On Generalized Quasi Einstein Manifold
Admitting $W_2$-Curvature Tensor

Shyamal Kumar Hui

Nikhil Banga Sikshan Mahavidyalaya
Bishnupur – 722122, Bankura
West Bengal, India
shyamal_hui@yahoo.co.in

Richard S. Lemence

Ochanomizu University
2-1-1 Otsuka, Bunkyo-ku, Tokyo 112-8610 Japan
Institute of Mathematics
College of Science, University of the Philippines
Diliman, Quezon City 1101 Philippines
rslemence@gmail.com

Abstract

A new curvature tensor, called $W_2$-curvature tensor, was defined by Pokhariyal and Mishra in [7]. In this paper, we study generalized quasi Einstein manifolds admitting $W_2$-curvature tensor. The existence of such manifolds is ensured by some interesting examples.

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1 Introduction

It is well known that a Riemannian manifold $(M^n, g)(n > 2)$ is Einstein if its Ricci tensor $S$ of type $(0, 2)$ is of the form $S = \alpha g$, where $\alpha$ is a constant, which turns into $S = \frac{r}{n} g$, $r$ being the (constant) scalar curvature of the manifold.

In [6], Karcher stated that a conformally flat perfect fluid space-time has
the geometric structure of quasi-constant curvature. A manifold of quasi-
constant curvature is a natural sub-class of quasi Einstein manifold [2]. Study
on quasi Einstein manifolds help us to have a deeper understanding of the
global characteristics of the universe including its topology [1]. Hence, The
notion of quasi Einstein manifolds was introduced by M. C. Chaki and R. K.
Maity [2]. A Riemannian manifold \((M^n, g)(n > 2)\) is said to be quasi Einstein
manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and satisfies
the following:

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y),
\]

(1)

where \(\alpha, \beta\) are scalars of which \(\beta \neq 0\) and \(A\) is a nowhere vanishing 1-form
defined by \(g(X, \rho) = A(X)\) for all vector fields \(X\); \(\rho\) being a unit vector field,
called the generator of the manifold. An \(n\)-dimensional manifold of this kind is
denoted by \((QE)_n\). The scalars \(\alpha, \beta\) are known as the associated scalars. The
quasi Einstein manifolds has also been studied by U. C. De and G. C. Ghosh
[3].

As a generalization of quasi Einstein manifold, in [4], U. C. De and G. C.
Ghosh introduced and studied the notion of generalized quasi Einstein man-
ifold. A Riemannian manifold \((M^n, g)(n \geq 3)\) is said to be generalized quasi
Einstein manifold if its Ricci tensor \(S\) of type \((0, 2)\) is not identically zero and
satisfies the following:

\[
S(X, Y) = \alpha g(X, Y) + \beta A(X)A(Y) + \gamma B(X)B(Y),
\]

(2)

where \(\alpha, \beta, \gamma\) are scalars of which \(\beta \neq 0, \gamma \neq 0\) and \(A, B\) are nowhere vanishing 1-forms such that \(g(X, \rho) = A(X)\), \(g(X, \mu) = B(X)\) for all vector fields \(X\).
The unit vectors \(\rho\) and \(\mu\) corresponding to the 1-forms \(A\) and \(B\) are orthogonal
to each other. Also \(\rho\) and \(\mu\) are known as the generators of the manifold.
Such an \(n\)-dimensional manifold is denoted by \(G(QE)_n\). The significance of
a \(G(QE)_n\) lies in the fact that a 4-dimensional semi-Riemannian manifold is
relevant to the study of a general relativistic fluid space-time admitting heat
flux [8]. The generalized quasi Einstein manifolds has also been studied by A.
A. Shaikh and S. K. Hui [9].

In Cosmology, space-time models are studied in order to represent the
different phases in the evolution of the Universe which can be divided into
three phases:

1. Initial Phase. The initial phase is just after the big bang when the
effects of both viscosity and heat flux were quite pronounced.

2. Intermediate Phase. The effect of viscosity was no longer significant
but the heat flux was still not negligible.
3. Final Phase. This phase extends to the present state of the universe.

In this phase, both the effects of viscosity and the heat flux have become negligible and the matter content of the universe may be assumed to be a perfect fluid.

The significance of the study of \(G(QE)_n\) and \((QE)_n\) lies in the fact that \(G(QE)_n\) space-time manifold represents the second phase while \((QE)_n\) the space-time manifold correspond to the third phase in the evolution of the universe [5]. One way of understanding the geometric properties of such manifolds is by studying the tensors these manifolds admit.

In 1970 G. P. Pokhariyal and R. S. Mishra [7] were introduced new tensor fields, called \(W_2\) and \(E\) tensor fields, in a Riemannian manifold and studied their properties. According to them [7], a \(W_2\)-curvature tensor on a manifold \((M^n, g)(n > 3)\) is defined by

\[
W_2(X, Y, Z, U) = R(X, Y, Z, U) + \frac{1}{n-1} \left[ g(X, Z)S(Y, U) - g(Y, Z)S(X, U) \right].
\]

The object of the present paper is to study \(W_2\)-curvature tensor field in a generalized quasi Einstein manifold.

The paper is organized as follows. Section 2 is concerned with preliminaries. Section 3 is devoted to the study of \(W_2\)-flat generalized quasi Einstein manifolds. It is shown that in a \(W_2\)-flat \(G(QE)_n(n > 3)\), either the associated scalars \(\beta\) and \(\gamma\) are equal or the vector fields \(\rho\) and \(\mu\) corresponding to the 1-forms \(A\) and \(B\) respectively are co-directional.

In section 4, we investigate the \(G(QE)_n(n > 3)\) satisfying the condition \(W_2 \cdot S = 0\) and it is proved that in this case, either the associated scalars \(\beta\) and \(\gamma\) are equal or the curvature tensor \(R\) satisfies a definite condition. Section 5 deals with study of \(W_2\)-flat Ricci-semisymmetric \(G(QE)_n(n > 3)\). In [4] it is proved that if a \(G(QE)_n\) is Ricci-semisymmetric then the associated scalars \(\beta\) and \(\gamma\) are equal. However, in section 5, it is shown that if in a \(W_2\)-flat \(G(QE)_n(n > 3)\), with non-zero eigenvalue of the Ricci-operator, is Ricci-semisymmetric, then either the associated scalars \(\beta\) and \(\gamma\) are equal or the vector fields \(\rho\) and \(\mu\) corresponding to the 1-forms \(A\) and \(B\) are co-directional.

The last section provides the existence of proper \(G(QE)_n\).

2 Preliminaries

In this section we will obtain some formulas of \(G(QE)_n\), which will be required in the sequel. Let \(\{e_i : i = 1, 2, \ldots, n\}\) be an orthonormal frame field at any point of the manifold. Then setting \(X = Y = e_i\) in (2) and taking summation over \(i, 1 \leq i \leq n\), we obtain

\[
r = n\alpha + \beta + \gamma,
\]
where $r$ is the scalar curvature of the manifold. Also, from (2), we have
\[ S(X, \rho) = (\alpha + \beta)A(X), \quad S(\rho, \rho) = \alpha + \beta, \]
and
\[ S(X, \mu) = (\alpha + \gamma)B(X), \quad S(\mu, \mu) = \alpha + \gamma \]
\[ S(\rho, \mu) = 0. \]
Let $Q$ be the Ricci-operator, i.e., $g(QX,Y) = S(X,Y)$ for all $X, Y$.

3 \hspace{0.5cm} W_2\text{-flat} \hspace{0.5cm} \text{generalized quasi Einstein manifolds}

Let us consider a Kenmotsu manifold $G(QE)_n(n > 3)$, which is $W_2$-flat. Then from (3), we get
\[ R(X,Y,Z,U) = \frac{1}{n-1} \left[ g(Y,Z)S(X,U) - g(X,Z)S(Y,U) \right]. \]
Using (2) in (8), we obtain
\[ R(X,Y,Z,U) = \frac{1}{n-1} \left\{ \alpha \{g(X,U)g(Y,Z) - g(Y,U)g(X,Z)\} \right. \\
+ \quad \left. \beta \{A(X)g(Y,Z) - A(Y)g(X,Z)\}A(U) \right. \\
+ \quad \left. \gamma \{B(X)g(Y,Z) - B(Y)g(X,Z)\}B(U) \right\}. \]
Setting $Z = \rho$ and $U = \mu$ in (9), we have
\[ R(X,Y,\rho,\mu) = \frac{1}{n-1} (\alpha + \gamma) \left[ A(Y)B(X) - A(X)B(Y) \right]. \]
Again plugging $Z = \mu$ and $U = \rho$ in (9), we get
\[ R(X,Y,\mu,\rho) = \frac{1}{n-1} (\alpha + \beta) \left[ A(X)B(Y) - A(Y)B(X) \right]. \]
From (10) and (11), it follows that
\[ (\beta - \gamma) \left[ A(X)B(Y) - A(Y)B(X) \right] = 0, \]
which implies that either $\beta = \gamma$ or
\[ A(X)B(Y) = A(Y)B(X), \]
that is, the vector fields $\rho$ and $\mu$ are co-directional. Thus we can state the following:

**Theorem 1.** In a $W_2$-flat $G(QE)_n(n > 3)$, either the associated scalars $\beta$ and $\gamma$ of the manifold are equal or the vector fields $\rho$ and $\mu$ corresponding to the 1-forms $A$ and $B$ respectively are co-directional.
4 $G(QE)_n(n > 3)$ satisfying the condition $W_2 \cdot S = 0$

Let us take a $G(QE)_n(n > 3)$ with $W_2 \cdot S = 0$. Then we get

$$S(W_2(X, Y)Z, U) + S(Z, W_2(X, Y)U) = 0.$$ (13)

Using (2) in (13), we get

$$(\alpha + \gamma)W_2(X, Y, \rho, \mu) + (\alpha + \beta)W_2(X, Y, \mu, \rho) = 0.$$ (15)

In view of (3), we have from (15) that

$$R(X, Y, \rho, \mu) = \frac{1}{n-1}(2\alpha + \beta + \gamma)\{A(Y)B(X) - A(X)B(Y)\},$$ (16)

provided $\gamma - \beta \neq 0$. This leads to the following:

**Theorem 2.** If a $G(QE)_n(n > 3)$ satisfies the condition $W_2 \cdot S = 0$, then either the associated scalars $\beta$ and $\gamma$ are equal or the curvature tensor $R$ of the manifold satisfies the property (16).

5 $W_2$-flat $G(QE)_n(n > 3)$ with $R(X, Y) \cdot S = 0$

Let us consider a $W_2$-flat $G(QE)_n(n > 3)$. Then we get the relation (8), i.e.,

$$R(X, Y)Z = \frac{1}{n-1}[g(Y, Z)QX - g(X, Z)QY].$$ (17)

Since the manifold satisfies $R(X, Y) \cdot S = 0$, we get

$$S(R(X, Y)Z, U) + S(Z, R(X, Y)U) = 0.$$ (18)

Using (17) in (18), we get

$$g(Y, Z)S(QX, U) - g(X, Z)S(QY, U) + g(Y, U)S(QX, Z) - g(X, U)S(QY, Z) = 0.$$ (19)

Let $\lambda$ be the eigenvalue of the endomorphism $Q$ corresponding to an eigenvector $X$. Then $QX = \lambda X$, i.e., $S(X, U) = \lambda g(X, U)$ and hence

$$S(QX, U) = \lambda S(X, U).$$ (20)
By virtue of (20), it follows from (19) that
\[ g(Y, Z)S(X, U) - g(X, Z)S(Y, U) + g(Y, U)S(X, Z) - g(X, U)S(Y, Z) = 0, \]
provided \( \lambda \neq 0 \). Setting \( Z = \rho \) and \( U = \mu \) in (21), we get
\[ (\beta - \gamma)\{A(X)B(Y) - A(Y)B(X)\} = 0. \]
(22)
From (22), we get either \( \beta = \gamma \) or the vector fields \( \rho \) and \( \mu \) are co-directional.

Thus we can state the following:

**Theorem 3.** If a \( W_2 \)-flat \( G(QE)_n(n > 3) \) with non-zero eigenvalue of the Ricci-operator \( Q \) is Ricci-semisymmetric, then either the associated scalars \( \beta \) and \( \gamma \) of the manifold are equal or the vector fields \( \rho \) and \( \mu \) corresponding to the 1-forms \( A \) and \( B \) respectively are co-directional.

### 6 Some Examples of \( G(QE)_n \)

**Example 6.1.** [4] If the Ricci tensor \( S \) of a Riemannian manifold satisfies the relation
\[ S(Y, Z)S(X, W) - S(X, Z)S(Y, W) = \sigma[g(Y, Z)g(X, W) - g(X, Z)g(Y, W)], \]
where \( \sigma \) is a non-zero scalar, then the manifold is a generalized quasi Einstein manifold.

**Example 6.2.** As a generalization of the manifold of quasi constant curvature, U. C. De and G. C. Ghosh [4] introduced the notion of the manifold of generalized quasi constant curvature. Also it is proved that [4] a manifold of generalized quasi constant curvature is a generalized quasi Einstein manifold.

**Example 6.3.** [4] A 2-quasi umbilical hypersurface of a Euclidean space is a generalized quasi Einstein manifold.

**Example 6.4.** [9] Let \( (M^4, g) \) be a Riemannian manifold endowed with the metric given by
\[ ds^2 = g_{ij}dx^idx^j = (1 + 2p)[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + (dx^4)^2], (i, j = 1, 2, 3, 4), \]
where \( p = \frac{x^1}{x^4} \) and \( k \) is a non-zero constant. Then \( (M^4, g) \) is a \( G(QE)_4 \) with non-vanishing scalar curvature which is not quasi-Einstein.

**Example 6.5.** [9] Let \( (M^4, g) \) be a Riemannian manifold endowed with the metric given by
\[ ds^2 = e^{2x^1}(dx^1)^2 + \sin^2 x^1[(dx^2)^2 + (dx^3)^2 + (dx^4)^2], \]
where \( 0 < x^1 < \frac{\pi}{2} \) but \( x^1 \neq \frac{\pi}{4} \). Then \( (M^4, g) \) is a \( G(QE)_4 \) with non-vanishing scalar curvature which is not quasi-Einstein.

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