A \subseteq \mathbb{P} \) we have \( \text{int}(\overline{A}) = \text{int}(A) = \emptyset \) and thus \( A \not\subseteq \text{int}(\overline{A}) \). Hence, \( A \not\subseteq \mathbb{P} \). Then, either \( A \subseteq \mathbb{Q} \) or \( A \cap \mathbb{P} \neq \emptyset \neq A \cap \mathbb{Q} \). Now, we show that \( A \) is an open subset of \( X \) for each case.

**Case 1.** Let \( A \subseteq \mathbb{Q} \).

Then, \( A \) is an open subset of \( X \).

**Case 2.** Let \( A \cap \mathbb{P} \neq \emptyset \neq A \cap \mathbb{Q} \).

For each \( x \in A \cap \mathbb{P} \), there is a sequence \( \{x_n : n \in \mathbb{N}\} \subset \mathbb{Q} \) such that \( x_n \to x \).

Let \( D_x(n) = \{x_j : j \geq n, n \in \mathbb{N}\} \cup \{x\} \) be a basic open neighborhood of \( x \) and let \( E = \bigcup_{x \in A \cap \mathbb{P}} D_x(n) \).

**Claim:** For each \( x \in A \cap \mathbb{P} \), there is a natural number \( m_x \) such that \( x \in D_x(m_x) \subseteq A \).

Suppose that there exists an \( x \in A \cap \mathbb{P} \) such that \( D_x(n) \not\subseteq A \) for each \( n \in \mathbb{N} \).

Without loss of generality, we may assume that \( D_x(n) \cap A = \{x\} \). This implies that \( \overline{A} = (E \setminus D_x(n)) \cup \{x\} \cup (A \cap \mathbb{Q}) \) and \( \text{int}(\overline{A}) = (E \setminus D_x(n)) \cup (A \cap \mathbb{Q}) \). So, we have \( x \not\in \text{int}(\overline{A}) \). But \( x \in A \). Thus, \( A \not\subseteq \text{int}(\overline{A}) \) and hence \( A \) is not p-open, which is a contradiction. Hence, there is an \( m_x \) such that \( x \in D_x(m_x) \subseteq A \) for each \( x \in A \cap \mathbb{P} \). Therefore, \( D = \bigcup_{x \in A \cap \mathbb{P}} D_x(m_x) \subseteq A \). Thus, \( D \) is an open subset of \( X \) and it is contained in \( A \). So, we have \( D \cup (A \cap \mathbb{Q}) \subseteq A \). Since \( A \subseteq D \cup (A \cap \mathbb{Q}) \), then \( A = D \cup (A \cap \mathbb{Q}) \). Therefore, \( A \) is open set in \( X \). □
Theorem 2.15 The Rational Sequence topology is an almost \( p \)-normal and not a \( \pi p \)-normal.

Proof. Since \( X \) is an almost normal, see [14], then it is an almost \( p \)-normal. Now, we need to show that \( X \) is not \( \pi p \)-normal. For that, suppose \( X \) is \( \pi p \)-normal. Let \( A \) and \( B \) be any disjoint closed subsets of \( X \) such that \( A \) is \( \pi \)-closed. By \( \pi p \)-normality of \( X \), there exist disjoint \( p \)-open subsets \( U \) and \( V \) of \( X \) such that \( A \subseteq U \) and \( B \subseteq V \). By the Lemma 2.14, \( U \) and \( V \) are disjoint open subsets of \( X \). Thus, \( A \) and \( B \) can be separated by two disjoint open subsets. Hence, \( X \) is \( \pi \)-normal, which is a contradiction as \( X \) is not \( \pi \)-normal, see [14]. Therefore, \( X \) is not \( \pi p \)-normal. \( \square \)

Observe that the Rational Sequence topology is an example of an almost \( p \)-normal Tychonoff space but not \( \pi p \)-normal.

3 Characterizations of \( \pi p \)-Normality and Preservation Theorems

Now, we give some characterizations of \( \pi p \)-normal spaces.

Theorem 3.1 For a space \( X \), the following are equivalent:

(a) \( X \) is \( \pi p \)-normal.

(b) For every pair of open sets \( U \) and \( V \), one of which is \( \pi \)-open, whose union is \( X \), there exist \( p \)-closed sets \( G \) and \( H \) such that \( G \subseteq U \), \( H \subseteq V \) and \( G \cup H = X \).

(c) For every closed set \( A \) and every \( \pi \)-open set \( B \) such that \( A \subseteq B \), there is a \( p \)-open set \( V \) such that \( A \subseteq V \subseteq p \text{cl}(V) \subseteq B \).

Proof. (a) \( \Longrightarrow \) (b). Let \( U \) and \( V \) be open sets in a \( \pi p \)-normal space \( X \) such that \( V \) is \( \pi \)-open and \( U \cup V = X \). Then, \( X \setminus U \) and \( X \setminus V \) are closed sets in \( X \) such that \( X \setminus V \) is \( \pi \)-closed and \( (X \setminus U) \cap (X \setminus V) = \emptyset \). By \( \pi p \)-normality of \( X \), there exist disjoint \( p \)-open sets \( U_1 \) and \( V_1 \) such that \( X \setminus U \subseteq U_1 \) and \( X \setminus V \subseteq V_1 \). Let \( G = X \setminus U_1 \) and \( H = X \setminus V_1 \). Then, \( G \) and \( H \) are \( p \)-closed sets of \( X \) such that \( G \subseteq U \), \( H \subseteq V \) and \( G \cup H = X \).

(b) \( \Longrightarrow \) (c). Let \( A \) be a closed set and let \( B \) be a \( \pi \)-open set of \( X \) such that \( A \subseteq B \). Then, \( A \cap (X \setminus B) = \emptyset \). Thus, \( (X \setminus A) \cup B = X \), where \( X \setminus A \) is open. By (b), there exist \( p \)-closed sets \( G \) and \( H \) of \( X \) such that \( G \subseteq X \setminus A \), \( H \subseteq B \) and \( G \cup H = X \). Thus, we obtain that \( A \subseteq X \setminus G \) and \( X \setminus G \subseteq H \). Let \( V = X \setminus G \), then \( V \) is \( p \)-open set of \( X \). Therefore, we have \( A \subseteq V \subseteq p \text{cl}(V) \subseteq B \).

(c) \( \Longrightarrow \) (a). Let \( A \) and \( B \) be any disjoint closed sets of \( X \) such that \( B \) is
π-closed. Since $A \cap B = \emptyset$, then $A \subseteq X \setminus B$ and $X \setminus B$ is π-open. By (c), there exists a p-open set $V$ such that $A \subseteq V \subseteq p\text{cl}(V) \subseteq X \setminus B$. Put $G = V$ and $H = X \setminus p\text{cl}(V)$. Then, $G$ and $H$ are disjoint p-open subsets of $X$ such that $A \subseteq G$ and $B \subseteq H$. Hence, $X$ is πp-normal. □

The following important proposition can be proved easily.

**Proposition 3.2** Let $f : X \rightarrow Y$ be a function, then:

(a) The image of p-open subset under an open continuous function is p-open.

(b) The inverse image of p-open (resp. p-closed) subset under an open continuous function is p-open.

(c) The image of p-closed subset under an open and a closed continuous surjective function is p-closed.

Now, we prove the following result.

**Theorem 3.3** The image of a πp-normal space under an open continuous injective function is πp-normal.

**Proof.** Let $X$ be a πp-normal space and let $f : X \rightarrow Y$ be an open continuous injective function. We need to show that $f(X)$ is πp-normal. Let $A$ and $B$ be two disjoint closed sets in $f(X)$ such that $A$ is π-closed. Then, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint closed sets in $X$ such that $f^{-1}(A)$ is π-closed. By πp-normality of $X$, there exist p-open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subseteq U$, $f^{-1}(B) \subseteq V$ and $U \cap V = \emptyset$. Since $f$ is an open continuous one-to-one function, we have $A \subseteq f(U)$, $B \subseteq f(V)$ and $f(U) \cap f(V) = \emptyset$. By the Proposition 3.2, $f(U)$ and $f(V)$ are disjoint p-open sets in $f(X)$ such that $A \subseteq f(U)$ and $B \subseteq f(V)$. Hence, $f(X)$ is πp-normal space. □

From the above Theorem, we obtain the following corollary.

**Corollary 3.4** πp-Normality is a topological property.

The following lemmas help us to show that πp-normality is a hereditary property with respect to closed domain subspaces.

**Lemma 3.5** If $M$ be a subspace of a space $X$ and $A \subseteq M$, then $\text{int}_M(A) = \text{int}_M(A) \cap \text{int}_X(M)$.

**Remark.** If $M$ be an open subspace of a space $X$ and $A \subseteq M$, then $\text{int}_M(\overline{A}^M) = \text{int}_X(\overline{A}^M) = \text{int}_X(\overline{A}) \cap M$.

**Lemma 3.6** Let $M$ be a closed domain subspace of a space $X$ and $A \subseteq M$. $A$ is p-closed set in $M$ if and only if $A$ is p-closed set in $X$. 


Proof. Let $M$ be a closed domain subspace of $X$ and $A \subseteq M$. Then, $M = \text{int}_X(M)^X$. Let $A$ be a $p$-closed set in $M$. Then, $\text{int}_M(A)^M \subseteq A$. Since $\text{int}_X(M)$ is dense in $M$, then $\text{int}_M(A)^M = \text{int}_M(A) \cap \text{int}_X(M)^M = \text{int}_X(A)^X \cap M = \text{int}_X(A)^X$ as $\text{int}_X(A)^X \subseteq M$. Thus, $\text{int}_X(A)^X \subseteq A$. Hence, $A$ is $p$-closed set in $X$.

Conversely, suppose that $A$ is $p$-closed set in $X$. Then, $\text{int}_X(A)^X \subseteq A$. Now, $\text{int}_M(A)^M = \text{int}_X(A)^X \cap M = \text{int}_X(A)^M = \text{int}_M(A) \cap \text{int}_X(M)^M = \text{int}_M(A)^M$. Thus, $\text{int}_M(A)^M \subseteq A$. Hence, $A$ is $p$-closed set in $M$. □

Lemma 3.7 Let $M$ be an open subspace of a space $X$ and $A \subseteq M$. $A$ is $p$-open set in $M$ if and only if $A$ is $p$-open set in $X$.

Proof. Let $A$ be a $p$-open set in $M$. Then, $A \subseteq \text{int}_M(A)^M = \text{int}_X(A)^M = \text{int}_X(A^X) \cap M \subseteq \text{int}_X(A^X)$ as $M$ is open. Thus, $A \subseteq \text{int}_X(A^X)$. Hence, $A$ is $p$-open set in $X$.

Conversely, suppose that $A$ is $p$-open set in $X$. Then, $A \subseteq \text{int}_X(A^X)$. Thus, we have $A \subseteq \text{int}_X(A^X) \cap M = \text{int}_M(A^M)$ as $M$ is open. Therefore, $A \subseteq \text{int}_M(A^M)$. Hence, $A$ is $p$-open in $M$. □

Lemma 3.8 Let $M$ be a closed domain subspace of a space $X$. If $A$ is a $p$-closed set in $X$, then $A \cap M$ is $p$-closed set in $M$.

Proof. Let $A$ be a $p$-closed set in $X$. Then, $\text{int}_X(A)^X \subseteq A$. We need to show that $A \cap M$ is $p$-closed set in $M$.

Now we have, $\text{int}_M(A \cap M)^M = \text{int}_M(A \cap M) \cap \text{int}_X(M)^M = \text{int}_X(A \cap M)^X \cap M = \text{int}_X(A \cap M)^X \cap M \subseteq \text{int}_X(A)^X \cap M \subseteq A \cap M$. Hence, $A \cap M$ is $p$-closed set in $M$. □

Lemma 3.9 Let $M$ be a closed domain subspace of a space $X$. If $A$ is a $p$-open set in $X$, then $A \cap M$ is $p$-open set in $M$.

Proof. Let $A$ be a $p$-open set in $X$. Then, $X \setminus A$ is $p$-closed set in $X$. Thus by the Lemma 3.8, the set $G = (X \setminus A) \cap M$ is $p$-closed set in $M$. Therefore, $M \setminus G$ is $p$-open set in $M$. But $M \setminus G = A \cap M$. Hence, $A \cap M$ is $p$-open set in $M$. □

Now, we prove the following result.

Theorem 3.10 A closed domain subspace of a $\pi p$-normal space is $\pi p$-normal.

Proof. Let $M$ be a closed domain subspace of a $\pi p$-normal space $X$. Let $A$ and $B$ be any disjoint closed sets in $M$ such that $B$ is $\pi$-closed. Then, $A$ and $B$ are disjoint closed sets in $X$ such that $B$ is $\pi$-closed set in $X$. By $\pi p$-normality of
$\pi p$-normal topological spaces

Let $X$, there exist disjoint $p$-open sets $U$ and $V$ of $X$ such that $A \subseteq U$ and $B \subseteq V$. By the Lemma 3.9 (also, since every closed domain is semi-open, then by the Lemma 6.3 in [3]), we obtain that $U \cap M$ and $V \cap M$ are disjoint $p$-open sets in $M$ such that $A \subseteq U \cap M$ and $B \subseteq V \cap M$. Hence, $M$ is $\pi p$-normal subspace.

Since every closed-and-open (clopen) subset is a closed domain, then we have the following corollary.

**Corollary 3.11** $\pi p$-Normality is a hereditary with respect to clopen subspaces.

Now, let us recall the following definitions.

**Definition 3.12** A function $f : X \rightarrow Y$ is said to be:

(a) $\pi$-continuous, see [1, 10], (resp. rc-continuous, see [5]) if $f^{-1}(F)$ is $\pi$-closed (resp. closed domain) set in $X$ for each closed (resp. closed domain) set $F$ in $Y$.

(b) almost $p$-irresolute, see [8], if for each $x \in X$ and each $p$-neighborhood $V$ of $f(x)$ in $Y$, $p\text{cl}(f^{-1}(V))$ is a $p$-neighborhood of $x$ in $X$.

(c) $Mp$-closed ($Mp$-open), see [8], if $f(U)$ is $p$-closed (resp. $p$-open) set in $Y$ for each $p$-closed (resp. $p$-open) set $U$ in $X$.

Now, we prove the following results on the invariance of $\pi p$-normality.

**Theorem 3.13** If $f : X \rightarrow Y$ is a continuous $Mp$-open rc-continuous and almost $p$-irresolute surjection from a $\pi p$-normal space $X$ onto a space $Y$, then $Y$ is $\pi p$-normal.

*Proof.* Let $A$ be a closed subset of $Y$ and $B$ be a $\pi$-open subset of $Y$ such that $A \subseteq B$. By continuity and rc-continuity of $f$, we obtain that $f^{-1}(A)$ is closed in $X$ and $f^{-1}(B)$ is $\pi$-open in $X$ such that $f^{-1}(A) \subseteq f^{-1}(B)$. Since $X$ is $\pi p$-normal, then by the Theorem 3.1, there exists a $p$-open set $U$ of $X$ such that $f^{-1}(A) \subseteq U \subseteq p\text{cl}(U) \subseteq f^{-1}(B)$. Then, $f(f^{-1}(A)) \subseteq f(U) \subseteq f(p\text{cl}(U)) \subseteq f(f^{-1}(B))$. Since $f$ is $Mp$-open almost $p$-irresolute surjection, we obtain that $A \subseteq f(U) \subseteq p\text{cl}(f(U)) \subseteq B$ and $f(U)$ is $p$-open set in $Y$. Hence by the Theorem 3.1, $Y$ is $\pi p$-normal space. □

**Theorem 3.14** If $f : X \rightarrow Y$ is an $Mp$-open $\pi$-continuous almost $p$-irresolute function from a $\pi p$-normal space $X$ onto a space $Y$, then $Y$ is $\pi p$-normal.

*Proof.* Let $A$ be a closed set of $Y$ and let $B$ be a $\pi$-open set in $Y$ such that $A \subseteq B$. Then by $\pi$-continuity of $f$, $f^{-1}(A)$ is $\pi$-closed (hence closed) and $f^{-1}(B)$ is $\pi$-open set in $X$ such that $f^{-1}(A) \subseteq f^{-1}(B)$. By $\pi p$-normality of $X$,
there exists a $p$-open set $U$ of $X$ such that $f^{-1}(A) \subseteq U \subseteq p\text{cl}(U) \subseteq f^{-1}(B)$. Since $f$ is $Mp$-open almost $p$-irresolute surjection, we obtain that $A \subseteq f(U) \subseteq p\text{cl}(f(U)) \subseteq B$, where $f(U)$ is $p$-open set in $Y$. Hence by the Theorem 3.1, $Y$ is $\pi p$-normal. □

**Theorem 3.15** If $f : X \rightarrow Y$ is an $Mp$-closed $\pi$-continuous function from a $\pi p$-normal space $X$ onto a space $Y$, then $Y$ is $\pi p$-normal.

**Proof.** The proof is routine and hence omitted. □

**Theorem 3.16** If $f : X \rightarrow Y$ is an a closed $\pi$-continuous surjection and $X$ is $\pi$-normal, then $Y$ is $\pi p$-normal.

**Proof.** Let $A$ and $B$ be disjoint closed sets in $Y$ such that $A$ is $\pi$-closed. By $\pi$-continuity of $f$, $f^{-1}(A)$ and $f^{-1}(B)$ are disjoint $\pi$-closed sets of $X$. Since $X$ is $\pi$-normal, there exist disjoint open sets $U$ and $V$ of $X$ such that $f^{-1}(A) \subseteq U$ and $f^{-1}(B) \subseteq V$. By the Proposition 6. in [9], there are disjoint $\alpha$-open sets $G$ and $H$ in $Y$ such that $A \subseteq G$ and $B \subseteq H$. Since every $\alpha$-open set is $p$-open, then $G$ and $H$ are disjoint $p$-open sets containing $A$ and $B$, respectively. Therefore, $Y$ is $\pi p$-normal. □

## 4 Conclusion

We introduced a weaker version of $p$-normality called $\pi p$-normality. We proved that it is a topological property and a hereditary property with respect to closed domain subspaces. We gave some characterizations and preservation theorems of it. Some counterexamples were given and some basic properties were presented.

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**References**


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