Common Fixed Point Theorems for Maps Altering Distance under a Contractive Condition of Integral Type for Pairs of Sumcompatible Maps

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Abstract

In this paper we obtain a unique common fixed point theorem under a contractive condition of integral type for four self maps using generalized altering distance function in four variables. which generalizes and improves the main theorems of [1].

1 Introduction

M.S.Khan [9] introduced the altering distances and used it for solving fixed points problems in metric spaces. Recently many authors, for example [3] and [10] used the altering distance function and obtained some fixed point theorems. Choudhury [6] in 2005 introduced generalized distance function in three variables and obtain a common fixed point theorem for a pair of self maps in a complete metric space. The main aim of this paper is to prove the existence and uniqueness of common fixed points of two pairs of compatible of sumcompatible mappings by using a
generalized distance function of four variables under a contractive condition of integral type. Recently, H.Bouhadjera and C.Godet Thbie [5] obtained common fixed point theorems for some sumcompatible maps type mappings, in this paper we show two OWC maps are subcompatible, however the converse is not true in general.

**Definition 1.1.** Let \((X, d)\) be a metric space. Maps \(f\) and \(g : X \to X\) are said to be subcompatible iff there exists a sequence \(\{x_n\}\) in \(X\) such that \(\lim_{n \to \infty} f_n = \lim_{n \to \infty} g_n = t, t \in X\) and which satisfy \(\lim_{n \to \infty} d(fgx_n;gfx_n) = 0\).

Obviously, two OWC maps are subcompatible, however the converse is not true in general. The example below shows that there exist subcompatible maps which are not OWC.

**Example 1.2.** Let \(X = [0, \infty)\) with the usual metric \(d\). Define \(f\) and \(g\) as follows:

\[
f(x) = x^2 \quad \text{and} \quad g(x) = \begin{cases} x + 2 & \text{if } x \in [0,4] \cup (9, \infty) \\ x + 12 & \text{if } x \in [4,9] \end{cases}
\]

(1.1)

Let \(\{x_n\}\) be a sequence in \(X\) defined by \(x_n = 2 + \frac{1}{n}\) for \(n = 1,2,3,\ldots\).

Then

\[\lim_{n \to \infty} fx_n = \lim_{n \to \infty} x_n^2 = 4 = \lim_{n \to \infty} gfx_n = \lim_{n \to \infty}(x_n + 2)\]  (1.2)

And

\[fgx_n = f(x_n + 2) = (x_n + 2)^2 \to 16 \text{ when } n \to \infty\]  (1.3)

\[gfx_n = g(x_n^2) = x_n^2 + 12 \to 16 \text{ when }\]  (1.4)

Thus, \(\lim_{n \to \infty} d(fgx_n;gfx_n) = 0\)

That is, \(f\) and \(g\) are subcompatible.

On the other hand, we have \(fx = gx\) iff \(x = 2\) and

\[fg(2) = f(4) = 4^2 = 16\]

\[gf(2) = g(4) = 4 + 2 = 6\]

Then, \(f(2) = 4 = g(2)\) but \(fg(2) = 16 \neq 6 = gf(2)\), hence \(f\) and \(g\) are not own.

**Definition 1.3.** Let \(\Psi_n\) denote the set of all the variables

(i) \(\psi\) is continuous;

(ii) \(\psi\) is monotone increasing in all the variables;

(iii) \(\psi(t_1, t_2, t_3, t_4, \ldots, t_n) = 0\) if and only if \(t_1 = t_2 = t_3 = t_4 = \cdots = t_n = 0\)

we define \(\phi(x) = \psi(x,x,x,x,\ldots)\) for \(x \in [0, \infty]\). Clear, \(\phi(x) = 0\) if only if \(x = 0\).
Example of $\psi$ are $\psi(t_1,t_2,t_3,t_4,...,t_n) = k \max\{t_1,t_2,t_3,...,t_n\}$, for $k > 0$ \hspace{1cm} (1.6)

$\psi(t_1,t_2,t_3,t_4,...,t_n) = t_1^{a_1} + t_2^{a_2} + ... + t_n^{a_n}$ ; $a_1,a_2,a_3,... \geq 1$ \hspace{1cm} (1.7)

2 Main Result

Theorem 2.1. Let $(X,d)$ be a complete metric space and $f,g,S,T : X \to X$ such that:

(i) $\int_0^{\phi_1(d(fx,gy))} \varphi(t)dt \leq \int_0^{\psi_1(d(Sx,Ty),d(Sx,fx),d(Ty,gy))} \frac{1}{2}(d(Sx,gy)+d(Ty,fx))) \varphi(t)dt - \int_0^{\psi_2(d(Sx,Ty),d(Sx,fx),d(Ty,gy))} \frac{1}{2}(d(Sx,gy)+d(Ty,fx))) \varphi(t)dt$

For all $x,y \in X$ where $\psi_1, \psi_2 \in \mathcal{U}_4$ and $\phi_1 = \psi(x,x,x,x) \in [0,\infty)$

(ii) One of four mapping $f,g,S$ and $T$ is continuous;

(iii) $(f,S)$ and $(g,T)$ are subcompatible.

(iv) pairs; $f(x) \subseteq T(x)$, $g(x) \subseteq S(x)$

(v) where $\varphi : R^+ \to R^+$ is a lebesgue integrable mapping which is sum able, non negative and such that for each $\varepsilon > 0$, $\int_0^{\varepsilon} \varphi(t)dt > 0$ then $f,g,S,T$ have a unique common fixed point in $X$.

Proof. Let $x_0 \in X$, be an arbitrary point. From (iv) construct the sequence $\{x_n\}$ and $\{y_n\}$ in $X$ such that:

$$fx_{2n} = Tx_{2n+1} = y_{2n}$$

$$gx_{2n+1} = Sx_{2n+2} = y_{n+1} \hspace{1cm} ; \hspace{1cm} n = 0,1,2,3,...$$

Let $a_n = d(y_n,y_{n+1})$. Putting $x = x_{2n}$, $y = x_{2n+1}$ in (i) we get

$$\int_0^{\phi_1(a_{2n})} \varphi(t)dt \leq \int_0^{\psi_1(a_{2n-1},a_{2n-1},a_{2n},a_{2n})} \varphi(t)dt - \int_0^{\psi_2(a_{2n-1},a_{2n-1},a_{2n},a_{2n})} \varphi(t)dt \hspace{1cm} (2.8)$$

If $a_{2n-1} \leq a_{2n}$ then

$$\int_0^{\phi_1(a_{2n})} \varphi(t)dt \leq \int_0^{\psi_1(a_{2n},a_{2n-1},a_{2n})} \varphi(t)dt - \int_0^{\psi_2(a_{2n},a_{2n},a_{2n},a_{2n})} \varphi(t)dt \hspace{1cm} (2.10)$$
\[ \int_0^{\psi_2(a_{2n-1}, a_{2n-1}, a_{2n-1})} \varphi(t) \, dt < \int_0^{\phi_1(a_{2n})} \varphi(t) \, dt \]  

(2.11)

Which is contradiction? Hence \( a_{2n} \leq a_{2n-1} \), \( n = 0, 1, 2, 3, 4, \ldots \)

Similarly by putting \( x = x_{2n+1}, y = x_{2n+1} \) in (i) we can show that \( a_{2n+1} \leq a_{2n} \), \( n = 0, 1, 2, 3, 4, \ldots \) thus \( a_{n+1} \leq a_n \), \( n = 0, 1, 2, 3, 4, \ldots \) so that \( \{a_n\} \) is a decreasing sequence of non-negative real numbers and hence convergent to some \( a \in R \)

Let \( b = \lim_{n \to \infty} \frac{1}{2} d(y_n, y_{n+1}) \).

Letting \( n \to \infty \) in (i) we get

\[ \int_0^{\phi_1(a)} \varphi(t) \, dt \leq \int_0^{\psi_1(a, a, a, a)} \varphi(t) \, dt - \int_0^{\psi_2(a, a, a, b)} \varphi(t) \, dt = \]

\[ \int_0^{\phi_1(a)} \varphi(t) \, dt - \int_0^{\psi_2(a, a, a, b)} \varphi(t) \, dt \]

Thus

\[ \psi_2(a, a, a, a, b) = 0 \]

so that \( a = b = 0 \) hence

\[ \lim_{n \to \infty} d(y_n, y_{n+1}) = 0 \]  

(2.12)

To show that \( \{y_n\} \) is Cauchy sequence, it is sufficient to show that the subsequence \( \{y_{2n}\} \) of \( \{y_n\} \) is Cauchy sequence in view of (2.12). If \( \{y_{2n}\} \) is not Cauchy. There exists \( \alpha > 0 \) and monotone increasing sequence of natural numbers \( \{2m(k)\} \) and \( \{2n(k)\} \) such that \( n(k) > m(k) \),

\[ d(y_{2m(k)}, y_{2n(k)}) \geq \alpha \quad \text{and} \quad d(y_{2m(k)}, y_{2n(k) - 2}) < \varepsilon \]  

(2.13)

From (2.13)

\[ \varepsilon \leq d(y_{2m(k)}, y_{2n(k)}) \]

\[ \leq d(y_{2m(k)}, y_{2n(k) - 2}) + d(y_{2n(k) - 2}, y_{2n(k) - 2}) + d(y_{2n(k) - 2}, y_{2n(k)}) \]  

(2.14)

\[ \leq \varepsilon + d(y_{2n(k) - 1}, y_{2n(k)}) + d(y_{2n(k) - 1}, y_{2n(k)}) \]

Letting \( k \to \infty \) using (2.12) we have
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\[ \lim_{n \to \infty} d\left(y_{2m(k)}, y_{2n(k)}\right) = \varepsilon \]  
(2.15)

Letting \( k \to \infty \) using (2.12), and (2.13) in

\[ |d\left(y_{2m(k)}, y_{2n(k)}\right) - d\left(y_{2m(k)}, y_{2n(k)+1}\right)| \leq d\left(y_{2n(k)}, y_{2n(k)+1}\right) \]  
(2.16)

We get

\[ \lim_{n \to \infty} d\left(y_{2n(k)+1}, y_{2m(k)}\right) = \varepsilon \]  
(2.17)

Letting \( k \to \infty \) and using (2.12), (2.13) in

\[ |d\left(y_{2m(k)-1}, y_{2n(k)}\right) - d\left(y_{2m(k)}, y_{2n(k)}\right)| \leq d\left(y_{2m(k)}, y_{2m(k)-1}\right) \]  

We get

\[ \lim_{n \to \infty} d\left(y_{2n(k)}, y_{2m(k)-1}\right) = \varepsilon \]  
(2.18)

Putting in \( x = x_{2m(k)} \), \( y = x_{2n(k)-1} \) in (i) we have

\[ \int_0^{\phi_1 d(y_{2m(k)}, y_{2n(k)+1})} \varphi(t) dt \leq \]

\[ \int_0^{\psi_1(d(y_{2m(k)-1}, y_{2n(k)}), d(y_{2m(k)}, y_{2n(k)}), d(y_{2n(k)+1}, y_{2n(k)}))} d\left(y_{2m(k)-1}, y_{2n(k)+1}\right) + d\left(y_{2m(k)}, y_{2n(k)}\right) \]

\[ - \int_0^{\psi_2(d(y_{2m(k)-1}, y_{2n(k)}), d(y_{2m(k)-1}, y_{2m(k)}), d(y_{2n(k)+1}, y_{2n(k)}))} d\left(y_{2m(k)-1}, y_{2n(k)+1}\right) + d\left(y_{2m(k)}, y_{2n(k)}\right) \]

\[ \varphi(t) dt \]

Letting \( k \to \infty \) and using (2.12), (2.13), (2.15), (2.16) and (2.18) we get

\[ \int_0^{\phi_1(\varepsilon)} \varphi(t) dt \leq \int_0^{\psi_1(\varepsilon, 0, 0, \varepsilon)} \varphi(t) dt - \int_0^{\psi_2(\varepsilon, 0, 0, \varepsilon)} \varphi(t) dt \]

\[ < \int_0^{\psi_1(\varepsilon, \varepsilon, \varepsilon, \varepsilon)} \varphi(t) dt = \int_0^{\phi_1(\varepsilon)} \varphi(t) dt \]
It is a contraction. Therefore \( \{y_{2n}\} \) is a Cauchy sequence hence is a Cauchy sequence from (3) since \( X \) is complex, there exists \( z \in X \) such that \( y_n \to z \) as \( n \to \infty \).

Case: suppose \( S \) is continuous. Then \( Sf x_{2n} \to Sz, S^2 x_{2n} \to Sz \) since \( (f, S) \) is subcompatible. We have \( fS x_{2n} \to Sz \).

Step (I): putting in \( x = S x_{2n} \), \( y = x_{2n+1} \) (i) we have:

\[
\int_0^{\Phi_1 d(fS x_{2n}, g x_{2n+1})} \varphi(t) dt \leq \\
\int_0^{\psi_1(d(S^2 x_{2n} T x_{2n+1}) d(S^2 x_{2n} fS x_{2n}), (T x_{2n+1} g x_{2n+1}), \frac{1}{2}(S^2 x_{2n} g x_{2n+1} + (T x_{2n+1} fS x_{2n})))} \varphi(t) dt \\
- \int_0^{\psi_2(d(S^2 x_{2n} T x_{2n+1}), (S^2 x_{2n} fS x_{2n}), (T x_{2n+1}, g x_{2n+1}), \frac{1}{2}(S^2 x_{2n} g x_{2n+1} + (T x_{2n+1} fS x_{2n})))} \varphi(t) dt
\]

Therefore:

\[
\int_0^{\Phi_1 d(Sz, z)} \varphi(t) dt \leq \int_0^{\psi_1(d(Sz, z), d(Sz, z), d(Sz, z))} \varphi(t) dt \leq \int_0^{\Phi_1 d(Sz, z)} \varphi(t) dt - \int_0^{\psi_1(d(Sz, z), 0, 0, d(Sz, z))} \varphi(t) dt < \int_0^{\Phi_1 d(Sz, z)} \varphi(t) dt
\]

It is contradiction if \( Sz \neq z \) hence \( Sz = z \)

Step (II): putting in \( x = z, y = x_{2n+1} \) (i) and letting \( n \to \infty \); we get \( fz = z \).

Step (III): since \( z = fz \in f(x) \subseteq T(x) \) there exists \( u \in X \) such that \( z = Tu \) putting \( x = x_{2n} \), \( y = u \) in (i), we get \( gz = z \) so that \( gz = Tz \) and hence \( gu = Tu \) since \( (g, T) \) is subcompatible we have \( gTu = T gu \) so that \( gz = Tz \)

Step (IV): putting \( x = z \), \( y = z \), in (i) we get \( g(z) = z \) so that \( gz = Tz \).

Thus \( z \) is a common fixed point of \( f, g, S \) and \( T \).

Case: suppose is continuous, then \( f^2 x_{2n} \to fz \), \( fS x_{2n} \to fz \) since is similarly we can show that \( z \) is a common fixed point of \( f, g, S \) and \( T \) when \( g \) or \( T \) is continuous as a previous two cases uniqueness of common fixed point follows easily from (i). \( \Box \)

**Example 2.2.** Let \( X = [0,1] \) with the usual metric \( (x,y) = |x-y| \). Define \( f, g, T, S : X \to X ; fx = \frac{x}{2} ; gy = \frac{y}{2} ; Sx = x ; Ty = y \).

Let \( \psi_1(t_1, t_2, t_3, t_4) = Max\{t_1, t_2, t_3, t_4\} ; \phi(t) = 2t ; \psi_2 = \psi_1 \frac{1}{2} \psi_1 \)
Then $\phi_i(t) = t \quad \forall t \in [0, \infty)$. 

$$
\left(\frac{x - y}{2}\right)^2 \leq \frac{1}{2} \max \left\{ \left| x - y \right|^2, \left| x - \frac{x_i}{2}\right|^2, \left| y - \frac{y_i}{2}\right|^2, \left(\frac{1}{2}\left[\left| x - \frac{y_i}{2}\right| + \left| y - \frac{x_i}{2}\right|\right]\right)^2 \right\}
$$

For all $x, y \in X$, it follows that the condition (i).

Let $x_n$ be any sequence in $X$ such that $fx_n \to t$ and $Sx_n \to t$ for some $t$ in $X$. Then $t = 0$; $d(fSx_n, Sfx_n) \to 0$. Hence $\{f, S\}$ subcompatible. We have common fixed point in $X$.

References


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